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EUCLIDEAN SUBMANIFOLDS WITH CONFORMAL CANONICAL VECTOR FIELD

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Abstract. The position vector field $\mathbf{x}$ is the most elementary and natural geometric object on a Euclidean submanifold M. The position vector field plays very important roles in mathematics as well as in physics. Similarly, the tangential component $\mathbf{x}^T$ of the position vector field is the most natural vector field tangent to the Euclidean submanifold M. We simply call the vector field $\mathbf{x}^T$ the *canonical vector field* of the Euclidean submanifold M.

In earlier articles [4,5,9,11,12], we investigated Euclidean submanifolds whose canonical vector fields are concurrent, concircular, torse-forming, conservative or incompressible. In this article we study Euclidean submanifolds with conformal canonical vector field. In particular, we characterize such submanifolds. Several applications are also given. In the last section we present three global results on complete Euclidean submanifolds with conformal canonical vector field.

1. Introduction

For an n-dimensional submanifold M in the Euclidean m-space $\mathbb{E}^m$, the most elementary and natural geometric object is the position vector field $\mathbf{x}$ of M. The position vector is a Euclidean vector $\mathbf{x} = \overrightarrow{OP}$ that represents the position of a point $P \in M$ in relation to an arbitrary reference origin $O \in \mathbb{E}^m$.

The position vector field plays important roles in physics, in particular in mechanics. For instance, in any equation of motion, the position vector $\mathbf{x}(t)$ is usually the most sought-after quantity because the position vector field defines the motion of a particle (i.e., a point mass): its location relative to a given coordinate system at some time variable t. The first and the second derivatives of the position vector field with respect to time t give the velocity and acceleration of the particle.

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For a Euclidean submanifold M of $\mathbb{E}^m$, there exists a natural decomposition of the position vector field $\mathbf{x}$ given by:

$$\mathbf{x} = \mathbf{x}^T + \mathbf{x}^N, \tag{1.1}$$

where $\mathbf{x}^T$ and $\mathbf{x}^N$ are the tangential and the normal components of $\mathbf{x}$, respectively. We denote by $|\mathbf{x}^T|$ and $|\mathbf{x}^N|$ the lengths of $\mathbf{x}^T$ and of $\mathbf{x}^N$, respectively.

A vector field v on a Riemannian manifold N is called a *torse-forming vector field* if it satisfies (cf. [19–21])

$$\nabla_X v = \varphi X + \alpha(X)v, \quad \forall X \in TN, \tag{1.2}$$

for some function φ and 1-form α on M, where ∇ denotes the Levi-Civita connection of M. In the case that α is identically zero, v is called a *concircular vector field*. In particular, if $\alpha = 0$ and $\varphi = 1$, then v is called a *concurrent vector field*.

In earlier articles, we have investigated Euclidean submanifolds whose canonical vector fields are concurrent [4, 5], concircular [12], torse-forming [11], conservative or incompressible [9]. See [7,8] for two recent surveys on several topics in differential geometry associated with position vector fields on Euclidean submanifolds.

A tangent vector field v on a Riemannian manifold (N, g) is called a *conformal vector field* if it satisfies

$$\mathcal{L}_v g = 2\varphi g, \tag{1.3}$$

where $\mathcal{L}$ denotes the Lie derivative of (N, g) and φ is called the *potential function* of v.

In this article we study Euclidean submanifolds with conformal canonical vector field. In particular, we characterize such submanifolds. Several applications are also given. In the last section we present three global results on complete Euclidean submanifolds with conformal canonical vector field.

2. Preliminaries

Let $x : M \to \mathbb{E}^m$ be an isometric immersion of a connected Riemannian manifold M into a Euclidean m-space $\mathbb{E}^m$. For each point $p \in M$, we denote by T_pM and $T_p^{\perp}M$ the tangent space and the normal space of M at p, respectively.

Let ∇ and $\tilde{\nabla}$ denote the Levi–Civita connections of M and $\mathbb{E}^m$, respectively. The formulas of Gauss and Weingarten are given respectively by (cf. [2, 3, 6])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for vector fields X, Y tangent to M and ξ normal to M, where h is the second fundamental form, D the normal connection and A the shape operator of M.

For each normal vector ξ at p, the shape operator A_ξ is a self-adjoint endomorphism of T_pM. The second fundamental form h and the shape operator A

are related by

$$g(A_\xi X, Y) = \tilde{g}(h(X,Y),\xi), \tag{2.3}$$

where g and $\tilde{g}$ denote the metric of M and the metric of the ambient Euclidean space, respectively.

The mean curvature vector H of an n-dimensional submanifold M is defined by

$$H = \frac{1}{n}\text{trace}\, h. \tag{2.4}$$

A submanifold M is called *totally umbilical* (respectively, *totally geodesic*) if its second fundamental form h satisfies

$$h(X,Y) = g(X,Y)H \tag{2.5}$$

identically (respectively, $h = 0$ identically).

A submanifold is said to be *umbilical with respect to a normal vector field* ξ if its second fundamental form h satisfies

$$\tilde{g}(h(X,Y),\xi) = \mu g(X,Y) \tag{2.6}$$

for some function μ. In particular, a submanifold M is called *pseudo-umbilical* if it is umbilical with respect to the mean curvature vector field H of M.

The Laplace operator Δ of M acting on smooth vector fields on a Riemannian n-manifold (M,g) is defined by

$$\Delta X = \sum_{i=1}^{n}\left(\nabla_{e_i}\nabla_{e_i}X - \nabla_{\nabla_{e_i}e_i}X\right), \tag{2.7}$$

where $\{e_1,\ldots,e_n\}$ is an orthonormal local frame of M.

3. Euclidean submanifolds with conformal canonical vector field

The following result characterizes all Euclidean submanifolds with conformal canonical vector field.

Theorem 3.1. *Let M be a submanifold of the Euclidean m-space $\mathbb{E}^m$. Then the canonical vector field $\mathbf{x}^T$ of M is a conformal vector field if and only if M is umbilical with respect to the normal component $\mathbf{x}^N$ of the position vector field $\mathbf{x}$.*

Proof. Let M be a submanifold of $\mathbb{E}^m$. Then, by using the fact that the position vector field is a concurrent vector, we derive from Gauss' and Weingarten's formulas that

$$Z = \tilde{\nabla}_Z\mathbf{x} = \nabla_Z\mathbf{x}^T + h(\mathbf{x}^T, Z) - A_{\mathbf{x}^N}Z + D_Z\mathbf{x}^N$$

for any vector Z tangent to M, where $\tilde{\nabla}$ and ∇ are the Levi-Civita connections of $\mathbb{E}^{n+1}$ and of M, respectively. By comparing the tangential and normal components of the last equation, we obtain

$$\nabla_Z\mathbf{x}^T = Z + A_{\mathbf{x}^N}Z, \tag{3.1}$$

$$h(\mathbf{x}^T, Z) = -D_Z\mathbf{x}^N. \tag{3.2}$$

On the other hand, it is well-known that the Lie derivative on M satisfies (see, e.g. [6, Page 18] or [22])

$$(\mathcal{L}_v g)(X,Y) = g(\nabla_X v, Y) + g(X, \nabla_Y v) \tag{3.3}$$

for any vector fields X, Y, v tangent to M.

After combining (3.1) and (3.3) we find

$$(\mathcal{L}_{\mathbf{x}^T} g)(X,Y) = 2g(X,Y) + g(A_{\mathbf{x}^N} X, Y) + g(X, A_{\mathbf{x}^N} Y). \tag{3.4}$$

Therefore, by applying (2.3) we obtain

$$(\mathcal{L}_{\mathbf{x}^T} g)(X,Y) = 2g(X,Y) + 2g(h(X,Y), \mathbf{x}^N) \tag{3.5}$$

for vector fields X, Y tangent to M.

Now, let us suppose that the canonical vector field $\mathbf{x}^T$ of the submanifold M is a conformal vector field. Then we have

$$\mathcal{L}_{\mathbf{x}^T} g = 2\varphi g \tag{3.6}$$

for a function φ.

From (3.5) and (3.6) we derive

$$g(h(X,Y), \mathbf{x}^N) = (\varphi - 1)g(X,Y), \tag{3.7}$$

which shows that M is umbilical with respect to the normal component $\mathbf{x}^N$ of the position vector field $\mathbf{x}$.

Conversely, let us assume that the submanifold M is umbilical with respect to the normal component $\mathbf{x}^N$ so that we have

$$g(h(X,Y), \mathbf{x}^N) = \eta g(X,Y) \tag{3.8}$$

for some function η. Then it follows from (3.5) and (3.8) that

$$(\mathcal{L}_{\mathbf{x}^T} g)(X,Y) = 2(\eta + 1)g(X,Y). \tag{3.9}$$

Thus the canonical vector field $\mathbf{x}^T$ is a conformal vector field on M. $\square$

Remark 3.1. By applying the same proof as Theorem 3.1, we also know that Theorem 3.1 remains true for space-like submanifolds of pseudo-Euclidean spaces.

A unit normal vector field ξ of a Euclidean submanifold M is called a *parallel* (resp., *nonparallel*) normal vector field if $D\xi = 0$ (resp., $D\xi \neq 0$) everywhere on M (cf. [2, 13, 14]).

An easy consequence of Theorem 3.1 is the following.

Corollary 3.1. *Let M be a submanifold of $\mathbb{E}^m$ with conformal canonical vector field. If $\mathbf{x}^N \neq 0$ and $\mathbf{x}^N/|\mathbf{x}^N|$ is a parallel normal vector field, then either*

(1) *M lies in a hyperplane $\mathbb{E}^{m-1}$ of $\mathbb{E}^m$, or*

(2) *M lies in hypersphere of S^{m-1} centered the origin of $\mathbb{E}^m$.*

Proof. Let M be a submanifold of $\mathbb{E}^m$ with conformal canonical vector field. If $\mathbf{x}^N \neq 0$ and $\mathbf{x}^N/|\mathbf{x}^N|$ is a parallel normal vector field, then it follows from Theorem 3.1 that M is umbilical with respect to the parallel unit normal vector field $\mathbf{x}^N/|\mathbf{x}^N|$ satisfying (3.8).

If η in (3.8) vanishes identically, then it is easy to verify that M lies in a hyperplane $\mathbb{E}^{m-1}$ of $\mathbb{E}^m$.

On the other hand, if $\eta \neq 0$, then it follows from [13, Theorem 3.3] that M lies in hypersphere of S^{m-1} centered the origin of $\mathbb{E}^m$. □

In the case that M is a Euclidean hypersurface of $\mathbb{E}^{n+1}$ we have:

Corollary 3.2. *Let M be a hypersurface of $\mathbb{E}^{n+1}$ with conformal canonical vector field. If $\mathbf{x}^N \neq 0$, then either*

(1) *M is an open portion of a hypersphere centered at the origin of $\mathbb{E}^{n+1}$ or*
(2) *M is an open portion of a hyperplane which does not contain the origin of $\mathbb{E}^{n+1}$.*

Proof. Let M be a hypersurface of $\mathbb{E}^{n+1}$. Suppose that the canonical vector field $\mathbf{x}^T$ of M is a conformal vector field. If $\mathbf{x}^N \neq 0$, then the unit normal vector field of M is a parallel normal vector field automatically. Hence Theorem 3.1 implies that M lies either in a hypersphere of S^n centered the origin of $\mathbb{E}^{n+1}$ or in a hyperplane of $\mathbb{E}^{n+1}$.

If the second case occurs, then the hyperplane does not contain the origin of $\mathbb{E}^{n+1}$; otherwise one has $\mathbf{x}^N = 0$ which is a contradiction. □

For Euclidean submanifolds of codimension 2, we have the following.

Corollary 3.3. *Let (M, g) be an n-dimensional submanifold of $\mathbb{E}^{n+2}$ with $n > 3$ and $\mathbf{x}^N \neq 0$. If the canonical vector field $\mathbf{x}^T$ of M is a conformal vector field, then we have*:

(1) *If $\frac{\mathbf{x}^N}{|\mathbf{x}^N|}$ is a parallel normal section, then (M, g) lies in either a hyperplane or in a hypersphere of $\mathbb{E}^{n+2}$.*
(2) *If $\frac{\mathbf{x}^N}{|\mathbf{x}^N|}$ is a nonparallel normal section, then (M, g) is a conformally flat space. Moreover, in this case M is the locus of $(n-1)$-spheres.*

Proof. Let (M, g) be an n-dimensional submanifold of $\mathbb{E}^{n+2}$ with $n > 3$ and $\mathbf{x}^N \neq 0$. If the canonical vector field $\mathbf{x}^T$ of M is a conformal vector field, it follows from Theorem 3.1 that M is umbilical with respect the normal direction $\mathbf{x}^N$.

If $\mathbf{x}^N/|\mathbf{x}^N|$ is a parallel normal section, Corollary 3.1 implies that M lies in a hyperplane or in a hypersphere of $\mathbb{E}^{n+2}$.

If $\mathbf{x}^N/|\mathbf{x}^N|$ is a nonparallel normal section, it follows from [14, Theorem 3] that (M, g) is a conformally flat space. Moreover, in this case it also follows from [14, Theorem 4] that the submanifold is a locus of $(n-1)$-spheres in $\mathbb{E}^{n+1}$. □

4. Application to Yamabe solitons

The Yamabe flow was introduced by R. Hamilton at the same time as the Ricci flow (cf. [16]). It deforms a given manifold by evolving its metric according to

$$\frac{\partial}{\partial t} g(t) = -R(t)g(t), \tag{4.1}$$

where $R(t)$ denotes the scalar curvature of the metric $g(t)$. Yamabe solitons correspond to self-similar solutions of the Yamabe flow.

A Riemannian manifold (M, g) is called a *Yamabe soliton* if it admits a vector field v such that

$$\frac{1}{2}\mathcal{L}_v g = (R - \lambda)g, \tag{4.2}$$

where λ is a real number. The vector field v as in the definition is called a *soliton vector field* for (M, g). We denote the Yamabe soliton satisfying (4.2) by (M, g, v, λ).

By applying Theorem 3.1 we have the following.

Corollary 4.1. *If a Euclidean submanifold (M, g) of $\mathbb{E}^m$ is a Yamabe soliton with the canonical vector field $\mathbf{x}^T$ as its soliton vector field, then $\mathbf{x}^T$ is a conformal vector field.*

Proof. Assume that Euclidean submanifold (M, g) of the Euclidean m-space $\mathbb{E}^m$ is a Yamabe soliton with its canonical vector field $\mathbf{x}^T$ as the soliton vector field. Then it follows from [10, Theorem 3.1] that the second fundamental form h of M satisfies

$$\tilde{g}(h(V, W), \mathbf{x}^N) = (R - \lambda - 1)g(V, W) \tag{4.3}$$

for vectors V, W tangent to M, where R is the scalar curvature of M and λ is a constant. Hence M is umbilical with respect to $\mathbf{x}^N$. Consequently, the canonical vector field $\mathbf{x}^T$ is a conformal vector field of M according to Theorem 3.1. □

5. Application to generalized self-similar submanifolds

Consider the mean curvature flow for an isometric immersion $x : M \to \mathbb{E}^m$, that is, consider a one-parameter family $x_t = x(\cdot, t)$ of immersions $x_t : M \to \mathbb{E}^m$ such that

$$\frac{d}{dt} x(p, t) = H(p, t), \;\; x(p, 0) = x(p), \;\; p \in M \tag{5.1}$$

is satisfied, where $H(p, t)$ is the mean curvature vector of M_t in $\mathbb{E}^m$ at $x(p, t)$.

An important class of solutions to the mean curvature flow equations are *self-similar shrinkers* which satisfy a system of quasi-linear elliptic PDEs of the second order, namely,

$$H = -\mathbf{x}^N, \tag{5.2}$$

where $\mathbf{x}^N$ is the normal component of the position vector field of $x : M \to \mathbb{E}^m$ as before. Self-shrinkers play an important role in the study of the mean curvature flow because they describe all possible blow up at a given singularity of a mean curvature flow.

In view of (5.2), we simply call a Euclidean submanifold M a *generalized self-similar submanifold* if it satisfies

$$\mathbf{x}^N = fH \tag{5.3}$$

for some function f.

Obviously, it follows from (5.3) that every Euclidean hypersurface is a generalized self-similar hypersurface automatically.

By applying Theorem 3.1 we have the following.

Corollary 5.1. *Let M be a generalized self-similar submanifold of the Euclidean m-space $\mathbb{E}^m$. Then the canonical vector field of M is a conformal vector field if and only if M is a pseudo-umbilical submanifold.*

Proof. Let M be a generalized self-similar submanifold of $\mathbb{E}^m$. Then we have (5.3). If the canonical vector field of M is a conformal vector field, then (3.7) holds for some function φ. Clearly, it follows from (3.7) and (5.3) that M is pseudo-umbilical.

Conversely, if M is pseudo-umbilical, then (5.3) implies that M is umbilical with respect $\mathbf{x}^N$. Hence Theorem 3.1 implies that the canonical vector field of M is a conformal vector field. ☐

6. Three global results on complete submanifolds with conformal canonical vector field

Recall that Euclidean submanifolds in this article are assumed to be connected (see Preliminaries). In this article, by a *complete submanifold* of $\mathbb{E}^m$ we mean a complete Riemannian manifold isometrically immersed in $\mathbb{E}^m$.

Theorem 6.1. *Suppose that the canonical vector field $\mathbf{x}^T$ on a complete submanifold M of $\mathbb{E}^m$ is non-parallel and conformal. If $\mathbf{x}^T$ satisfies*

$$\Delta \mathbf{x}^T = -\lambda \mathbf{x}^T$$

for a non-negative constant λ, then either M is isometric to an n-sphere $S^n(c)$ or to the Euclidean space $\mathbb{E}^n$ with $n = \dim M$.

Proof. Suppose that the canonical vector field $\mathbf{x}^T$ is a non-parallel, conformal vector field satisfying

$$\mathcal{L}_{\mathbf{x}^T} g = 2\varphi g \tag{6.1}$$

for some function φ. Using equation (3.1), we compute the curvature tensor of the submanifold as

$$R(X,Y)\mathbf{x}^T = \left(\nabla A_{\mathbf{x}^N}\right)(X,Y) - \left(\nabla A_{\mathbf{x}^N}\right)(Y,X),$$

where the covariant derivative

$$\left(\nabla A_{\mathbf{x}^N}\right)(X,Y)=\nabla_X A_{\mathbf{x}^N}Y-A_{\mathbf{x}^N}\nabla_X Y.$$

Using (6.1) and equation (3.7), we compute

$$R(X,Y)\mathbf{x}^T=(X\varphi)Y-(Y\varphi)X.$$

Taking inner product with $\mathbf{x}^T$ in above equation, we get

$$(X\varphi)g(Y,\mathbf{x}^T)=(Y\varphi)g(X,\mathbf{x}^T),$$

that is, $(X\varphi)\mathbf{x}^T=g(X,\mathbf{x}^T)\nabla\varphi$, where $\nabla\varphi$ is the gradient of the function φ. The last relation shows that the vector fields $\nabla\varphi$ and $\mathbf{x}^T$ are parallel. Hence there exists a smooth function β such that

$$\nabla\varphi=\beta\mathbf{x}^T. \tag{6.2}$$

Now, using (3.1), we compute

$$\Delta\mathbf{x}^T=\sum_i\left(\nabla A_{\mathbf{x}^N}\right)(e_i,e_i),$$

which in view of equation (3.7) gives $\Delta\mathbf{x}^T=\nabla\varphi$, which in view of (6.2), yields $\Delta\mathbf{x}^T=\beta\mathbf{x}^T$. Using the condition in the statement, we get $\beta=-\lambda$. Thus equation (6.2) gives

$$\nabla\varphi=-\lambda\mathbf{x}^T, \tag{6.3}$$

which in view of (3.1), gives

$$\nabla_X\nabla\varphi=-\lambda\left(X+A_{\mathbf{x}^N}X\right)=-\lambda\varphi X, \tag{6.4}$$

where we have used (3.8). If φ is not a constant, then equation (6.3) insures that λ is a positive constant (since $\mathbf{x}^T\neq 0$ being a non-parallel vector). Thus, equation (6.4) is Obata's differential equation, which proves that M is isometric to $S^n(\sqrt{\lambda})$ (cf. [17]).

If φ is a constant, then the function

$$f=\frac{1}{2}|\mathbf{x}^T|^2,$$

on using equations (3.1) and (3.7) gives

$$Xf=g(X+A_{\mathbf{x}^N}X,\mathbf{x}^T)=\varphi g(X,\mathbf{x}^T),$$

that is, the gradient ∇f is given by

$$\nabla f=\varphi\mathbf{x}^T. \tag{6.5}$$

Hence, the Hessian H_f of the function f is given by

$$H_f(X,Y)=\varphi^2 g(X,Y). \tag{6.6}$$

Note that if f is a constant function, equation (6.5) would imply either the constant $\varphi=0$ or $\mathbf{x}^T=0$, and both in view of equations (3.1) and (3.7) will imply that $\mathbf{x}^T$ is a parallel vector field, which is contrary to our assumption in the hypothesis. Hence f is a non-constant function that satisfies equation (6.6) for nonzero constant φ^2 implies that M is isometric to the Euclidean space $\mathbb{E}^n$ (cf. [18, Theorem 1]). $\square$

Next, we use the potential function φ of the conformal canonical vector field $\mathbf{x}^T$ and the support function f in the definition (5.3) of generalized self-similar submanifold in proving the next result.

Theorem 6.2. *Let M be a generalized self-similar complete submanifold of the Euclidean space $\mathbb{E}^m$. If the canonical vector field $\mathbf{x}^T$ is a conformal vector field satisfying*

$$Ric(\mathbf{x}^T, \mathbf{x}^T) + \frac{n}{2}\left[\mathbf{x}^T\varphi + |H|^2(\mathbf{x}^T f)\right] \geq 0, \;\; n = \dim M,$$

then either M is isometric to the Euclidean n-space $\mathbb{E}^n$ or it is a submanifold of constant mean curvature of a hypersphere $S^{m-1}(c)$ of $\mathbb{E}^m$.

Proof. Equation (3.7) gives $ng(H, \mathbf{x}^N) = n(\varphi - 1)$, which in view of equation (5.3) yields

$$\varphi = 1 + f|H|^2. \tag{6.7}$$

Taking covariant derivative in equation (5.3) and using (3.2), we get

$$-h(X, \mathbf{x}^T) = (Xf)H + fD_X H. \tag{6.8}$$

Now, using equations (6.7) and (6.8), we have

$$\begin{aligned} X\varphi &= (Xf)|H|^2 + 2fg\,(D_X H, H) \\ &= -(Xf)|H|^2 - 2g(H, h(X, \mathbf{x}^T)). \end{aligned} \tag{6.9}$$

Recall that the expression for Ricci tensor of a submanifold derived from Gauss' equation gives

$$Ric(\mathbf{x}^T, \mathbf{x}^T) = ng(H, h(\mathbf{x}^T, \mathbf{x}^T)) - \sum_i \left\|h(e_i, \mathbf{x}^T)\right\|^2, \tag{6.10}$$

where $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal frame on M.

Inserting equation (6.10) in equation (6.9) gives

$$\mathbf{x}^T\varphi + |H|^2(\mathbf{x}^T f) + \frac{2}{n}Ric(\mathbf{x}^T, \mathbf{x}^T) = -\frac{2}{n}\sum_i \left\|h(e_i, \mathbf{x}^T)\right\|^2,$$

which in view of the condition in the hypothesis gives

$$h(X, \mathbf{x}^T) = 0 \tag{6.11}$$

for X tangent to M. Now, using equation (6.11), we find

$$Ric(X, \mathbf{x}^T) = 0. \tag{6.12}$$

However, using (3.1) and (3.7), we have $R(X, Y)\mathbf{x}^T = (X\varphi)Y - (Y\varphi)X$, which gives

$$Ric(Y, \mathbf{x}^T) = -(n-1)(Y\varphi). \tag{6.13}$$

Thus, in view of equation (6.12), φ is a constant. Therefore equation (6.9) implies that

$$(Xf)|H|^2 = 0. \tag{6.14}$$

Now, define a function

$$F = \frac{1}{2}|\mathbf{x}^T|^2,$$

which has gradient $\nabla F = \varphi \mathbf{x}^T$ and Hessian

$$H_F(X,Y) = \varphi^2 g(X,Y).$$

If F is not a constant, then as $\nabla F = \varphi \mathbf{x}^T$, constant φ^2 is nonzero, then M is isometric to the Euclidean space $\mathbb{E}^n$ (cf. [18]).

If F is a constant, then $|\mathbf{x}^T|$ is constant and equations (3.2) and (6.11) give $|\mathbf{x}^N|$ is constant. Consequently, $|\mathbf{x}|$ is a constant and this proves M is a submanifold of the hypersphere $S^{m-1}(c)$. Now, equation (6.14) gives $(Xf)|H|^2 = 0$, so either $H = \mathbf{0}$ or f is a constant.

Now, we claim that f is a nonzero constant, for if $f = 0$, then equation (5.3) will give $\mathbf{x}^N = 0$, which by equation (3.1) implies $\nabla_X \mathbf{x}^T = X$, and as $|\mathbf{x}^T|$ is a constant, we get $g(X, \mathbf{x}^T) = 0$ for any smooth vector field X tangent to M, that is, $\mathbf{x}^T = 0$, that is, $\mathbf{x} = 0$ and it is a contradiction. Therefore f is a nonzero constant. Consequently, equation (5.3) implies that $|H|$ is constant. □

Recall that a normal vector field ξ to a Euclidean submanifold M is said to be *parallel along a smooth curve* $\gamma : I \to M$ if $D_{\gamma'}\xi \equiv 0$. Also, a smooth function $f : M \to R$ is *constant along* γ if $\gamma' f \equiv 0$.

For a totally geodesic n-space $\mathbb{E}^n$ of $\mathbb{E}^m$, it is known that the canonical vector field $\mathbf{x}^T$ is a concurrent vector field satisfying

$$\nabla_X \mathbf{x}^T = X. \tag{6.15}$$

Hence the canonical vector field $\mathbf{x}^T$ is a non-parallel vector field. Also, it follows from (3.3) and (6.15) that $\mathcal{L}_{\mathbf{x}^T} g = 2g$. Thus the canonical vector field $\mathbf{x}^T$ is a conformal vector field with constant potential $\varphi = 1$. Furthermore, the mean curvature vector field H of $\mathbb{E}^n$ is zero vector which is trivially a parallel normal vector field.

Conversely, we prove the following.

Theorem 6.3. *Let M be a complete submanifold of $\mathbb{E}^m$ whose canonical vector field $\mathbf{x}^T$ is non-parallel and conformal. If the potential function φ of $\mathbf{x}^T$ is constant along the integral curves of $\mathbf{x}^T$ and the mean curvature vector field H of M is parallel along the integral curves of $\mathbf{x}^T$, then M is isometric to a Euclidean space.*

Proof. Suppose that the potential function φ of $\mathbf{x}^T$ is constant along the integral curves of $\mathbf{x}^T$ and that the mean curvature vector field H of M is parallel along the integral curves of $\mathbf{x}^T$. Then we have $\mathbf{x}^T\varphi = 0$ and $D_{\mathbf{x}^T}H = 0$. Then, by applying (6.13), we get

$$Ric(\mathbf{x}^T, \mathbf{x}^T) = -(n-1)\mathbf{x}^T\varphi = 0. \tag{6.16}$$

Also, equation (3.7) implies $g(H, \mathbf{x}^N) = \varphi - 1$, which, in view of the fact that H is parallel along the integral curves of $\mathbf{x}^T$, the equation (3.2) gives

$$\mathbf{x}^T\varphi = -g(H, h(\mathbf{x}^T, \mathbf{x}^T)).$$

Since φ is constant along integral curves of $\mathbf{x}^T$, equation (6.16) and above equation yield

$$g(H, h(\mathbf{x}^T, \mathbf{x}^T)) = 0. \tag{6.17}$$

Using equations (6.16) and (6.17) in equation (6.10) gives

$$h(X, \mathbf{x}^T) = 0$$

for any X tangent to M. The above equation implies $Ric(X, \mathbf{x}^T) = 0$ for X tangent to M, which proves $X\varphi = 0$. Hence φ is a constant.

Now, define the function $f = \frac{1}{2}\left|\mathbf{x}^T\right|^2$, which in view of (3.1) and (3.7) gives the gradient ∇f and the Hessian of f as

$$\nabla f = \varphi\mathbf{x}^T, \quad H_f(X, Y) = \varphi^2 g(X, Y). \tag{6.18}$$

If f is constant, then (6.18) implies either constant $\varphi = 0$ or $\mathbf{x}^T = 0$ and both of these in view of equations (3.1), (3.7) will imply that $\mathbf{x}^T$ is a parallel vector field which is contrary to our assumption. Hence f must be a nonconstant function satisfying the Hessian condition in (6.18) with nonzero constant φ. Consequently, M is isometric to a Euclidean space (cf. [18, Theorem 1]). □

Remark 6.1. For further global results on compact Euclidean submanifolds with conformal canonical vector fields, see [1, 15].

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