Bull. Korean Math. Soc. **55** (2018), No. 6, pp. 1845–1858
https://doi.org/10.4134/BKMS.b171112
pISSN: 1015-8634 / eISSN: 2234-3016

THE GENERALIZED FERMAT TYPE DIFFERENCE EQUATIONS

Kai Liu, Lei Ma, and Xiaoyang Zhai

Abstract. This paper is to consider the generalized Fermat difference equations with different types which ever considered by Li [14], Ishizaki and Korhonen [9], Zhang [26] and Liu [15–18], respectively. Some new observations and results on these equations will be given.

1. Introductions

It is well known that the equations

$$f(z)^n + g(z)^n = 1 \tag{1}$$

regard as the Fermat diophantine equations $x^n + y^n = 1$ over functional fields, where $n \geq 2$ is a positive integer. Montel [20] proved that (1) has no transcendental entire solutions when $n \geq 3$. Gross [3] showed that (1) has no transcendental meromorphic solutions when $n \geq 4$. If $n = 2$, then Gross [4] proved that (1) has the entire solutions $f(z) = \sin(h(z))$ and $g(z) = \cos(h(z))$, where $h(z)$ is any entire function, no other solutions exist. If $n = 3$, then Baker [1] obtained that the nonconstant meromorphic solutions $f(z) = F(\omega(z))$ and $g(z) = cG(\omega(z))$, where c is a cube-root unity, $\omega(z)$ is an entire function and $F(z) = \frac{1+3^{-\frac{1}{2}}\varphi'(z)}{2\varphi(z)}$ and $G(z) = \frac{1-3^{-\frac{1}{2}}\varphi'(z)}{2\varphi(z)}$, where $\varphi(z)$ denotes the Weierstrass function with periods ω_1 and ω_2 defined as

$$\varphi(z) := \frac{1}{z^2} + \sum_{\mu,\nu\in\mathbb{Z};\mu^2+\nu^2\neq 0} \left\{ \frac{1}{(z+\mu\omega_1+\nu\omega_2)^2} - \frac{1}{(\mu\omega_1+\nu\omega_2)^2} \right\}$$

and satisfies the equation $(\varphi')^2 = 4\varphi^3 - 1$ for suitable ω_1 and ω_2.

Received December 24, 2017; Accepted April 3, 2018.
2010 *Mathematics Subject Classification.* Primary 30D35, 39A05.
Key words and phrases. Fermat difference equation, meromorphic function, finite order.
This work was partial supported by the NSFC (No.11661052), the NSF of Jiangxi (No. 20161BAB211005), the outstanding youth scientist foundation plan of Jiangxi (No. 20171BCB23003) and the Innovation Found for the Postgraduates of Nanchang University (No.cx2016148).

One of recent interests on Fermat functional equations is to get the precise forms of meromorphic solutions on Fermat differential equations or Fermat difference equations. Yang and Li [23, Theorem 1] considered the transcendental entire solutions of the Fermat differential equations

$$f(z)^2 + f'(z)^2 = 1.$$

Liu [15, Proposition 5.1] considered the transcendental entire solutions with finite order on Fermat difference equations

$$f(z)^2 + f(z+c)^2 = 1. \tag{2}$$

Liu, Cao and Cao [16] proceed to consider the precise expressions on the transcendental entire solutions on (2), Liu and Yang [18] obtained the existence on the transcendental meromorphic solutions with finite order or infinite order, see some examples in [18]. Recent progresses on Fermat difference equations are to consider the existence on the transcendental meromorphic solutions on

$$f(z)^3 + f(z+c)^3 = 1, \tag{3}$$

see Korhonen and Zhang [11], Lü and Han [19], they proved that there are no transcendental meromorphic solutions with hyper-order less than one. In addition, Li [13], Ishizaki and Korhonen [9] and Zhang [26] considered the properties of transcendental solutions of the generalizations of (2) with different types, the details will be stated in the following. We will proceed to consider and give some new observations and results on these equations.

We assume that the reader is familiar with standard symbols and results on Nevanlinna theory [24].

2. The lower estimates on $F_S(k)$

Let n, k be positive integers. The equations

$$f_1(z)^k + f_2(z)^k + \cdots + f_n(z)^k = 1 \tag{4}$$

are called the generalized Fermat type equations. Some results on the existence of meromorphic solutions when $n = 3, 4$ can be found in [5]. Recent progresses on the case of $n = 3$ are given by Ng and Yeung [21], they proved that there are no transcendental meromorphic solutions when $k = 8$ and there are no transcendental entire solutions when $k = 6$. It is open for the case that $n = 3$ and $k = 7$ for meromorphic solutions of (4).

Let $F_S(k)$ denote the smallest number n when (4) admits nonconstant solutions in S, where S is one of the following functions class, the linear polynomials L, the polynomials P, entire functions E, rational functions R and meromorphic functions M. Obviously, $F_S(k)$ depends on k and S. The best lower estimates can be found [6, Theorem 4.1] which is a collection of the results in the references of [6]. The estimates can be stated as follows.

Theorem A. *Suppose that $k \geq 2$ and $n \geq 2$. Let $f_1(z), f_2(z), \ldots, f_n(z)$ be nonconstant functions in S satisfying* (4), *the best lower estimates for* (4) *as follows*:

$$F_P(k) > \frac{1}{2} + \sqrt{k + \frac{1}{4}}, \tag{5}$$

$$F_R(k) > \sqrt{k+1}, \tag{6}$$

$$F_M(k) \geq \sqrt{k+1}, \tag{7}$$

$$F_E(k) \geq \frac{1}{2} + \sqrt{k + \frac{1}{4}}. \tag{8}$$

In addition, Kim [10] applied linear algebra theory to obtain the estimate on $F_L(k)$.

Theorem B ([10, Theorem 2.2.1])**.** *Let $n \geq 2$ and $k \geq 2$ be integers. Then*

$$F_L(k) = k + 1.$$

The difference monomial $x(x-1)\cdots(x-k+1)$ is a discrete version of x^k. Similar correspondence occurs frequently in the theory of difference equations as well, for example $f(z)f(z+c)\cdots f(z+(k-1)c)$ is a discrete version of $f(z)^k$. Recently, Li [13] considered a difference version of (4) as follows

$$\sum_{i=1}^{n} f_i(z)f_i(z+c)\cdots f_i(z+(k-1)c) = 1, \tag{9}$$

where $c = 1$. For the statement of Li's results, some definitions should be recalled firstly.

Definition 2.1. Let f and g be meromorphic functions and a be a complex number. Let $z_n (n = 1, 2, \ldots)$ be zeros of $f - a$. If $z_n (n = 1, 2, \ldots)$ are also zeros of $g - a$ (ignoring multiplicity), we denote $f = a \Rightarrow g = a$. Let $\nu(n)$ be the multiplicity of the zeros z_n. If $z_n (n = 1, 2, \ldots)$ are also $\nu(n) (n = 1, 2, \ldots)$ multiple zeros of $g - a$ at least, then we write $f = a \to g = a$. If $f = a \rightleftharpoons g = a$, then it is said that f and g share a CM. If $f = a \Leftrightarrow g = a$, then it is said that f and g share a IM. If $f = a \to f(z+c) = a$ except for at most finitely many a-points of f, then it is said that a is an exceptional paired value of f with the separation c, which is defined in [7].

Let $\widetilde{M}$ be the collection of all nonconstant meromorphic functions of hyper-order < 1 such that any finite collection $\{f_1, \ldots, f_k\} \subset \widetilde{M}$ satisfies the following properties:

(i) f_i and $1/f_j (i, j = 1, \ldots, k, i \neq j)$ have no common zeros;
(ii) $f_i = \infty \rightleftharpoons f_i(z+c) = \infty$ for all $i = 1, \ldots, k$;
(iii) 0 is an exceptional paired value of f_i for all $i = 1, \ldots, k$.

Let $\widetilde{E}$ be the collection of all nonconstant entire functions of hyper-order < 1 such that any finite collection $\{f_1, \ldots, f_k\} \subset \widetilde{E}$ satisfies $f_i(z) = 0 \Rightarrow f_i(z+c) = 0$ for all $i = 1, \ldots, k$.

Let $\widetilde{R}$ be the collection of all nonconstant rational functions that satisfies the property that the zeros and the poles are of multiplicity positive integer multiple of n.

Let $\widetilde{P}$ be the collection of all nonconstant polynomial functions such that the zeros are of multiplicity no less than n.

Li [13] obtained the best lower estimates $F_{\widetilde{S}}(k)$ for (9), which can be collected as follows.

Theorem C. *Suppose that $k \geq 2$ and $n \geq 2$. Let $f_1(z), f_2(z), \ldots, f_n(z)$ be nonconstant functions in S satisfying* (9). *Then*

$$F_{\widetilde{P}}(k) > \frac{1}{2} + \sqrt{k + \frac{1}{4}}, \tag{10}$$

$$F_{\widetilde{R}}(k) > \sqrt{k+1}, \tag{11}$$

$$F_{\widetilde{M}}(k) \geq \sqrt{k+1}, \tag{12}$$

$$F_{\widetilde{E}}(k) \geq \frac{1}{2} + \sqrt{k + \frac{1}{4}}. \tag{13}$$

Remark 2.2. The conditions (ii) above can not be removed in $\widetilde{M}$, which can be seen by $f(z) = \frac{e^{\pi iz}+1}{e^{\pi iz}-1}$, thus $f(z+1) = \frac{e^{\pi iz}-1}{e^{\pi iz}+1}$ thus $f(z)f(z+1) = 1$. Here $k = 2$ and $n = 1$.

Observing the proofs of Theorem C in [13], the idea is to use an important result [5, Theorem 3.2]. In this paper, we provide firstly some other results on the generalized Fermat type equations and show our observations on the conditions (i), (ii), (iii). Toda [22] investigated the generalized Fermat type equations

$$a_1(z)f_1(z)^{k_1} + a_1(z)f_2(z)^{k_2} + \cdots + a_n(z)f_n(z)^{k_n} = 1, \tag{14}$$

where $a_1(z), a_2(z), \ldots, a_n(z)$ are nonzero meromorphic functions such that $T(r, a_j(z)) = o(T(r, f_j(z)))$, $j = 1, 2, \ldots n$ and obtained the theorem below.

Theorem D ([22, Theorem 1])**.** *Let $f_1, f_2, \ldots, f_n$ be nonconstant entire functions. If f_j and a_j satisfy* (14) *for positive integers $k_1, \ldots k_n$, then*

$$\sum_{j=1}^{n} \frac{1}{k_j} \geq \frac{1}{n-1}.$$

In 2002, Yu and Yang [25] improved Theorem D to meromorphic functions and proved the following result.

Theorem E. *Let $f_1, f_2, \dots, f_n$ be nonconstant meromorphic functions. If f_j and a_j satisfy* (14) *for some positive integers $k_1, \dots k_n$, then*

$$\sum_{j=1}^{n} \frac{1}{k_j} \geq \frac{1}{4(n-1)}.$$

Li and Yang [14] obtained the following result, which can be used to improve Theorem D and Theorem E.

Lemma 2.3. *Let n be a positive integer and $n \geq 2$. Let $F_1, F_2, \dots, F_n$ be n linearly independent non-constant meromorphic functions. If*

$$\sum_{i=1}^{n} F_i \equiv 1,$$

then

$$T(r, F_1) \leq \sum_{i=1}^{n} (n-1)\overline{N}(r, \frac{1}{F_i}) + A_n \sum_{i=1}^{n} \overline{N}(r, F_i) + o(T(r)),$$

where

$$A_n := \begin{cases} \frac{1}{2}, & n = 2, \\ \frac{2n-3}{3}, & n = 3, 4, 5, \\ \frac{2n+1-2\sqrt{2n}}{2}, & n \geq 6. \end{cases} \tag{15}$$

Lahiri and Yu [12] obtained the result below using Lemma 2.3.

Theorem F. *Let $f_1, f_2, \dots, f_n$ be nonconstant meromorphic functions. If f_j and a_j satisfy* (14) *for some positive integers $k_1, \dots k_n$, then*

$$\sum_{j=1}^{n} \frac{1}{k_j} \geq \frac{1}{n-1+A_n}.$$

Let us observe the above three conditions (i), (ii), (iii). In fact, the function $f(z)$ that satisfies (ii) and (iii) implies that

$$N\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f).$$

In addition, for meromorphic functions with hyper order less than one,

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f)$$

follows by Halburd, Korhonen and Tohge [8]. So the function $f(z)$ in $\widetilde{M}$ satisfies

$$T\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f). \tag{16}$$

Similarly, the functions with $f_i(z) = 0 \to f_i(z+c) = 0$ in $\widetilde{E}$ satisfy (16). Hence, we replace the functions class $\widetilde{M}$ and $\widetilde{E}$ with $\widehat{M}$ and $\widehat{E}$, where $\widehat{M}$ denotes the class of meromorphic functions satisfying (16) and $\widehat{E}$ denotes the class of entire functions satisfying (16).

Remark 2.4. The condition (i) is removed in $\widehat{M}$. The class $\widehat{E}$ indeed extend the class $\widetilde{E}$. For example $f_1(z) = i\sin(e^z) \in \widehat{E}$ and $f_2(z) = \cos(e^z) \in \widehat{E}$ are functions of hyper-order one. If $e^c = -1$, then $f_1(z)$ and $f_2(z)$ do not belong to $\widetilde{E}$, but satisfy (16). It is easy to see that $f_1(z)$ and $f_2(z)$ solve $f_1(z)f_1(z+c) + f_2(z)f_2(z+c) = 1$.

Proposition 2.5. *Suppose that $k \geq 2$ and $n \geq 2$. Let $f_1(z), f_2(z), \ldots, f_n(z)$ be nonconstant functions in S satisfying* (9). *Then*

$$F_{\widehat{E}}(k) \geq \frac{1}{2} + \sqrt{k + \frac{1}{4}}, \quad F_{\widehat{M}}(k) \geq \frac{1}{2} + \sqrt{\frac{k+1}{4}}.$$

Proof. We change (9) into another form

$$\sum_{i=1}^{n} a_i(z) f_i(z)^k \prod_{j=1}^{k-1} \frac{f_i(z+jc)}{f_i(z)} = 1. \tag{17}$$

Considering the collection $\{f_1, \ldots, f_k\} \subset \widehat{E}$, we have

$$T\left(r, a_i(z) \prod_{j=1}^{k-1} \frac{f_i(z+jc)}{f_i(z)}\right) = S(r, f).$$

Using Theorem D, we get $\frac{n}{k} \geq \frac{1}{n-1}$, that is $F_{\widehat{E}}(k) \geq \frac{1}{2} + \sqrt{k + \frac{1}{4}}$. Using Theorem E for $\widehat{M}$, we have $\frac{n}{k} \geq \frac{1}{4(n-1)}$, thus $F_{\widehat{M}}(k) \geq \frac{1}{2} + \sqrt{\frac{k+1}{4}}$. □

In fact, we also can give a generalization of (9) as follows:

$$\sum_{i=1}^{k} a_i(z) \prod_{j=0}^{n_j} f_i(z+jc) = a(z), \tag{18}$$

where $a_i(z)$, $i = 0, 1, \ldots, k$ are nonzero small functions with respect to $f(z)$. If all $n_j = n$, then (18) reduces to (9). If f_j belongs to $\widehat{M}$, then (18) can be written as

$$\sum_{i=1}^{k} a_i(z) f_i(z)^{n_j} \prod_{j=0}^{n_j} \frac{f_i(z+jc)}{f_i(z)} = a(z). \tag{19}$$

Theorem F implies that $\sum_{j=1}^{k} \frac{1}{n_j} \geq \frac{1}{k-1+A_k}$.

This method also can be used to consider the transcendental meromorphic solutions on other generalized Fermat difference equations, for example

$$\sum_{i=1}^{k} a_i(z) \prod_{j=0}^{n_j} [b_j(z) f_i(z+jc) - c_j(z) f(z)] = a(z),$$

where $a(z), a_i(z), b_j(z), c_j(z)$ are small functions with respect to $f(z)$.

Remark 2.6. Considering $F_L(k)$, we see that the equation

$$f_1(z)f_1(z+c) + f_2(z)f_2(z+c) = 1$$

has no linear solutions, but

$$f_1(z)f_1(z+1) + f_2(z)f_2(z+1) + f_3(z)f_3(z+1) = 1$$

exists linear solutions, for example $f_1(z) = \frac{z+\frac{1}{\sqrt{2}}}{2}$, $f_2(z) = \frac{\frac{-i}{\sqrt{2}}z+\frac{2-i}{2}}{2}$ and $f_3(z) = \frac{\frac{i}{\sqrt{2}}z+\frac{2+i}{2}}{2}$ satisfy the above equation. It seems that for the case of linear polynomials, $F_L(k) = k+1$ for (9). Unfortunately, we have not succeed in proving it.

3. Generalized Fermat equations with three terms

In this section, we consider another generalized Fermat difference equations

$$f(z+1)^2 - A_1(z)f(z)f(z+1) + f(z)^2 = B_1(z), \tag{20}$$

where $A_1(z)$ and $B_1(z)$ are nonzero meromorphic functions. The equation (20) is also considered in a recent paper by Ishizaki and Korhonen [9] with different expression as follows

$$(\Delta f(z))^2 = A(z)(f(z)f(z+1) - B(z)), \tag{21}$$

where $A(z), B(z)$ are meromorphic functions. It is easy to see that $A_1(z) = 2 + A(z)$ and $B_1(z) = -A(z)B(z)$ in (20).

Firstly, we recall some results on $A_1(z)$ and $B_1(z)$ are constants in (20). Obviously, if $A_1(z) = \pm 2$ and $B_1(z) = B_1$, then (20) can be written as

$$[f(z+1) \pm f(z)]^2 = B_1,$$

thus $f(z+1) \pm f(z) = \pm\sqrt{B_1}$. If $f(z)+f(z+1) = \sqrt{B_1}$, then $f(z)$ may be a finite order entire function, such as $f(z) = e^{\pi iz} + \frac{\sqrt{B_1}}{2}$, $f(z)$ may also be an infinite order entire function, such as $f(z) = \sin(e^{\pi iz}) + \frac{\sqrt{B_1}}{2}$. If $f(z) - f(z+1) = \sqrt{B_1}$, then $f(z)$ may be a finite order entire function, such as $f(z) = e^{2\pi iz} - z\sqrt{B_1}$, $f(z)$ may also be an infinite order entire function, such as $f(z) = \cos(e^{\pi iz}) - z\sqrt{B_1}$. If $A_1(z) = -2\alpha$, $\alpha \neq \pm 1$ and $B_1(z) = 1$, Liu and Yang [18] obtained the transcendental meromorphic solutions of (20) should satisfy $f(z) = \frac{\alpha_1 - \alpha_2\beta(z)^2}{(\alpha_1-\alpha_2)\beta(z)}$ and $f(z+c) = \frac{1-\beta(z)^2}{(\alpha_1-\alpha_2)\beta(z)}$ where $\alpha_1 = \frac{1}{\alpha_2}$, $\alpha_1+\alpha_2 = 2\alpha$ and $\beta(z)\beta(z+c) = -\frac{1}{\alpha_2}$ or $\alpha_2\beta(z+c) = \beta(z)$. See some examples in [18].

Ishizaki and Korhonen [9] proved that the difference equation (21) possesses a continuous limit to the differential equation $(w')^2 = A(z)(w^2-1)$ and obtained the following Theorem G. We will give a short proof of Theorem G from the point of view of (20).

Theorem G ([9, Proposition 4.1]). *Suppose that $A(z)$ and $B(z)$ are periodic functions of period 1 in (21) and suppose that (21) possesses a meromorphic solution $f(z)$. Then either $f(z)$ is a periodic function of period 2, or $f(z)$ satisfies a linear difference equation of second order*

$$\Delta^2 f(z) - A(z)\Delta f(z) - A(z)f(z) = 0.$$

Proof. Shifting forward the equation (20), we have

$$f(z+2)^2 - A_1(z+1)f(z+1)f(z+2) + f(z+1)^2 = B_1(z+1). \tag{22}$$

Since $A(z)$ and $B(z)$ are periodic functions of period 1, then $A_1(z)$ and $B_1(z)$ are also periodic functions of period 1. From (20) and (22), we have

$$[f(z+2)-f(z)][f(z+2)+f(z)] = A_1(z)f(z+1)[f(z+2)-f(z)].$$

Thus, we have that either $f(z)$ is a periodic function of period 2 or $f(z)$ satisfies a linear difference equation

$$f(z+2) - A_1(z)f(z+1) + f(z) = 0,$$

that is

$$\Delta^2 f(z) - A(z)\Delta f(z) - A(z)f(z) = 0. \qquad \square$$

There are some examples in [9] to show that the orders of growth on two transcendental meromorphic solutions $f_1(z)$ and $f_2(z)$ do not satisfy

$$T(r,f_1) = T(r,f_2)(1+o(1)) \quad \text{as } r\to\infty,\ r\notin \mathbf{E},$$

where $\mathbf{E}$ is an exceptional set with finite logarithmic measure. The examples themselves are also interesting. We will give more details on these examples from the point of view of (20).

Example 3.1 ([9, Example 2.1]). Taking $A(z) = -4\sin^2\frac{a}{2}$ and $B(z) = \cos^2\frac{a}{2}$ in (21), that is

$$(\Delta f(z))^2 = -4\sin^2\frac{a}{2}(f(z)f(z+1) - \cos^2\frac{a}{2}).$$

Thus we can rewrite the above equation as follows

$$f(z+1)^2 - (2-4\sin^2\frac{a}{2})f(z)f(z+1) + f(z)^2 = 4\sin^2\frac{a}{2}\cos^2\frac{a}{2}, \tag{23}$$

which also can be written as

$$g(z+1)^2 - 2\cos a g(z)g(z+1) + g(z)^2 = 1, \tag{24}$$

where $g(z) = \frac{f(z)}{\sin a}$ and a is nonzero constant. Using the result in [18, p. 321], we have that all meromorphic solutions of (24) can be written as

$$g(z) = \frac{\alpha_1 - \alpha_2\beta(z)^2}{(\alpha_1 - \alpha_2)\beta(z)},$$

where $\beta(z)\beta(z+1) = -\frac{1}{\alpha_2}$ or $\alpha_2\beta(z+1) = \beta(z)$ and $\alpha_1 = \cos a + i\sin a$ and $\alpha_2 = \cos a - i\sin a$, $\beta(z)$ is a meromorphic function. Here $\beta(z)$ can take transcendental meromorphic solutions with finite order or infinite order. Then $f(z)$ can be transcendental meromorphic solutions with finite order or infinite order.

Example 3.2 ([9, Example 2.2])**.** The transcendental meromorphic function $f(z) = \frac{1+be^{\pi iz}-e^{2\pi iz}}{e^{2\pi iz}-1}$ satisfies the equation $(\triangle_c f)^2 = -4(f(z)f(z+1)-1)$, which also can be written as $(f(z+1)+f(z))^2 = 4$. We can get that all meromorphic solutions should satisfy $f(z+1)+f(z) = 2$ or $f(z+1)+f(z) = -2$. Thus $f(z)$ should be a periodic function with period 2.

Example 3.3 ([9, Example 2.3])**.** If $\beta(z) \neq 0, \pm 2$ is an arbitrary periodic function of period 1. We see that

$$(\triangle_c f)^2 = -\frac{4\beta(z)^2}{\beta(z)^2-4}(f(z)f(z+1)-1)$$

which also can be written as

$$f(z+1)^2 + \frac{2\beta(z)^2+8}{\beta(z)^2-4}f(z)f(z+1) + f(z)^2 = \frac{4\beta(z)^2}{\beta(z)^2-4}.$$

It also can be written as

$$\left(f(z+1) + \frac{\beta(z)+2}{\beta(z)-2}f(z)\right)\left(f(z+1) + \frac{\beta(z)-2}{\beta(z)+2}f(z)\right) = \frac{4\beta(z)^2}{\beta(z)^2-4}.$$

We will try to solve the above equation in the following

$$\begin{cases} f(z+1) + \dfrac{\beta(z)+2}{\beta(z)-2}f(z) = \dfrac{2\beta(z)}{\beta(z)-2}h(z), \\ f(z+1) + \dfrac{\beta(z)-2}{\beta(z)+2}f(z) = \dfrac{2\beta(z)}{\beta(z)+2}\dfrac{1}{h(z)}, \end{cases} \tag{25}$$

where $h(z)$ is a nonconstant meromorphic function. From (25), we have

$$f(z) = \frac{h(z)(\beta(z)+2) - (\beta(z)-2)\frac{1}{h(z)}}{4}. \tag{26}$$

We also have

$$f(z+1) = \frac{h(z)(\beta(z)-2) - (\beta(z)+2)\frac{1}{h(z)}}{-4}. \tag{27}$$

Combining (27) with the shift of (26), we have

$$\frac{h(z+1)}{h(z)} = \frac{2-\beta(z)}{2+\beta(z)}. \tag{28}$$

Since that $\beta(z)$ is an arbitrary periodic function of period 1, then

$$h(z) = \tau(z) \left(\frac{2-\beta(z)}{2+\beta(z)} \right)^z,$$

where $\tau(z)$ is an arbitrary periodic function of period 1.

Example 3.4 ([9, Example 2.4])**.** See the equation

$$(\triangle_c f)^2 = \frac{1}{z(z+1)} \left(f(z)f(z+1) - \frac{(1+2z)^2}{4z(z+1)} \right),$$

which can be written as

$$f(z+1)^2 - \left(2 + \frac{1}{z(z+1)}\right) f(z)f(z+1) + f(z)^2 = \frac{-(1+2z)^2}{4z^2(z+1)^2}.$$

It also can be decomposed into

$$\left(f(z+1) - \frac{z+1}{z} f(z) \right) \left(f(z+1) - \frac{z}{z+1} f(z) \right) = \frac{-(1+2z)^2}{4z^2(z+1)^2}.$$

We will try to solve the above equation as follows

$$\begin{cases} f(z+1) - \dfrac{z+1}{z} f(z) = \dfrac{1+2z}{2z^2} h(z), \\ f(z+1) - \dfrac{z}{z+1} f(z) = \dfrac{-1-2z}{2(z+1)^2} \dfrac{1}{h(z)}, \end{cases} \tag{29}$$

where $h(z)$ is a nonconstant meromorphic function. Furthermore, we can obtain

$$f(z) = \frac{-h(z-1) - \frac{1}{h(z-1)}}{2},$$

where $h(z)$ satisfies $h(z+1) = \frac{z+1}{z+2} h(z)$ or $h(z+1)h(z) = \frac{z+1}{z+2}$. In [9, Example 2.4], the special solution $f(z) = \frac{z^2+Q(z)^2}{2Q(z)z}$ where $Q(z)$ is a periodic function of period 1. In fact, we can take $h(z) = -\frac{Q(z)}{z+1}$ that satisfies $h(z+1) = \frac{z+1}{z+2} h(z)$.

From Example 1 to Example 4, we see that $A_1(z)$ in (23) always satisfies

$$\begin{cases} A_a(z) + A_b(z) = A_1(z), \\ \quad A_a(z)A_b(z) = 1, \end{cases} \tag{30}$$

where $A_a(z)$ and $A_b(z)$ are meromorphic functions. It is natural to consider the existence of meromorphic solutions on a general equation as follows

$$f(z+1)^2 + 2\alpha(z)f(z+1)f(z) + f(z)^2 = \beta(z), \tag{31}$$

where $\alpha(z)$ and $\beta(z)$ are nonconstant meromorphic functions.

Remark 3.5. For a special case $\beta(z) = 1 - \alpha(z)^2$ where $\alpha(z)$ is a periodic function with period 1, $\sqrt{\frac{\beta}{1-\alpha}}$ and $\sqrt{\frac{\beta}{1+\alpha}}$ are entire functions, the transcendental entire function

$$f(z) = \sqrt{\frac{\beta}{2}}\left(\frac{\sin(\frac{\pi z}{2})}{\sqrt{1+\alpha}} + \frac{\cos(\frac{\pi z}{2})}{\sqrt{1-\alpha}}\right)$$

solves the equation (31).

For our further studying, we raise the following question.

Question. How to describe the transcendental meromorphic solutions for any meromorphic functions $\alpha(z)$ and $\beta(z)$?

4. Some results on $f(z)^2 + (\triangle_c f)^2 = \beta^2$

In this section, we consider a modification of (2), that is

$$f(z)^2 + (\triangle_c f)^2 = \beta^2, \tag{32}$$

which also be considered in [26]. In fact, Liu [15] has considered the equation $f(z)^2+(\triangle_c f)^2 = 1$ to show that there is no transcendental entire solutions with finite order. Recently, Zhang [26] considered the existence of transcendental entire solutions of (32) and raised the following conjecture.

Conjecture. *Let f be a transcendental entire function with finite order. Suppose that f is a solution of* (32) *where β is a small function of f. Then $\beta \equiv 0$.*

Related to the above conjecture, we give the following observations. Liu [17, Theorem 2.3] also considered another difference equations as follows

$$f(z)^2 + P(z)^2(\triangle_c f)^2 = Q(z), \tag{33}$$

and proved that there is no transcendental entire function with finite order if $P(z)$ and $Q(z)$ are nonzero polynomials. Thus, if β is a polynomial in (32), then $\beta \equiv 0$. See the proof in Liu [17, Theorem 2.3], the idea is to decompose the left hand side of (33). Here, we proceed to consider the above conjecture and obtain the following result.

Theorem 4.1. *If β satisfies one the following conditions*:

(1) *β is a nonzero constant*;

(2) *β is a nonzero periodic function with period c*;

(3) *β is an entire function without zeros,*

then there is no any transcendental entire function with finite order that satisfies (32)*, where β is a small function with respect to $f(z)$.*

Proof. Let $f(z)$ be a transcendental entire solution with finite order of (32).

Case (1): If β is a nonzero constant, from (32), then

$$(\triangle_c f)^2 = -[f(z)^2 - \beta^2].$$

Thus, $\triangle_c f$ should have infinite many zeros, otherwise $f - \beta$ and $f + \beta$ has finitely many zeros, which is a contradiction with $f(z)$ is an entire function.

In fact, we have at least one of $f-\beta$ and $f+\beta$ has infinitely many zeros. If both $f-\beta$ and $f+\beta$ have infinitely many zeros, then the zeros should have multiplicities at least two. Thus, we have

$$\overline{N}(r,\frac{1}{f\pm\beta})\leq\frac{1}{2}N(r,\frac{1}{f\pm\beta})\leq\frac{1}{2}T(r,f).$$

Shifting backward the equation (32), we have

$$2f(z-c)^2-2f(z)f(z-c)=\beta^2-f(z)^2=(\triangle_c f)^2,$$

hence

$$2f(z-c)[f(z-c)-f(z)]=[f(z+c)-f(z)]^2. \tag{34}$$

If $f(z-c)$ has finitely many zeros, combining the second main theorem with [2, Theorem 2.2], then

$$\begin{aligned}2T(r,f)&\leq N(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f+\beta}\right)+\overline{N}\left(r,\frac{1}{f-\beta}\right)+S(r,f)\\&\leq\overline{N}\left(r,\frac{1}{f(z-c)}\right)+\overline{N}\left(r,\frac{1}{f+\beta}\right)+\overline{N}\left(r,\frac{1}{f-\beta}\right)+S(r,f)\\&\leq T(r,f(z))+S(r,f),\end{aligned} \tag{35}$$

which is a contradiction. If one of $f-\beta$ and $f+\beta$ have infinitely many zeros, the inequality (35) also holds. Thus $f(z-c)$ has infinitely many zeros. Furthermore, from (34), we have $f(z)$, $f(z-c)$ and $f(z+c)-f(z)$ share 0 IM, otherwise, if $f(z)$ and $f(z-c)$ has no common zeros, then the zeros multiplicities of $f(z-c)$ should at least two, then (35) is replaced by $2T(r,f)\leq\frac{3}{2}T(r,f)+S(r,f)$, which is also impossible. However, $f(z)$, $f(z+c)-f(z)$ share the value 0 IM is impossible from (32).

Case (2): Using the above method, we see that if β is a nonzero small periodic function with respect to $f(z)$, we also have $f(z)$, $f(z-c)$ and $f(z+c)-f(z)$ should share 0 IM. Thus $\overline{N}(r,\frac{1}{f(z)})\leq\overline{N}(r,\frac{1}{\beta})+S(r,f)$. Thus, we also get a contradiction from (35).

Case (3): We consider that β is a transcendental entire function without zeros. Thus $\beta(z)=e^{h(z)}$. It is easy to see that (32) can be written as

$$f(z+c)^2-2f(z+c)f(z)+2f(z)^2=\beta^2. \tag{36}$$

The equation (36) implies that

$$f(z+c)=\frac{\beta(z)}{\sqrt{2}}(U+V),\quad \sqrt{2}f(z)=\frac{\beta(z)}{\sqrt{2}}(U-V),$$

where U,V are entire functions provided that β has no zeros and satisfy the following Fermat equation

$$(1-\frac{\sqrt{2}}{2}t)U^2+(1+\frac{\sqrt{2}}{2}t)V^2=1 \tag{37}$$

and $t^2 = 1$. So we have

$$U = \frac{\sin(\alpha(z))}{\sqrt{1 - \frac{\sqrt{2}}{2}t}} = t_1 \sin(\alpha(z)), \quad V = \frac{\cos(\alpha(z))}{\sqrt{1 + \frac{\sqrt{2}}{2}t}} = t_2 \cos(\alpha(z)).$$

Then

$$f(z) = \frac{\beta(z)}{2}(t_1 \sin(\alpha(z) - t_2 \cos(\alpha(z)) = \frac{\beta(z)}{2}(\sin(\alpha(z) + B))$$

and

$$f(z+c) = \frac{\beta(z)}{\sqrt{2}}(t_1 \sin(\alpha(z) + t_2 \cos(\alpha(z)) = \frac{\beta(z)}{\sqrt{2}}(\sin(\alpha(z) + D)).$$

Thus, we have

$$\frac{\beta(z+c)}{2}(\sin(\alpha(z+c) + B)) = \frac{\beta(z)}{\sqrt{2}}(\sin(\alpha(z) + D)).$$

Furthermore, we have

$$\frac{\beta(z+c)}{\beta(z)}\left(e^{i(\alpha(z+c)+\alpha(z)+B+D)} - e^{-i(\alpha(z+c)-\alpha(z)+B-D)}\right) - \sqrt{2}e^{2i(\alpha(z)+D))}$$
$$= -\sqrt{2}.$$

Using Borel Lemma [24, Theorem 1.62] to above equation, it implies that $\alpha(z)$ should be a linear polynomial. Since $\beta(z)$ is a small function with respect to $f(z)$, then $h(z)$ should be a constant, a contradiction. Thus, we have the proof of Theorem 4.1. $\square$

Acknowledgements. The authors would like to thank the referee for his/her helpful suggestions and comments.

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Kai Liu
Department of Mathematics
Nanchang University
Nanchang, Jiangxi, 330031, P. R. China
Email address: liukai418@126.com, liukai@ncu.edu.cn

Lei Ma
Department of Mathematics
Nanchang University
Nanchang, Jiangxi, 330031, P. R. China
Email address: malei92201@126.com

Xiaoyang Zhai
Dongfang College
Shandong University of Finance and Economics
Taian, Shandong, 271000, P. R. China
Email address: zhaixiaoyang@126.com