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SOME PROPERTIES OF THE BERNOULLI NUMBERS OF THE SECOND KIND AND THEIR GENERATING FUNCTION

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Abstract. In the paper, the authors find a common solution to three series of differential equations related to the generating function of the Bernoulli numbers of the second kind and present a recurrence relation, an explicit formula in terms of the Stirling numbers of the first kind, and a determinantal expression for the Bernoulli numbers of the second kind.

1. Introduction

In number theory, the Bernoulli numbers of the second kind b_n can be generated by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} b_n x^n. \tag{1}$$

They are also known as the Cauchy numbers of the first kind, the Gregory coefficients, or logarithmic numbers. The first few Bernoulli numbers of the second kind b_n are

$$b_0 = 1, \quad b_1 = \frac{1}{2}, \quad b_2 = -\frac{1}{12}, \quad b_3 = \frac{1}{24}, \quad b_4 = -\frac{19}{720}, \quad b_5 = \frac{3}{160}.$$

Before stating main results of this paper, we recall some known results published in recent years about the Bernoulli numbers of the second kind b_n as follows.

In [6, p. 2], the Bernoulli numbers of the second kind b_n for $n \ge 0$ were expressed as

$$b_n = \frac{1}{n!}\sum_{k=0}^{n} \frac{s(n,k)}{k+1} \tag{2}$$

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in terms of the Stirling numbers of the first kind $s(n,k)$ which can be generated by

$$\frac{[\ln(1+x)]^k}{k!}=\sum_{n=k}^{\infty}s(n,k)\frac{x^n}{n!},\quad |x|<1.$$

In [8, Theorem 3.1 and Corollary 2.3], by induction and other techniques, the Bernoulli numbers of the second kind b_n for $n\ge 2$ were expressed by

$$b_n=(-1)^n\left[\frac{1}{(n+1)!}+\frac{1}{n!}\sum_{k=2}^{n}\frac{a_{n,k}-na_{n-1,k}}{k!}\right]$$

and the Stirling numbers of the first kind $s(n,k)$ for $n\ge k\ge 1$ were expressed by

$$s(n,k)=(-1)^{n+k}(n-1)!\sum_{\ell_1=1}^{n-1}\frac{1}{\ell_1}\sum_{\ell_2=1}^{\ell_1-1}\frac{1}{\ell_2}\cdots\sum_{\ell_{k-2}=1}^{\ell_{k-3}-1}\frac{1}{\ell_{k-2}}\sum_{\ell_{k-1}=1}^{\ell_{k-2}-1}\frac{1}{\ell_{k-1}},$$

where $a_{n,2}=(n-1)!$ and, for $n+1\ge i\ge 3$,

$$a_{n,i}=(i-1)!(n-1)!\sum_{\ell_1=1}^{n-1}\frac{1}{\ell_1}\sum_{\ell_2=1}^{\ell_1-1}\frac{1}{\ell_2}\cdots\sum_{\ell_{i-3}=1}^{\ell_{i-4}-1}\frac{1}{\ell_{i-3}}\sum_{\ell_{i-2}=1}^{\ell_{i-3}-1}\frac{1}{\ell_{i-2}}.$$

In [10, Theorem 1.1], basing on some results in [8], the Bernoulli numbers of the second kind b_n for $n\ge 2$ were similarly expressed as

$$b_n=\frac{1}{n!}\sum_{k=1}^{n-1}(-1)^k\frac{s(n-1,k)}{(k+1)(k+2)}. \tag{3}$$

The Cauchy numbers of the second kind c_k can be generated [3, p. 294] by

$$\frac{-t}{(1-t)\ln(1-t)}=\sum_{n=0}^{\infty}c_n\frac{t^n}{n!}$$

which is equivalent to

$$\frac{t}{\ln(1+t)}=c_0+\sum_{n=1}^{\infty}(-1)^n(c_n-nc_{n-1})\frac{t^n}{n!}.$$

Hence, it follows that $b_0=c_0=1$,

$$b_n=(-1)^n\left[\frac{c_n}{n!}-\frac{c_{n-1}}{(n-1)!}\right],\quad c_n=n!\sum_{k=0}^{n}(-1)^kb_k,\quad n\in\mathbb{N}.$$

In [7, Theorem 2.1], the Cauchy numbers of the second kind c_n were represented by the integral

$$c_n=n!\int_0^{\infty}\frac{\mathrm{d}\,u}{u[\pi^2+(\ln u)^2](1+u)^n},\quad n\ge 0. \tag{4}$$

In [32, Theorem 1], making use of the Cauchy integral formula in the theory of complex functions, the Bernoulli numbers of the second kind b_n for $n \in \mathbb{N}$ were represented by the integral

$$\begin{aligned} b_n &= (-1)^{n+1} \int_1^\infty \frac{1}{\{[\ln(t-1)]^2 + \pi^2\} t^n} \,\mathrm{d}\, t \\ &= (-1)^{n+1} \int_0^\infty \frac{1}{[(\ln u)^2 + \pi^2](1+u)^n} \,\mathrm{d}\, u. \end{aligned} \tag{5}$$

Consequently, for $n \in \mathbb{N}$, any one of the integral representations (4) and (5) can be derived from another one.

With the help of (4) and (5) and by some properties of completely monotonic functions (the Laplace transforms), some determinantal inequalities, some product inequalities, the complete monotonicity, and the logarithmic convexity for the Bernoulli numbers of the second kind b_n and for the Cauchy numbers of the second kind c_n were established in [7, 32] respectively. For examples,

(1) the product inequalities

$$\left| \prod_{\ell=1}^{m} \lambda_\ell! b_{\lambda_\ell+1} \right| \le \left| \prod_{\ell=1}^{m} \mu_\ell! b_{\mu_\ell+1} \right|, \quad m \in \mathbb{N}$$

hold for all m-tuples $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_m)$ of nonnegative integers such that

$$\sum_{\ell=1}^{k} \lambda_{[\ell]} \le \sum_{\ell=1}^{k} \mu_{[\ell]}, \quad k = 1, 2, \dots, m-1$$

and $\sum_{\ell=1}^{m} \lambda_\ell = \sum_{\ell=1}^{m} \mu_\ell$, where $\lambda_{[1]} \ge \lambda_{[2]} \ge \dots \ge \lambda_{[n]}$ and $\mu_{[1]} \ge \mu_{[2]} \ge \dots \ge \mu_{[m]}$ are respectively the components of λ and μ in decreasing order;

(2) the infinite sequence $\{c_n\}_{n\ge 0}$ is logarithmically convex.

In this paper, we will find a common solution to three series of differential equations related to the generating function $\frac{x}{\ln(1+x)}$ of the Bernoulli numbers of the second kind b_n and present a recurrence relation, an explicit formula in terms of the Stirling numbers of the first kind $s(n,k)$, and a determinantal expression for the Bernoulli numbers of the second kind b_n.

Our main results can be summarized as the following three theorems.

Theorem 1. *For all $n \ge 2$, the nonlinear ordinary differential equations*

$$F^{(n)}(x) = \frac{(-1)^n}{(1+x)^n} \frac{F(x)}{x} \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} (n-r-1)! (1+x)^r F^{(r)}(x) \tag{6}$$

have the same solution $F(x) = \frac{x}{\ln(1+x)}$.

Theorem 2. *For $n \in \mathbb{N}$, the Bernoulli numbers of the second kind b_n satisfy the recurrence relation*

$$b_n = (-1)^{n+1} \sum_{r=0}^{n-1} \frac{(-1)^r}{n-r+1} b_r. \tag{7}$$

Theorem 3. *For $n \ge 0$, the Bernoulli numbers of the second kind b_n can be represented by*

$$b_n = \frac{1}{n!} \sum_{k=0}^{n} \sum_{m=0}^{k} (-1)^m \binom{k}{m} \frac{s(n+m,m)}{\binom{n+m}{m}} \tag{8}$$

and

$$b_n = \frac{(-1)^n}{n!} \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \binom{1}{0}\frac{-1!}{2} & 1 & \cdots & 0 & 0 \\ 0 & \binom{2}{0}\frac{2!}{3} & \binom{2}{1}\frac{-1!}{2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \binom{n-2}{0}\frac{(-1)^{n-2}(n-2)!}{n-1} & \binom{n-2}{1}\frac{(-1)^{n-3}(n-3)!}{n-2} & \cdots & 1 & 0 \\ 0 & \binom{n-1}{0}\frac{(-1)^{n-1}(n-1)!}{n} & \binom{n-1}{1}\frac{(-1)^{n-2}(n-2)!}{n-1} & \cdots & \binom{n-1}{n-2}\frac{-1!}{2} & 1 \\ 0 & \binom{n}{0}\frac{(-1)^{n}n!}{n+1} & \binom{n}{1}\frac{(-1)^{n-1}(n-1)!}{n} & \cdots & \binom{n}{n-2}\frac{2!}{3} & \binom{n}{n-1}\frac{-1!}{2} \end{vmatrix}. \tag{9}$$

The differential equations

$$\sum_{r=0}^{n} \frac{(-1)^r}{r!} \left[\int_0^1 \frac{u^{n-r}}{(1+xu)^{n-r+1}} \,\mathrm{d}\, u\right] F^{(r)}(x) = 0, \quad n \ge 1 \tag{10}$$

have the same solution $F(x) = \frac{x}{\ln(1+x)}$.

2. Lemmas

In order to prove our main results, we recall several lemmas below.

Lemma 1. *Let $p = p(x)$ and $q = q(x) \ne 0$ be two differentiable functions. Then*

$$\left[\frac{p(x)}{q(x)}\right]^{(k)} = \frac{(-1)^k}{q^{k+1}} \begin{vmatrix} p & q & 0 & \cdots & 0 & 0 \\ p' & q' & q & \cdots & 0 & 0 \\ p'' & q'' & \binom{2}{1}q' & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p^{(k-2)} & q^{(k-2)} & \binom{k-2}{1}q^{(k-3)} & \cdots & q & 0 \\ p^{(k-1)} & q^{(k-1)} & \binom{k-1}{1}q^{(k-2)} & \cdots & \binom{k-1}{k-2}q' & q \\ p^{(k)} & q^{(k)} & \binom{k}{1}q^{(k-1)} & \cdots & \binom{k}{k-2}q'' & \binom{k}{k-1}q' \end{vmatrix} \tag{11}$$

for $k \ge 0$. In other words, the formula (11) can be rewritten as

$$\frac{\mathrm{d}^k}{\mathrm{d}\, x^k}\left[\frac{p(x)}{q(x)}\right] = \frac{(-1)^k}{q^{k+1}(x)} \big|W_{(k+1)\times(k+1)}(x)\big|, \tag{12}$$

where $|W_{(k+1)\times(k+1)}(x)|$ *denotes the determinant of the* $(k+1)\times(k+1)$ *matrix*

$$W_{(k+1)\times(k+1)}(x)=\begin{pmatrix}U_{(k+1)\times 1}(x) & V_{(k+1)\times k}(x)\end{pmatrix},$$

the quantity $U_{(k+1)\times 1}(x)$ *is a* $(k+1)\times 1$ *matrix whose elements* $u_{\ell,1}(x)=p^{(\ell-1)}(x)$ *for* $1\le \ell\le k+1$*, and* $V_{(k+1)\times k}(x)$ *is a* $(k+1)\times k$ *matrix whose elements*

$$v_{i,j}(x)=\begin{cases}\dbinom{i-1}{j-1}q^{(i-j)}(x), & i-j\ge 0\\ 0, & i-j<0\end{cases}$$

for $1\le i\le k+1$ *and* $1\le j\le k$*.*

Proof. This is a reformulation of a formula in [1, p. 40, Entry 5]. □

Lemma 2 ([2, p. 222, Theorem] and [34, Remark 3])**.** *Let* $M_0=1$ *and*

$$M_n=\begin{vmatrix} m_{1,1} & m_{1,2} & 0 & \cdots & 0 & 0\\ m_{2,1} & m_{2,2} & m_{2,3} & \cdots & 0 & 0\\ m_{3,1} & m_{3,2} & m_{3,3} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ m_{n-2,1} & m_{n-2,2} & m_{n-2,3} & \cdots & m_{n-2,n-1} & 0\\ m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & m_{n-1,n-1} & m_{n-1,n}\\ m_{n,1} & m_{n,2} & m_{n,3} & \cdots & m_{n,n-1} & m_{n,n}\end{vmatrix}$$

for $n\in\mathbb{N}$*. Then the sequence* M_n *for* $n\ge 0$ *satisfies* $M_1=m_{1,1}$ *and*

$$M_n=m_{n,n}M_{n-1}+\sum_{r=1}^{n-1}(-1)^{n-r}m_{n,r}\left(\prod_{j=r}^{n-1}m_{j,j+1}\right)M_{r-1},\quad n\ge 2. \tag{13}$$

Lemma 3 ([3, p. 134, Theorem A] and [3, p. 139, Theorem C])**.** *For* $n\ge k\ge 0$*, the Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by* $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$*, are defined by*

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n,\ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^n i\ell_i=n\\ \sum_{i=1}^n \ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ *by*

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}f\circ h(t)=\sum_{k=0}^n f^{(k)}(h(t))\,\mathrm{B}_{n,k}\big(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\big). \tag{14}$$

Lemma 4 ([3, p. 135])**.** *For complex numbers* a *and* b*, we have*

$$\mathrm{B}_{n,k}\big(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\big)=a^kb^n\,\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}). \tag{15}$$

Lemma 5 ([11, Theorem 1.1])**.** *For $n \ge k \ge 0$, we have*
(16)
$$\mathrm{B}_{n,k}\left(\frac{1!}{2}, \frac{2!}{3}, \dots, \frac{(n-k+1)!}{n-k+2}\right) = (-1)^{n-k}\frac{1}{k!}\sum_{m=0}^{k}(-1)^m\binom{k}{m}\frac{s(n+m,m)}{\binom{n+m}{m}}.$$

3. Proofs of main results

We are now in a position to prove our main results as follows.

Proof of Theorem 1. For $n \ge 2$, by the formulas (11) or (12) in Lemma 1, we have
$$\left[\frac{x}{\ln(1+x)}\right]^{(n)} = \frac{(-1)^n}{\ln^{n+1}(1+x)}$$
$$\times \begin{vmatrix} x & \ln(1+x) & 0 & \cdots & 0 & 0 \\ 1 & \frac{1}{1+x} & \ln(1+x) & \cdots & 0 & 0 \\ 0 & -\frac{1}{(1+x)^2} & \binom{2}{1}\frac{1}{1+x} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{(-1)^{n-3}(n-3)!}{(1+x)^{n-2}} & \binom{n-2}{1}\frac{(-1)^{n-4}(n-4)!}{(1+x)^{n-3}} & \cdots & \ln(1+x) & 0 \\ 0 & \frac{(-1)^{n-2}(n-2)!}{(1+x)^{n-1}} & \binom{n-1}{1}\frac{(-1)^{n-3}(n-3)!}{(1+x)^{n-2}} & \cdots & \binom{n-1}{n-2}\frac{1}{1+x} & \ln(1+x) \\ 0 & \frac{(-1)^{n-1}(n-1)!}{(1+x)^{n}} & \binom{n}{1}\frac{(-1)^{n-2}(n-2)!}{(1+x)^{n-1}} & \cdots & -\binom{n}{n-2}\frac{1}{(1+x)^2} & \binom{n}{n-1}\frac{1}{1+x} \end{vmatrix}.$$
Denote the above $(n+1)\times(n+1)$ determinant by M_{n+1}. By the recurrence relation (13), we have
$$\begin{aligned}
&\frac{(-1)^n}{\ln^{n+1}(1+x)}M_{n+1} \\
&= \binom{n}{n-1}\frac{1}{1+x}\frac{(-1)^n}{\ln^{n+1}(1+x)}M_n \\
&\quad + \sum_{r=2}^{n}\binom{n}{r-2}\frac{(n-r+1)!}{(1+x)^{n-r+2}}\frac{(-1)^n}{\ln^{n+1}(1+x)}\ln^{n-r+1}(1+x)M_{r-1},
\end{aligned}$$
$$\begin{aligned}
\left[\frac{x}{\ln(1+x)}\right]^{(n)} = &-\binom{n}{n-1}\frac{1}{1+x}\frac{1}{\ln(1+x)}\left[\frac{x}{\ln(1+x)}\right]^{(n-1)} \\
&+ \sum_{r=2}^{n}\binom{n}{r-2}\frac{(n-r+1)!}{(1+x)^{n-r+2}}\frac{(-1)^{n+r}}{\ln(1+x)}\left[\frac{x}{\ln(1+x)}\right]^{(r-2)},
\end{aligned}$$
$$\left[\frac{x}{\ln(1+x)}\right]^{(n)} = \frac{(-1)^n}{\ln(1+x)}\sum_{r=2}^{n+1}(-1)^r\binom{n}{r-2}\frac{(n-r+1)!}{(1+x)^{n-r+2}}\left[\frac{x}{\ln(1+x)}\right]^{(r-2)}.$$
The last equation above can be rewritten as the nonlinear ordinary differential equations (6). The proof of Theorem 1 is complete. □

Proof of Theorem 2. Taking the limit $x \to 0$ in the last equation in the proof of Theorem 1 and making use of the L'Hôpital rule give

$$
\begin{aligned}
&\lim_{x\to0}\left[\frac{x}{\ln(1+x)}\right]^{(n)}\\
&= \lim_{x\to0}\frac{(-1)^n}{\ln(1+x)}\sum_{r=2}^{n+1}\binom{n}{r-2}\\
&\quad\times\frac{(-1)^r(n-r+1)!}{(1+x)^{n-r+2}}\left[\frac{x}{\ln(1+x)}\right]^{(r-2)}\\
&= (-1)^n\lim_{x\to0}(1+x)\sum_{r=2}^{n+1}(-1)^r\binom{n}{r-2}(n-r+1)!\\
&\quad\times\left(\frac{1}{(1+x)^{n-r+2}}\left[\frac{x}{\ln(1+x)}\right]^{(r-1)}-\frac{n-r+2}{(1+x)^{n-r+3}}\left[\frac{x}{\ln(1+x)}\right]^{(r-2)}\right)\\
&= (-1)^n\lim_{x\to0}(1+x)\sum_{r=2}^{n+1}(-1)^r\binom{n}{r-2}(n-r+1)!\\
&\quad\times\left(\lim_{x\to0}\frac{1}{(1+x)^{n-r+2}}\left[\frac{x}{\ln(1+x)}\right]^{(r-1)}-\lim_{x\to0}\frac{n-r+2}{(1+x)^{n-r+3}}\left[\frac{x}{\ln(1+x)}\right]^{(r-2)}\right).
\end{aligned}
$$

Further considering the generating function (1) yields

$$
n!b_n=\sum_{r=2}^{n+1}(-1)^{n+r}\binom{n}{r-2}(n-r+1)![(r-1)!b_{r-1}-(n-r+2)(r-2)!b_{r-2}]
$$

which can be rearranged as

$$
\begin{aligned}
b_n &= (-1)^n\sum_{r=2}^{n+1}(-1)^r\binom{n}{r-2}\left[\frac{1}{\binom{n}{r-1}}b_{r-1}-\frac{1}{\binom{n}{r-2}}b_{r-2}\right]\\
&= (-1)^n\sum_{r=2}^{n+1}(-1)^r\left(\frac{r-1}{n-r+2}b_{r-1}-b_{r-2}\right)\\
&= (-1)^n\left[\sum_{r=2}^{n+1}(-1)^r\frac{r-1}{n-r+2}b_{r-1}-\sum_{r=2}^{n+1}(-1)^r b_{r-2}\right]\\
&= (-1)^n\left[\sum_{r=1}^{n}(-1)^{r+1}\frac{r}{n-r+1}b_r-\sum_{r=0}^{n-1}(-1)^r b_r\right]\\
&= (-1)^n\left[(-1)^{n+1}nb_n+\sum_{r=1}^{n-1}(-1)^{r+1}\frac{n+1}{n-r+1}b_r-b_0\right].
\end{aligned}
$$

Therefore, we obtain the recurrence relation (7). The proof of Theorem 2 is complete. $\square$

Proof of Theorem 3. It is not difficult to see that the generating function of the Bernoulli numbers of the second kind b_k can be written as $\frac{t}{\ln(1+t)} = \frac{1}{\frac{\ln(1+t)}{t}}$ and

$$\left[\frac{\ln(1+t)}{t}\right]^{(\ell)} = \left(\frac{1}{t}\int_0^t \frac{1}{1+u}\,\mathrm{d}\,u\right)^{(\ell)} = \left(\int_0^1 \frac{1}{1+tu}\,\mathrm{d}\,u\right)^{(\ell)}$$
$$= \int_0^1 \left(\frac{1}{1+tu}\right)^{(\ell)}\mathrm{d}\,u = \int_0^1 \frac{(-1)^\ell \ell! u^\ell}{(1+tu)^{\ell+1}}\,\mathrm{d}\,u \to (-1)^\ell \frac{\ell!}{\ell+1}$$

as $t \to 0$. Making use of (14), (15), and (16) in sequence gives

$$\begin{aligned}
\left[\frac{t}{\ln(1+t)}\right]^{(n)} &= \sum_{k=0}^{n}\left(\frac{1}{v}\right)^{(k)} \mathrm{B}_{n,k}\big(v'(t), v''(t), \dots, v^{(n-k+1)}(t)\big)\\
&= \sum_{k=0}^{n}\frac{(-1)^k k!}{v^{k+1}} \mathrm{B}_{n,k}\big(v'(t), v''(t), \dots, v^{(n-k+1)}(t)\big)\\
\to\ & \sum_{k=0}^{n}(-1)^k k!\,\mathrm{B}_{n,k}\left(-\frac{1!}{2}, \frac{2!}{3}, \dots, \frac{(-1)^{n-k+1}(n-k+1)!}{n-k+2}\right)\\
&= (-1)^n\sum_{k=0}^{n}(-1)^k k!\,\mathrm{B}_{n,k}\left(\frac{1!}{2}, \frac{2!}{3}, \dots, \frac{(n-k+1)!}{n-k+2}\right)\\
&= (-1)^n\sum_{k=0}^{n}(-1)^k k!(-1)^{n-k}\frac{1}{k!}\sum_{m=0}^{k}(-1)^m\binom{k}{m}\frac{s(n+m,m)}{\binom{n+m}{m}}\\
&= \sum_{k=0}^{n}\sum_{m=0}^{k}(-1)^m\binom{k}{m}\frac{s(n+m,m)}{\binom{n+m}{m}}
\end{aligned}$$

as $t \to 0$, where $v = v(t) = \frac{\ln(1+t)}{t}$. Hence, the formula (8) is proved.

Making use of the formulas (11) or (12) in Theorem 1 yields

$$\left[\frac{t}{\ln(1+t)}\right]^{(n)} = \left[\frac{1}{\frac{\ln(1+t)}{t}}\right]^{(n)} \tag{17}$$
$$= \frac{(-1)^n}{v^{n+1}(t)}\begin{vmatrix}
1 & v(t) & 0 & \cdots & 0 & 0\\
0 & v'(t) & v(t) & \cdots & 0 & 0\\
0 & v''(t) & \binom{2}{1}v'(t) & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & v^{(n-2)}(t) & \binom{n-2}{1}v^{(n-3)}(t) & \cdots & v(t) & 0\\
0 & v^{(n-1)}(t) & \binom{n-1}{1}v^{(n-2)}(t) & \cdots & \binom{n-1}{n-2}v'(t) & v(t)\\
0 & v^{(n)}(t) & \binom{n}{1}v^{(n-1)}(t) & \cdots & \binom{n}{n-2}v''(t) & \binom{n}{n-1}v'(t)
\end{vmatrix},$$

where $v=v(t)=\frac{\ln(1+t)}{t}$. Letting t tend to 0 on both sides arrives at

$$n!b_n=(-1)^n\begin{vmatrix}1 & 1 & 0 & \cdots & 0 & 0\\ 0 & -\frac{1}{2} & 1 & \cdots & 0 & 0\\ 0 & \frac{2}{3} & -1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & (-1)^{n-2}\frac{(n-2)!}{n-1} & \binom{n-2}{1}(-1)^{n-3}\frac{(n-3)!}{n-2} & \cdots & 1 & 0\\ 0 & (-1)^{n-1}\frac{(n-1)!}{n} & \binom{n-1}{1}(-1)^{n-2}\frac{(n-2)!}{n-1} & \cdots & -\frac{n-1}{2} & 1\\ 0 & (-1)^{n}\frac{n!}{n+1} & \binom{n}{1}(-1)^{n-1}\frac{(n-1)!}{n} & \cdots & \frac{2}{3}\binom{n}{n-2} & -\frac{n}{2}\end{vmatrix}$$

for $n\ge 0$. The determinantal expression (9) follows readily.

Denoting the $(n+1)\times(n+1)$ determinant in (17) by M_{n+1} and employing the recurrence relation (13) reveal

$$M_{n+1}=\binom{n}{n-1}v'(t)M_n+\sum_{r=2}^{n}(-1)^{n-r+1}\binom{n}{r-2}v^{(n-r+2)}(t)v^{n-r+1}(t)M_{r-1}$$

for $n\ge 2$, which can be rearranged as

$$\begin{aligned}\frac{(-1)^n}{v^{n+1}(t)}M_{n+1}&=-\binom{n}{n-1}\frac{v'(t)}{v(x)}\frac{(-1)^{n-1}}{v^n(t)}M_n\\&\quad-\sum_{r=2}^{n}\binom{n}{r-2}\frac{v^{(n-r+2)}(t)}{v(t)}\frac{(-1)^{r-2}}{v^{r-1}(t)}M_{r-1},\\ \left[\frac{t}{\ln(1+t)}\right]^{(n)}&=-\binom{n}{n-1}\frac{v'(t)}{v(x)}\left[\frac{t}{\ln(1+t)}\right]^{(n)}\\&\quad-\sum_{r=2}^{n}\binom{n}{r-2}\frac{v^{(n-r+2)}(t)}{v(t)}\left[\frac{t}{\ln(1+t)}\right]^{(r-2)},\\ \left[\frac{t}{\ln(1+t)}\right]^{(n)}&=-\binom{n}{n-1}\frac{v'(t)}{v(x)}\left[\frac{t}{\ln(1+t)}\right]^{(n-1)}\\&\quad-\sum_{r=0}^{n-2}\binom{n}{r}\frac{v^{(n-r)}(t)}{v(t)}\left[\frac{t}{\ln(1+t)}\right]^{(r)}.\end{aligned}$$

In a word, we have

$$\sum_{r=0}^{n}\binom{n}{r}v^{(n-r)}(t)\left[\frac{t}{\ln(1+t)}\right]^{(r)}=0$$

which can be reformulated as (10). Further taking $t\to 0$ in the above equation acquires

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^{n-r}\frac{(n-r)!}{n-r+1}r!b_r=0$$

which recovers (7). The proof of Theorem 3 is complete. □

4. Remarks

Finally we give several remarks on our main results and lemmas.

Remark 1. About the differential equations (6) and (10), there are some related references such as [4, 15, 19, 22, 23, 25, 30, 31, 36] worth to reading.

Remark 2. The formula (8) is essentially different from (2) and (3), because the Stirling numbers of the second kind $s(n, k)$ are used diagonally in (8) but horizontally in (2) and (3). For more information, please refer to the papers [11, 12].

Remark 3. About the determinantal expression (9), we note that the formula (12) in Lemma 1 has been applied in the papers [5, 9, 13, 14, 16–18, 20, 21, 24, 26–29, 34, 35] to express the Apostol–Bernoulli polynomials, the Cauchy product of central Delannoy numbers, the Bernoulli polynomials, the Schröder numbers, the (generalized) Fibonacci polynomials, the Catalan numbers, derangement numbers, and the Euler numbers and polynomials in terms of the Hessenberg and tridiagonal determinants. This implies that Lemma 1 is effectual to express some mathematical quantities in terms of the Hessenberg and tridiagonal determinants.

Remark 4. This paper is a slightly modified version of the preprint [33].

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