Bull. Korean Math. Soc. **55** (2018), No. 6, pp. 1755–1771
https://doi.org/10.4134/BKMS.b171049
pISSN: 1015-8634 / eISSN: 2234-3016

PROPERTIES ON q-DIFFERENCE RICCATI EQUATION

Zhi-Bo Huang and Ran-Ran Zhang

Abstract. In this paper, we investigate a certain type of q-difference Riccati equation in the complex plane. We prove that q-difference Riccati equation possesses a one parameter family of meromorphic solutions if it has three distinct meromorphic solutions. Furthermore, we find that all meromorphic solutions of q-difference Riccati equation and corresponding second order linear q-difference equation can be expressed by q-gamma function if this q-difference Riccati equation admits two distinct rational solutions and $q \in \mathbb{C}$ such that $0 < |q| < 1$. The growth and value distribution of differences of meromorphic solutions of q-difference Riccati equation are also treated.

1. Introduction

Let $q \in \mathbb{C}$ such that $0 < |q| < 1$. It is well known that q-gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},$$

where $(a;q)_\infty = \prod_{k=0}^{\infty}\left(1-aq^k\right)$. Here we take the principal vales of q^x and $(1-q)^x$. Thus, $\Gamma_q(x)$ is a meromorphic function with poles at $x = -n \pm 2\pi ik/\log q$, where k and n are non-negative integers, see [1]. Define

$$\gamma_q(z) := (1-q)^{x-1}\Gamma_q(x), \quad z = q^x,$$

and $\gamma_q(0) := (q;q)_\infty$, we obtain that $\gamma_q(z)$ is a meromorphic function of zero order with no zeros, having its poles at $\left\{q^{-k}\right\}_{k=0}^{\infty}$.

We easily conclude that the first order linear q-difference equation

$$h(qz) = (1-z)h(z)$$

is solved by the function $\gamma_q(z)$.

We then consider a general first order linear q-difference equation

$$h(qz) = a(z)h(z), \tag{1.1}$$

Received December 2, 2017; Revised February 25, 2018; Accepted March 16, 2018.
2010 *Mathematics Subject Classification.* Primary 39B32; Secondary 30D35.
Key words and phrases. q-difference Riccati equation, q-difference equation, q-gamma function.

where $a(z)$ is a rational function. If $a(z) \equiv a$ is a constant, equation (1.1) is solvable in terms of rational functions if and only if $\log_q a$ is an integer. If $a(z)$ is a nonconstant rational function, let α_i, $i = 1, 2, \ldots, n$ and β_j, $j = 1, 2, \ldots, m$ be the zeros and poles of $a(z)$, respectively, repeated according to their multiplicities. Then $a(z)$ can be written in the form

$$a(z) = \frac{c(1 - z/\alpha_1)(1 - z/\alpha_2) \cdots (1 - z/\alpha_n)}{(1 - z/\beta_1)(1 - z/\beta_2) \cdots (1 - z/\beta_m)},$$

where $c \neq 0$ is a complex number depending on $a(z)$. Thus, equation (1.1) is solved by

$$h(z) = z^{\log_q c} \frac{\gamma_q(z/\alpha_1)\gamma_q(z/\alpha_2) \cdots \gamma_q(z/\alpha_n)}{\gamma_q(z/\beta_1)\gamma_q(z/\beta_2) \cdots \gamma_q(z/\beta_m)}, \tag{1.2}$$

which is meromorphic if and only if $\log_q c$ is an integer.

In this paper, we are concerned with the q-difference Riccati equation

$$f(qz) = \frac{A(z) + f(z)}{1 - (q-1)zf(z)}, \tag{1.3}$$

and second order linear q-difference equation

$$\Delta_q^2 y(z) + \frac{A(z)}{(q-1)z} y(z) = 0, \tag{1.4}$$

where $q \in \mathbb{C}\backslash\{0\}, |q| \neq 1$, $A(z)$ is a meromorphic function. Here, for a meromorphic function $y(z)$, the q-difference operator Δ_q is defined by $\Delta_q y(z) = \frac{y(qz)-y(z)}{(q-1)z}$ and $\Delta_q^{n+1} y(z) = \Delta_q \left(\Delta_q^n y(z)\right)$, $n = 1, 2, \ldots$ [1, p. 488].

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's value distribution theory of meromorphic function, see, e.g., [14].

Recently, a number of papers, see e.g. [6,7,10,12,13,15,17,19,20,22], focused on complex difference equations and difference analogues of Nevanlinna theory. q-difference counterparts are also investigated [3,19,23]. But there are only few papers concerning with the properties of meromorphic solutions of q-difference equations, see e.g. [16, 18, 21, 24].

The remainder of the paper is organized as follows. A system of solutions of q-difference Riccati equation (1.3) is stated in Section 2. Section 3 contains the relationships between q-difference equations and q-gamma function. The growth and value distribution of differences on solutions of q-difference Riccati equation (1.3) is investigated in Section 4.

2. A system of solutions of q-difference Riccati equation

Let f_1, f_2, f_3 be distinct meromorphic solutions of differential Riccati equation

$$w'(z) + w(z)^2 + A(z) = 0. \tag{2.1}$$

Then (2.1) possesses a one parameter family of meromorphic solutions $(f_c)_{c\in\mathbb{C}}$, see e.g., [2, pp. 371–373].

Ishizaki extended this property to the difference Riccati equation

$$\Delta f(z) + \frac{f(z)^2 + A(z)}{f(z) - 1} = 0, \tag{2.2}$$

and obtained the following difference analogue of this property as follows.

Theorem 2.A ([17, Proposition 2.1]). *Suppose that* (2.2) *possesses three distinct meromorphic solutions* $f_1(z)$, $f_2(z)$ *and* $f_3(z)$. *Then any meromorphic solution* $f(z)$ *of* (2.2) *can be represented by*

$$f(z) = \frac{f_1(z)f_2(z) - f_2(z)f_3(z) - f_1(z)f_2(z)Q(z) + f_1(z)f_3(z)Q(z)}{f_1(z) - f_3(z) - f_2(z)Q(z) + f_3(z)Q(z)}, \tag{2.3}$$

where $Q(z)$ *is a periodic function of period* 1. *Conversely, if for any periodic function* $Q(z)$ *of period* 1, *we define a function* $f(z)$ *by* (2.3), *then* $f(z)$ *is a meromorphic solution of* (2.2).

We then extend this property to the q-difference Riccati equation (1.3) and obtained a q-difference analogue as follows.

Theorem 2.1. *Suppose that* (1.3) *possesses three distinct meromorphic solutions* $f_1(z)$, $f_2(z)$ *and* $f_3(z)$. *Then a meromorphic solution* $f(z)$ *of* (1.3) *can be represented by*

$$f(z) = \frac{f_1(z)f_3(z) - f_1(z)f_2(z) + f_1(z)f_2(z)\phi(z) - f_2(z)f_3(z)\phi(z)}{f_3(z) - f_2(z) - f_3(z)\phi(z) + f_1(z)\phi(z)}, \tag{2.4}$$

where $\phi(z)$ *is a meromorphic function satisfying* $\phi(qz) = \phi(z)$. *Conversely, if for any meromorphic function* $\phi(z)$ *satisfying* $\phi(qz) = \phi(z)$, *we define a function* $f(z)$ *by* (2.4), *then* $f(z)$ *is a meromorphic solution of* (1.3).

Proof. Using a similar proof of Theorem 2 in [16]. Let $h_j(z)$, $j = 1, 2, 3, 4$ be distinct meromorphic functions. We define a cross ratio of $h_j(z)$, $j = 1, 2, 3, 4$ by

$$R(h_1, h_2, h_3, h_4; z) := \frac{h_4(z) - h_1(z)}{h_4(z) - h_2(z)} : \frac{h_3(z) - h_1(z)}{h_3(z) - h_2(z)}.$$

We first show that $f(z)$, distinct from $f_1(z), f_2(z)$ and $f_3(z)$, is a meromorphic solution of (1.3) if and only if $R(qz) = R(z)$, where $R(z) = R(f_1, f_2, f_3, f; z)$. Thus, we conclude from (1.3) that

$$\begin{aligned} R(qz) &= \frac{f(qz) - f_1(qz)}{f(qz) - f_2(qz)} : \frac{f_3(qz) - f_1(qz)}{f_3(qz) - f_2(qz)} \\ &= \frac{\frac{[(q-1)zA(z)+1][f(z)-f_1(z)]}{[1-(q-1)zf(z)][1-(q-1)zf_1(z)]}}{\frac{[(q-1)zA(z)+1][f(z)-f_2(z)]}{[1-(q-1)zf(z)][1-(q-1)zf_2(z)]}} : \frac{\frac{[(q-1)zA(z)+1][f_3(z)-f_1(z)]}{[1-(q-1)zf_3(z)][1-(q-1)zf_1(z)]}}{\frac{[(q-1)zA(z)+1][f_3(z)-f_2(z)]}{[1-(q-1)zf_3(z)][1-(q-1)zf_2(z)]}} \\ &= \frac{f(z) - f_1(z)}{f(z) - f_2(z)} : \frac{f_3(z) - f_1(z)}{f_3(z) - f_2(z)} = R(z). \end{aligned}$$

On the other hand, if $R(qz) = R(z)$, then

$$\frac{f(qz) - \frac{A(z)+f_1(z)}{1-(q-1)zf_1(z)}}{f(qz) - \frac{A(z)+f_2(z)}{1-(q-1)zf_2(z)}} : \frac{\frac{A(z)+f_3(z)}{1-(q-1)zf_3(z)} - \frac{A(z)+f_1(z)}{1-(q-1)zf_1(z)}}{\frac{A(z)+f_3(z)}{1-(q-1)zf_3(z)} - \frac{A(z)+f_2(z)}{1-(q-1)zf_2(z)}}$$
$$= \frac{f(z)-f_1(z)}{f(z)-f_2(z)} : \frac{f_3(z)-f_1(z)}{f_3(z)-f_2(z)},$$

and so,

$$\frac{f(qz) - \frac{A(z)+f_1(z)}{1-(q-1)zf_1(z)}}{f(qz) - \frac{A(z)+f_2(z)}{1-(q-1)zf_2(z)}} : \frac{\frac{[(q-1)zA(z)-1](f_3(z)-f_1(z))}{[1-(q-1)zf_3(z)][1-(q-1)zf_1(z)]}}{\frac{[(q-1)zA(z)-1](f_3(z)-f_2(z))}{[1-(q-1)zf_3(z)][1-(q-1)zf_2(z)]}}$$
$$= \frac{f(z)-f_1(z)}{f(z)-f_2(z)} : \frac{f_3(z)-f_1(z)}{f_3(z)-f_2(z)}. \tag{2.5}$$

We then conclude from (2.5) that $f(qz) = \frac{A(z)+f(z)}{1-(q-1)zf(z)}$, which shows that $f(z)$ satisfies (1.3).

Thus, for any meromorphic function $\phi(z)$ satisfying $\phi(qz) = \phi(z)$, we define $f(z)$ by

$$R(f_1, f_2, f_3, f; z) = \phi(z).$$

Then $f(z)$ is represented by (2.4), and also satisfies (1.3). The proof of Theorem 2.1 is completed. □

It is difficult for us to detect the properties of meromorphic solutions since the parameter function $Q(z)$ in Theorem 2.A and $\phi(z)$ in Theorem 2.1 appear more than one time. Furthermore, we note that $f(z) \neq f_2(z)$ in Theorem 2.1. This shows that the representation of (2.4) cannot represent all meromorphic solutions of q-difference Riccati equation (1.3). Thus, we can use a new method used in [8, Theorem 8.3.4], and prove a family of solutions of q-difference Riccati equation (1.3).

Theorem 2.2. *Let $q \in \mathbb{C}\backslash\{0\}$, $|q| \neq 1$, and $A(z)$ be a meromorphic function with $A(z) \neq -\frac{1}{(q-1)z}$. If q-difference Riccati equation* (1.3) *possesses three distinct meromorphic solutions $f_0(z)$, $f_1(z)$ and $f_2(z)$, then all meromorphic solutions of q-difference Riccati equation* (1.3) *constitute a one parameter family*

$$\left\{ f_0(z),\ f(z) = \frac{(f_1(z)-f_0(z))(f_2(z)-f_0(z))}{\phi(z)(f_2(z)-f_1(z)) + (f_2(z)-f_0(z))} + f_0(z) \right\}, \tag{2.6}$$

where $\phi(z)$ is any constant in $\mathbb{C}$, or any non-zero meromorphic function with $\phi(qz) = \phi(z)$; as $\phi(z) \equiv 0, f(z) = f_1(z)$; as $\phi(z) \equiv -1, f(z) = f_2(z)$.

In particular, if $\phi(z)$ is any constant in $\mathbb{C}$, we obtain:

Corollary 2.1. *Let $q \in \mathbb{C}\backslash\{0\}$, $|q| \neq 1$, and $A(z)$ be a meromorphic function with $A(z) \neq -\frac{1}{(q-1)z}$. If q-difference Riccati equation* (1.3) *possesses three distinct rational solutions $f_0(z)$, $f_1(z)$ and $f_2(z)$, then q-difference Riccati equation* (1.3) *has infinitely many rational solutions.*

We now list some preliminaries to prove Theorem 2.2.

Lemma 2.1. *Let $q \in \mathbb{C}\backslash\{0\}$, $|q| \neq 1$, and $A(z)$ be a meromorphic function with $A(z) \not\equiv -\frac{1}{(q-1)z}$. If $f(z)$ is a meromorphic solution of q-difference Riccati equation* (1.3), *then*

$$1-(q-1)zf(z) \not\equiv 0 \text{ and } 1+(q-1)zf(qz) \not\equiv 0.$$

Proof. If $1-(q-1)zf(z) \equiv 0$, then $f(z) = \frac{1}{(q-1)z}$. Now substituting $f(z) = \frac{1}{(q-1)z}$ into (1.3), and noting that $A(z) \not\equiv -\frac{1}{(q-1)z}$, we conclude that

$$\frac{1}{(q-1)qz} = \frac{A(z)+\frac{1}{(q-1)z}}{1-(q-1)z\cdot\frac{1}{(q-1)z}} = \frac{(q-1)zA(z)+1}{0} = \infty.$$

This yields that $q=0$ or $q=1$, a contradiction.

If $1+(q-1)zf(qz) \equiv 0$, then $f(qz) = -\frac{1}{(q-1)z}$ and $f(z) = -\frac{q}{(q-1)z}$. Now substituting these into (1.3), we deduce that $A(z) = -\frac{1}{(q-1)z}$, a contradiction. □

Lemma 2.2. *Let $q \in \mathbb{C}\backslash\{0\}$, $|q| \neq 1$, $A_1(z)$ and $A_0(z)$ be nonzero meromorphic functions. If q-difference equation*

$$A_1(z)y(qz)+A_0(z)y(z)=0 \tag{2.7}$$

has a nonzero meromorphic solution $y_0(z)$, then all meromorphic solutions of (2.7) *constitute a one parameter family*

$$\{y(z)=\phi(z)y_0(z)\},$$

where $\phi(z)$ is any constant in $\mathbb{C}$, or any nonzero meromorphic function with $\phi(qz)=\phi(z)$.

Proof. Since $y_0(z)$ is a nonzero meromorphic solution of (2.7), we easily conclude that $y(z)=\phi(z)y_0(z)$ is also a meromorphic solution of (2.7) for any constant $\phi(z)$ in $\mathbb{C}$, or any non-zero meromorphic function with $\phi(qz)=\phi(z)$.

On the other hand, if $y(z)$ is also a meromorphic solution of (2.7), we conclude from (2.7) that

$$\frac{y(qz)}{y_0(qz)} \equiv \frac{y(z)}{y_0(z)}.$$

Set $\phi(z) = \frac{y(z)}{y_0(z)}$. Then $\phi(z)$ is a constant in $\mathbb{C}$, or a nonzero meromorphic function with $\phi(qz)=\phi(z)$. This shows that $y(z)=\phi(z)y_0(z)$. □

We now give the proof of Theorem 2.2.

Proof of Theorem 2.2. Since $f_0(z)$, $f_1(z)$ and $f_2(z)$ are three distinct meromorphic solutions of q-difference Riccati equation (1.3), we set

$$u_j(z) = \frac{1}{f_j(z)-f_0(z)}\ ,\ j=1,2. \tag{2.8}$$

Obviously, $u_1(z) \not\equiv u_2(z)$ and $f_j(z) = \frac{1}{u_j(z)}+f_0(z),\ j=1,2.$

Now, substituting $f_j(z) = \frac{1}{u_j(z)} + f_0(z),\ j = 1, 2$ into (1.3), and noting that $f_0(z)$ is also a meromorphic solution of (1.3), we conclude that

$$[1 + (q-1)zf_0(qz)]u_j(qz) - [1 - (q-1)zf_0(z)]u_j(z) + (q-1)z = 0.$$

Set

$$\alpha_1(z) = 1 + (q-1)zf_0(qz) \text{ and } \alpha_0(z) = (q-1)zf_0(z) - 1.$$

Then we deduce from Lemma 2.1 that $\alpha_1(z) \not\equiv 0$ and $\alpha_0(z) \not\equiv 0$, and $u_j(z),\ j = 1, 2$ are two distinct meromorphic solutions of q-difference equation

$$\alpha_1(z)u(qz) + \alpha_0(z)u(z) + (q-1)z = 0. \tag{2.9}$$

Thus, $u_0(z) = u_1(z) - u_2(z)$ is a nonzero meromorphic solution of q-difference equation

$$\alpha_1(z)u(qz) + \alpha_0(z)u(z) = 0, \tag{2.10}$$

which is a corresponding linear homogeneous q-difference equation of (2.9).

Therefore, we deduce from Lemma 2.2 that all meromorphic solutions of (2.10) constitute a one parameter family

$$H(y(z)) = \{y(z) = \phi(z)u_0(z)\},$$

where $\phi(z)$ is any constant in $\mathbb{C}$, or any non-zero meromorphic function with $\phi(qz) = \phi(z)$. This yields that q-difference equation (2.9) has a general solution

$$\begin{aligned} u(z) &= y(z) + u_1(z) = \phi(z)u_0(z) + u_1(z) \\ &= \phi(z)[u_1(z) - u_2(z)] + u_1(z) \\ &= \frac{\phi(z)[f_2(z) - f_1(z)]}{[f_1(z) - f_0(z)][f_2(z) - f_0(z)]} + \frac{1}{f_1(z) - f_0(z)}. \end{aligned} \tag{2.11}$$

We now suppose that $f(z)(\not\equiv f_0(z))$ is a meromorphic solution of (1.3), and conclude from the argumentation of (2.9) that $u(z) = \frac{1}{f(z) - f_0(z)}$ is also a meromorphic solution of (2.9). Thus, we deduce from (2.11) that there exists a constant $\phi(z)$ in $\mathbb{C}$, or any non-zero meromorphic function $\phi(z)$ with $\phi(qz) = \phi(z)$ such that

$$\frac{1}{f(z) - f_0(z)} = \frac{\phi(z)[f_2(z) - f_1(z)]}{[f_1(z) - f_0(z)][f_2(z) - f_0(z)]} + \frac{1}{f_1(z) - f_0(z)}.$$

Therefore, we obtain that

$$f(z) = \frac{[f_1(z) - f_0(z)][f_2(z) - f_0(z)]}{\phi(z)[f_2(z) - f_1(z)] + [f_2(z) - f_0(z)]} + f_0(z), \tag{2.12}$$

where $\phi(z)$ is any constant in $\mathbb{C}$, or any non-zero meromorphic function with $\phi(qz) = \phi(z)$. This shows that any meromorphic solution $f(z)(\not\equiv f_0(z))$ of (1.3) has the form (2.12).

We then affirm that any meromorphic function $f(z)(\not\equiv f_0(z))$ denoted by (2.12) must be a meromorphic solution of (1.3). In fact, we conclude from (2.11) and (2.12) that

$$f(z) = \frac{1}{u(z)} + f_0(z), \tag{2.13}$$

where $u(z)$ satisfies the q-difference equation (2.9). Thus, we further conclude from (2.9), (1.3) and the assumption that $f_0(z)$ is a meromorphic solution of (1.3), that

$$\begin{aligned} f(qz) &= \frac{1}{u(qz)} + f_0(qz) \\ &= \frac{\alpha_1(z)}{-\alpha_0(z)u(z) - (q-1)z} + f_0(qz) \\ &= \frac{1 + (q-1)zf_0(qz)}{[1-(q-1)zf_0(z)]u(z) - (q-1)z} + f_0(qz) \\ &= \frac{1 + [1-(q-1)zf_0(z)]u(z)f_0(qz)}{[1-(q-1)zf_0(z)]u(z) - (q-1)z} \\ &= \frac{1 + [1-(q-1)zf_0(z)]u(z) \cdot \frac{A(z)+f_0(z)}{1-(q-1)zf_0(z)}}{[1-(q-1)zf_0(z)]u(z) - (q-1)z} \\ &= \frac{1 + [A(z)+f_0(z)]u(z)}{[1-(q-1)zf_0(z)]u(z) - (q-1)z}. \end{aligned} \tag{2.14}$$

On the other hand, we can obtain from (2.13) that

$$\begin{aligned} \frac{A(z)+f(z)}{1-(q-1)zf(z)} &= \frac{A(z) + \frac{1}{u(z)} + f_0(z)}{1-(q-1)z\left[\frac{1}{u(z)} + f_0(z)\right]} \\ &= \frac{1 + [A(z)+f_0(z)]u(z)}{[1-(q-1)zf_0(z)]u(z) - (q-1)z}. \end{aligned} \tag{2.15}$$

Therefore, we deduce from (2.14) and (2.15) that

$$f(qz) = \frac{A(z)+f(z)}{1-(q-1)zf(z)},$$

which shows that meromorphic function $f(z)(\not\equiv f_0(z))$ denoted by (2.12) is a meromorphic solution of (1.3). The proof of Theorem 2.2 is completed. □

If q-difference Riccati equation (1.3) possesses a rational solution $f_0(z)$ such that $f_0(z) \not\to 0$ as $z \to \infty$, we further obtain:

Theorem 2.3. *Let $q \in \mathbb{C}\backslash\{0\}$, $|q| \neq 1$, and $A(z)$ be a meromorphic function. If q-difference Riccati equation* (1.3) *possesses a rational solution $f_0(z)$ such that $f_0(z) \not\to 0$ as $z \to \infty$, then q-difference Riccati equation* (1.3) *has at most two rational solutions.*

Proof. Contrary to the assumption, we suppose that (1.3) has three distinct rational solutions $f_0(z)$, $f_1(z)$ and $f_2(z)$, where $f_0(z) \not\to 0$ as $z \to \infty$.

Set

$$u_j(z) = \frac{1}{f_j(z) - f_0(z)}\ ,\ j = 1, 2.$$

Obviously, $u_1(z)$ and $u_2(z)$ are rational functions with $u_1(z) \not\equiv u_2(z)$, and $f_j(z) = \frac{1}{u_j(z)} + f_0(z),\ j = 1, 2$.

Now, substituting $f_j(z) = \frac{1}{u_j(z)} + f_0(z),\ j = 1, 2$ into (1.3), and noting that $f_0(z)$ is also a rational solution of (1.3), we conclude that

$$[1 + (q - 1)zf_0(qz)]u_j(qz) - [1 - (q - 1)zf_0(z)]u_j(z) + (q - 1)z = 0,\ j = 1, 2.$$

This shows that $u_1(z)$ and $u_2(z)$ are two distinct rational solutions of q-difference equation

$$[1 + (q - 1)zf_0(qz)]u(qz) - [1 - (q - 1)zf_0(z)]u(z) + (q - 1)z = 0, \tag{2.16}$$

and $u_0(z) = u_1(z) - u_2(z)$ is a nonzero rational solution of

$$[1 + (q - 1)zf_0(qz)]u(qz) - [1 - (q - 1)zf_0(z)]u(z) = 0. \tag{2.17}$$

Since $f_0(z)(\not\to 0, z \to \infty)$ and $u_0(z)$ are both rational functions, we can set

$$f_0(z) = \frac{P(z)}{Q(z)} \text{ and } u_0(z) = \frac{U(z)}{V(z)},$$

where $P(z), Q(z), U(z)$ and $V(z)$ are nonzero polynomials with $\deg P(z) \geq \deg Q(z)$.

Now substituting $f_0(z) = \frac{P(z)}{Q(z)}$ and $u_0(z) = \frac{U(z)}{V(z)}$ into (2.17), we conclude that

$$\begin{aligned} &Q(z)Q(qz)U(qz)V(z) + (q - 1)zP(qz)Q(z)U(qz)V(z) \\ &- Q(z)Q(qz)U(z)V(qz) + (q - 1)zP(z)Q(qz)U(z)V(qz) = 0. \end{aligned} \tag{2.18}$$

We can obtain that

$$\begin{aligned} \deg\{Q(z)Q(qz)U(qz)V(z)\} &= \deg\{Q(z)Q(qz)U(z)V(qz)\} \\ &< \deg\{(q - 1)zP(qz)Q(z)U(qz)V(z)\} \\ &= \deg\{(q - 1)zP(z)Q(qz)U(z)V(qz)\}, \end{aligned}$$

and at most one of the coefficient of power $z^{\deg\{Q(z)Q(qz)U(qz)V(z)\}}$ and the coefficient of power $z^{\deg\{(q-1)zP(qz)Q(z)U(qz)V(z)\}}$ is zero. These all show that the degree of left hand side of (2.18) is great than 1, and yield a contradiction.

Thus, (1.3) has at most two rational solutions. The proof of Theorem 2.3 is completed. □

We now present two examples to show that Theorem 2.3 remain valid.

Example 2.1. Let $q = \frac{1}{2}$. Then ration function $f_0(z) = 2z + 4$ solves the q-difference Riccati equation

$$f\left(\frac{1}{2}z\right) = \frac{z^3 + 6z^2 + 7z + f(z)}{1 + \frac{z}{2}f(z)} \tag{2.19}$$

of type (1.3), and $f_0(z) = 2z+4 \to \infty$ as $z \to \infty$. Suppose that $f_1(z)(\not\equiv f_0(z))$ is another rational solution of (2.19). Set $u_1(z) = \frac{1}{f_1(z)-f_0(z)}$. Then we conclude that $u_1(z)$ satisfies the q-difference equation

$$(z^2 + 4z - 2)u\left(\frac{z}{2}\right) + 2(z^2 + 2z + 1)u(z) + z = 0. \tag{2.20}$$

According to the proof of Theorem 2.2, we note that all meromorphic solutions $f(z)$ (except exceptional solution $f_0(z)$) of (2.19) and all solutions $u(z) = \frac{1}{f(z)-f_0(z)}$ of (2.20) are one-one corresponding.

However, the q-difference equation, which is the corresponding homogeneous difference equation of (2.20),

$$(z^2 + 4z - 2)u\left(\frac{z}{2}\right) + 2(z^2 + 2z + 1)u(z) = 0, \tag{2.21}$$

has no nonzero rational solution. Otherwise, suppose that $u(z) = \frac{P(z)}{Q(z)}$ is a nonzero rational solution of (2.21), where $P(z)$ and $Q(z)$ are nonzero polynomials with degree $\deg P(z) = p$ and $\deg Q(z) = q$ respectively. Then we conclude from (2.21) that

$$(z^2 + 4z - 2)P\left(\frac{z}{2}\right)Q(z) + 2(z^2 + 2z + 1)P(z)Q\left(\frac{z}{2}\right) = 0. \tag{2.22}$$

We can easily deduce that the degree of left hand side of (2.22) is great than 2 since $P(z)$ and $Q(z)$ are nonzero polynomials, and yields a contradiction. Hence, we obtain that (2.19) has at most two rational solutions $f_0(z) = 2z+4$ and $f_1(z) = \frac{1}{u_1(z)} + 2z + 4$.

Example 2.2. Let $q = \frac{1}{2}$ and $A(z) = \frac{2(z+1)(z+2)}{z(z^2-3z-2)}$. Then the ration function $f_0(z) = \frac{z-1}{z+1}$ solves the q-difference Riccati equation

$$f\left(\frac{1}{2}z\right) = \frac{A(z) + f(z)}{1 + \frac{z}{2}f(z)} \tag{2.23}$$

of type (1.3), and $f_0(z) = \frac{z-1}{z+1} \to 1$ as $z \to \infty$. Suppose that $f_1(z)(\not\equiv f_0(z))$ is another rational solution of (2.23). Set $u_1(z) = \frac{1}{f_1(z)-f_0(z)}$. Then we conclude that $u_1(z)$ satisfies the q-difference equation

$$(z^3 - 3z^2 - 8z - 4)u\left(\frac{z}{2}\right) + (z^3 + 3z^2 + 4z + 4)u(z) + z(z^2 + 3z + 2) = 0. \tag{2.24}$$

By using similar calculation of Example 2.1, the q-difference equation, which is the corresponding homogeneous difference equation of (2.24),

$$(z^3 - 3z^2 - 8z - 4)u\left(\frac{z}{2}\right) + (z^3 + 3z^2 + 4z + 4)u(z) = 0, \tag{2.25}$$

has no nonzero rational solution. Thus, we obtain that (2.23) has at most two rational solutions $f_0(z) = \frac{z-1}{z+1}$ and $f_1(z) = \frac{1}{u_1(z)} + \frac{z-1}{z+1}$.

3. Relationships between *q*-difference equation and *q*-gamma function

In this section, we focus on the relationships between q-difference equation and q-gamma function, and firstly obtain the following result.

Theorem 3.1. *Let $q \in \mathbb{C}$ with $0 < |q| < 1$. Suppose that q-difference Riccati equation* (1.3) *possesses two distinct rational solutions $f_1(z)$ and $f_2(z)$. Then all meromorphic solutions of q-difference Riccati equation* (1.3) *are concerned with q-gamma function.*

Proof. Since $f_1(z)$ and $f_2(z)$ are two distinct rational solutions of (1.3), we construct a Möbius translation

$$f(z) = \frac{f_1(z)h(z) + f_2(z)}{h(z) + 1}. \tag{3.1}$$

Substituting (3.1) into (1.3), we conclude that

$$h(qz) = \frac{1 - (q-1)zf_1(z)}{1 - (q-1)zf_2(z)}h(z), \tag{3.2}$$

which is type of (1.1). Thus, the meromorphic solution $h(z)$ of (3.2) has the form (1.2), which is concerned with q-gamma function. The proof of Theorem 3.1 is completed. □

Now, we give an example to give a presentation for Theorem 3.1.

Example 3.1. Let $q = -\frac{1}{2}$, $A(z) = -\frac{6z}{(z+1)(z-2)}$ in (1.3). Then the functions

$$f_1(z) = \frac{1}{z+1} \text{ and } f_2(z) = \frac{-2}{z+1} \tag{3.3}$$

satisfy the q-difference Riccati equation (1.3). Then, by using the transformation (3.1), we can switch q-difference Riccati equation (1.3) into the type (3.2) and conclude that

$$h\left(-\frac{1}{2}z\right) = \frac{\left(1 - \frac{z}{-\frac{2}{3}}\right)}{\left(1 - \frac{z}{\frac{1}{2}}\right)}h(z),$$

which is type of (1.1), and so

$$h(z) = \frac{\gamma_{-\frac{1}{2}}\left(\frac{z}{-\frac{2}{3}}\right)}{\gamma_{-\frac{1}{2}}\left(\frac{z}{\frac{1}{2}}\right)} = \frac{\gamma_{-\frac{1}{2}}\left(-\frac{3z}{2}\right)}{\gamma_{-\frac{1}{2}}(2z)}. \tag{3.4}$$

We then conclude from (3.1) and (3.4) that

$$f(z)=\frac{\gamma_{-\frac{1}{2}}\left(-\frac{3z}{2}\right)-2\gamma_{-\frac{1}{2}}(2z)}{(z+1)\left(\gamma_{-\frac{1}{2}}\left(-\frac{3z}{2}\right)+\gamma_{-\frac{1}{2}}(2z)\right)},$$

which is concerned with q-gamma function.

We second show that solutions of second order q-difference equation (1.4) are also concerning with q-gamma function. Thus, we investigate the passage between q-difference Riccati equation (1.3) and second order q-difference equation (1.4), and obtain the following result.

Theorem 3.2. *The passage between q-difference Riccati equation* (1.3) *and second order q-difference equation* (1.4) *is*

$$f(z)=-\frac{\Delta_q y(z)}{y(z)}=-\frac{y(qz)-y(z)}{(q-1)zy(z)}. \tag{3.5}$$

Proof. We first prove that $f(z)$ defined as (3.5) is a meromorphic solution of (1.3) if $y(z)$ is a nontrivial meromorphic solution of (1.4). In fact, we conclude from (3.5) that

$$\begin{aligned}\Delta_q^2 y(z)&=\Delta_q(\Delta_q y(z))\\&=\Delta_q(-f(z)y(z))\\&=\frac{-f(qz)y(qz)+f(z)y(z)}{(q-1)z}\\&=\frac{-f(qz)[y(z)-(q-1)zf(z)y(z)]+f(z)y(z)}{(q-1)z}\\&=\frac{-f(qz)y(z)[1-(q-1)zf(z)]+f(z)y(z)}{(q-1)z}.\end{aligned} \tag{3.6}$$

Thus, we deduce from (1.4) and (3.6) that

$$-f(qz)y(z)[1-(q-1)zf(z)]+f(z)y(z)=-A(z)y(z),$$

which implies the desired form of equation (1.3).

We second prove that a meromorphic function $y(z)$ satisfying (3.5) is a meromorphic solution of (1.4) if $f(z)$ defined as (3.5) is a meromorphic solution of (1.3).

In fact, we conclude from (3.6) and (1.3) that

$$\begin{aligned}\Delta_q^2 y(z)&=\frac{-f(qz)y(z)[1-(q-1)zf(z)]+f(z)y(z)}{(q-1)z}\\&=\frac{-\frac{A(z)+f(z)}{1-(q-1)zf(z)}y(z)[1-(q-1)zf(z)]+f(z)y(z)}{(q-1)z}\\&=-\frac{A(z)}{(q-1)z}y(z),\end{aligned}$$

which implies the desired form of (1.4). □

Thus, we deduce from Theorems 3.1 and 3.2 that:

Theorem 3.3. *Let* $q \in \mathbb{C}$ *with* $0 < |q| < 1$. *Suppose that* q*-difference Riccati equation* (1.3) *possesses two distinct rational solutions* $f_1(z)$ *and* $f_2(z)$. *Then all meromorphic solutions of second order* q*-difference equation* (1.4) *are concerned with* q*-gamma function.*

4. Value distribution of solutions of q-difference Riccati equations

If $g(z)$ is a transcendental meromorphic solution of equation

$$g(qz) = R(z, g(z)), \tag{4.1}$$

where $q \in \mathbb{C}, |q| > 1$, and the coefficients of $R(z, g(z))$ are small functions relative to $g(z)$, Gundersen et al. [11] showed that the order of growth of equation (4.1) is equal to $\log \deg_g(R)/\log|q|$, where $\deg_g(R)$ is the degree of irreducible rational function $R(z, g(z))$ in $g(z)$, which means that all transcendental meromorphic solutions of q-difference Riccati equation (1.3) have zero order when $q \in \mathbb{C}$, $|q| > 1$.

On the other hand, second order q-difference equation (1.4) is equivalent to the second order linear q-difference equation

$$y\left(q^2 z\right) - (q+1)y(qz) + q[1 + (q-1)zA(z)]y(z) = 0. \tag{4.2}$$

Bergweiler et al. [4] pointed out that all transcendental meromorphic solutions of equation (4.2) satisfy $T(r, f) = O((\log r)^2)$ if $q \in \mathbb{C}$ and $0 < |q| < 1$. This indicates that all transcendental meromorphic solutions of equation (1.4) satisfy $T(r, f) = O((\log r)^2)$ if $q \in \mathbb{C}$ and $0 < |q| < 1$. Since (3.5) is a passage between (1.3) and (1.4), we deduce that all transcendental meromorphic solutions of q-difference Riccati equation (1.3) are of zero order if $q \in \mathbb{C}$ and $0 < |q| < 1$. Thus, we obtain the following result.

Theorem 4.1. *All transcendental meromorphic solutions of* q*-difference Riccati equation* (1.3) *are of zero order for all* $q \in \mathbb{C}\backslash\{0\}$ *and* $|q| \neq 1$.

Bergweiler et al. [5] first investigated the existence of zeros of $\Delta_c f(z) = f(z+c) - f(z)$ and $\frac{\Delta_c f(z)}{f(z)}$, where $c \in \mathbb{C}\backslash\{0\}$, and obtained many profound results (see [5, Theorem 1.2-Theorem 1.4 and Theorem 1.6]). Chen and Shon [9] then extended these results of [5] and proved a number of results concerned with the existence of zeros and fixed points of $\Delta_c f(z) = f(z+c) - f(z)$ and $\frac{\Delta_c f(z)}{f(z)}$, where $c \in \mathbb{C}\backslash\{0\}$ (see [9, Theorem 1-Theorem 6]). Zhang and Chen [22] further considered the difference Riccati equation

$$f(z+1) = \frac{p(z+1)w(z) + q(z)}{w(z) + p(z)}, \tag{4.3}$$

where $p(z)$ and $q(z)$ are small functions relative to $f(z)$, and obtained the following results.

Theorem 4.A ([22, Theorem 1.1]). *Let $p(z)$, $q(z)$ be meromorphic functions of finite order, and let $[p(z+1)f(z)+q(z)]/[f(z)+p(z)]$ be an irreducible function in $f(z)$. Suppose that $f(z)$ is an admissible finite order meromorphic solution of* (4.3). *Set $\Delta f(z) = f(z+1) - f(z)$. Then*

(i) $\lambda\left(\frac{1}{\Delta f(z)}\right) = \sigma(\Delta f(z)) = \sigma(f)$, $\lambda\left(\frac{1}{\Delta f(z)/f(z)}\right) = \sigma\left(\frac{\Delta f(z)}{f(z)}\right) = \sigma(f)$;

(ii) *If $q(z) \not\equiv 0$, then* $\lambda\left(\frac{1}{f}\right) = \lambda(f) = \sigma(f)$;

(iii) *If $p(z) \equiv p$ is a constant and $q(z) \equiv s(z)^2$, where $s(z)$ is a non-constant rational function, then $f(z)$ has no Borel exceptional value and* $\lambda(\Delta f(z)) = \lambda\left(\frac{\Delta f(z)}{f}\right) = \sigma(f)$.

We now consider the value distribution of differences of transcendental meromorphic solutions of q-difference Riccati equation (1.3) as follows.

Theorem 4.2. *Let $A(z)$ be a non-constant rational function, $q \in \mathbb{C}\backslash\{0\}$ and $|q| \neq 1$. Suppose that $f(z)$ is a transcendental meromorphic solution of q-difference Riccati equation* (1.3). *Set $\Delta_q f(z) = \frac{f(qz)-f(z)}{(q-1)z}$.*

(1) *If $A(z) \neq -\frac{1}{(q-1)z}$, then $\Delta_q f(z)$ and $\frac{\Delta_q f(z)}{f(z)}$ have infinitely many poles*;

(2) *If $A(z) = -(q-1)zs(z)^2$, where $s(z)$ is a non-constant rational function, then $\Delta_q f(z) = \frac{f(qz)-f(z)}{(q-1)z}$ and $\frac{\Delta_q f(z)}{f(z)}$ have infinitely many zeros.*

Remark 4.1. The similar result of Theorem 4.2(2) has been obtained in [18]. For the completeness, we list it again.

In order to prove Theorem 4.2, we need some lemmas.

Lemma 4.1 ([3, Theorem 1.2]). *Let $f(z)$ be a non-constant zero-order meromorphic function, and $q \in \mathbb{C}\backslash\{0\}$. Then*

$$m\left(r, \frac{f(qz)}{f(z)}\right) = o(T(r,f))$$

on a set of logarithmic density 1.

Lemma 4.2 ([19, Theorem 2.5]). *Let $f(z)$ be a transcendental meromorphic solution of order zero of a q-difference equation of the form*

$$U_q(z,f)P_q(z,f) = Q_q(z,f),$$

where $U_q(z,f)P_q(z,f)$ and $Q_q(z,f)$ are q-difference polynomials such that the total degree $\deg U_q(z,f) = n$ in $f(z)$ and its q-shifts, $\deg Q_q(z,f) \leq n$. Moreover, we assume that $U_q(z,f)$ contains just one term of maximal total degree in $f(z)$ and its q-shifts. Then

$$m(r, P_q(z,f)) = o(T(r,f))$$

on a set of logarithmic density 1.

Lemma 4.3 ([3, Theorem 2.2])**.** *Let $f(z)$ be a nonconstant zero order meromorphic solution of*

$$P(z, f) = 0,$$

where $P(z, f)$ is a q-difference polynomial in $f(z)$. If $P(z, \alpha) \not\equiv 0$ for a small function $\alpha(z)$ relative to $f(z)$, then

$$m\left(r, \frac{1}{f-\alpha}\right) = o(T(r, f))$$

on a set of logarithmic density 1.

Proof of Theorem 4.2. (1) Since $f(z)$ is a transcendental meromorphic solution of q-difference Riccati equation (1.3), we deduce from (1.3) that

$$(q-1)zf(qz)f(z) = f(qz) - f(z) - A(z). \tag{4.4}$$

Thus, we conclude from Lemma 4.1, Lemma 4.2 and (4.4) that

$$\begin{aligned} m(r, f) &\leq m\left(r, \frac{f(z)}{f(qz)}\right) + m(r, f(qz)) + o(T(r, f)) \\ &= o(T(r, f)) \end{aligned}$$

on a set of logarithmic density 1, and so

$$N(r, f) = T(r, f) + o(T(r, f)), \tag{4.5}$$

$$\begin{aligned} m(r, \Delta_q f(z)) &\leq m\left(r, \frac{\Delta_q f(z)}{f(z)}\right) + m(r, f(z)) + o(T(r, f)) \\ &= o(T(r, f)) \end{aligned} \tag{4.6}$$

on a set of logarithmic density 1.

We further conclude from (1.3) that

$$\Delta_q f(z) = \frac{1}{(q-1)z} \cdot \frac{A(z) + (q-1)zf(z)^2}{1-(q-1)zf(z)}. \tag{4.7}$$

Thus, we apply Valiron-Mohon'ko Theorem to (4.7) and get that

$$T(r, \Delta_q f(z)) = 2T(r, f) + o(T(r, f)). \tag{4.8}$$

We then obtain from (4.6) and (4.8) that

$$N(r, \Delta_q f(z)) = 2T(r, f) + o(T(r, f))$$

on a set of logarithmic density 1, and so $\Delta_q f(z)$ has infinitely many poles.

We note that

$$\begin{aligned} N\left(r, \frac{\Delta_q f(z)}{f(z)}\right) &\geq N(r, \Delta_q f(z)) - N(r, f) \\ &= T(r, f) + o(T(r, f)) \end{aligned}$$

on a set of logarithmic density 1, and so $\frac{\Delta_q f(z)}{f(z)}$ has infinitely many poles.

(2) Let

$$P(z,f)=(q-1)zf(z)f(qz)-f(qz)+f(z)+A(z). \tag{4.9}$$

We then affirm that $P(z,s(z))\not\equiv 0$ or $P(z,-s(z))\not\equiv 0$. Otherwise, if $P(z,s(z))\equiv 0$ and $P(z,-s(z))\equiv 0$, we can obtain from (4.9) that

$$s(qz)=s(z).$$

This is impossible since $s(z)$ is a non-constant rational function. Without loss of generality, we assume that $P(z,s(z))\not\equiv 0$. Thus, we obtain from Lemma 4.3, (1.3) and (4.9) that

$$m\left(r,\frac{1}{f(z)-s(z)}\right)=o(T(r,f))$$

on a set of logarithmic density 1, and so

$$N\left(r,\frac{1}{f(z)-s(z)}\right)=T(r,f)+o(T(r,f)) \tag{4.10}$$

on a set of logarithmic density 1.

Since $A(z)=-(q-1)zs(z)^2$, we can conclude from (1.3) and (4.7) that

$$\Delta_q f(z)=\frac{[f(z)+s(z)][f(z)-s(z)]}{1-(q-1)zf(z)}. \tag{4.11}$$

If $f(z_0)-s(z_0)=1-(q-1)z_0f(z_0)=0$, then $(q-1)z_0s(z_0)=0$. If $f(z_0)-s(z_0)=0$ and $f(z_0)+s(z_0)=\infty$, then $s(z_0)=\infty$. Thus, we deduce from (4.10) and (4.11) that

$$\begin{aligned}N\left(r,\frac{1}{\Delta_q f(z)}\right)&=N\left(r,\frac{1-(q-1)zf(z)}{[f(z)+s(z)][f(z)-s(z)]}\right)\\&\geq N\left(r,\frac{1}{f(z)-s(z)}\right)\\&=T(r,f)+o(T(r,f))\end{aligned}$$

on a set of logarithmic density 1. This shows that $\frac{1}{\Delta_q f(z)}$ has infinitely many zeros.

We now obtain from (4.11) that

$$\frac{\Delta_q f(z)}{f(z)}=\frac{[f(z)+s(z)][f(z)-s(z)]}{[1-(q-1)zf(z)]f(z)}. \tag{4.12}$$

By combining (4.5) and (4.12), and using similar method that $\frac{1}{\Delta_q f(z)}$ has infinitely many zeros, we can conclude that $\frac{\Delta_q f(z)}{f(z)}$ has infinitely many zeros. The proof of Theorem 4.2 is completed. □

Acknowledgements. The authors would like to thank the referees for making helpful remarks and valuable suggestions to improve this paper. The first author is partly supported by Guangdong National Natural Science Foundation(No.2014A030313422), and the second author is partly supported by Guangdong National Natural Science Foundation(No.2016A030313745).

References

[1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications, **71**, Cambridge University Press, Cambridge, 1999.

[2] S. B. Bank, G. G. Gundersen, and I. Laine, *Meromorphic solutions of the Riccati differential equation*, Ann. Acad. Sci. Fenn. Ser. A I Math. **6** (1981), no. 2, 369–398 (1982).

[3] D. C. Barnett, R. G. Halburd, W. Morgan, and R. J. Korhonen, *Nevanlinna theory for the q-difference operator and meromorphic solutions of q-difference equations*, Proc. Roy. Soc. Edinburgh Sect. A **137** (2007), no. 3, 457–474.

[4] W. Bergweiler, K. Ishizaki, and N. Yanagihara, *Meromorphic solutions of some functional equations*, Methods Appl. Anal. **5** (1998), no. 3, 248–258. (Correction: Methods Appl. Anal. **6** (1999), no. 4, 617–618).

[5] W. Bergweiler and J. K. Langley, *Zeros of differences of meromorphic functions*, Math. Proc. Cambridge Philos. Soc. **142** (2007), no. 1, 133–147.

[6] Z. X. Chen, *On growth, zeros and poles of meromorphic solutions of linear and nonlinear difference equations*, Sci. China Math. **54** (2011), no. 10, 2123–2133.

[7] ______, *On properties of meromorphic solutions for difference equations concerning gamma function*, J. Math. Anal. Appl. **406** (2013), no. 1, 147–157.

[8] ______, *Complex Differences and Difference equations*, Scince Press, Beijing, 2014.

[9] Z. X. Chen and K. H. Shon, *On zeros and fixed points of differences of meromorphic functions*, J. Math. Anal. Appl. **344** (2008), no. 1, 373–383.

[10] Y.-M. Chiang and S.-J. Feng, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, Ramanujan J. **16** (2008), no. 1, 105–129.

[11] G. G. Gundersen, J. Heittokangas, I. Laine, J. Rieppo, and D. G. Yang, *Meromorphic solutions of generalized Schröder equations*, Aequationes Math. **63** (2002), no. 1-2, 110–135.

[12] R. G. Halburd and R. J. Korhonen, *Difference analogue of the lemma on the logarithmic derivative with applications to difference equations*, J. Math. Anal. Appl. **314** (2006), no. 2, 477–487.

[13] ______, *Nevanlinna theory for the difference operator*, Ann. Acad. Sci. Fenn. Math. **31** (2006), no. 2, 463–478.

[14] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[15] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo, and K. Tohge, *Complex difference equations of Malmquist type*, Comput. Methods Funct. Theory **1** (2001), no. 1, [On table of contents: 2002], 27–39.

[16] Z.-B. Huang, *On q-difference Riccati equations and second-order linear q-difference equations*, J. Complex Anal. **2013**, Art. ID 938579, 10 pp.

[17] K. Ishizaki, *On difference Riccati equations and second order linear difference equations*, Aequationes Math. **81** (2011), no. 1-2, 185–198.

[18] Y. Jiang and Z. Chen, *On solutions of q-difference Riccati equations with rational coefficients*, Appl. Anal. Discrete Math. **7** (2013), no. 2, 314–326.

[19] I. Laine and C.-C. Yang, *Clunie theorems for difference and q-difference polynomials*, J. Lond. Math. Soc. (2) **76** (2007), no. 3, 556–566.

[20] J. Wang, *Growth and poles of meromorphic solutions of some difference equations*, J. Math. Anal. Appl. **379** (2011), no. 1, 367–377.

[21] Z.-T. Wen, *Finite logarithmic order solutions of linear q-difference equations*, Bull. Korean Math. Soc. **51** (2014), no. 1, 83–98.

[22] R. Zhang and Z. Chen, *On meromorphic solutions of Riccati and linear difference equations*, Acta Math. Sci. Ser. B Engl. Ed. **33** (2013), no. 5, 1243–1254.

[23] J. Zhang and R. Korhonen, *On the Nevanlinna characteristic of $f(qz)$ and its applications*, J. Math. Anal. Appl. **369** (2010), no. 2, 537–544.

[24] X. Zheng and Z. Chen, *On properties of q-difference equations*, Acta Math. Sci. Ser. B Engl. Ed. **32** (2012), no. 2, 724–734.

Zhi-Bo Huang
School of Mathematical Sciences
South China Normal University
Guangzhou, 510631, P. R. China
Email address: huangzhibo@scnu.edu.cn; zhibo.huang@ucl.ac.uk

Ran-Ran Zhang
Department of Mathematics
Guangdong University of Education
Guangzhou, 510303, P. R. China
Email address: zhangranran@gdei.edu.cn