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A CLASS OF EDGE IDEALS WITH REGULARITY AT MOST FOUR

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Abstract. If a graph G is both claw-free and gap-free, then E. Nevo showed that the Castelnuovo-Mumford regularity of the associated edge ideal $I(G)$ is at most three. Later Dao, Huneke and Schwieg gave a simpler proof of this result. In this paper we introduce a class of edge ideals with Castelnuovo-Munmford regularity at most four.

1. Introduction

Let I be a homogeneous ideal in the polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$. Suppose that the minimal free resolution of I is given by

$$0 \longrightarrow \bigoplus_j S(-j)^{\beta_{p,j}} \longrightarrow \cdots \longrightarrow \bigoplus_j S(-j)^{\beta_{1,j}} \longrightarrow \bigoplus_j S(-j)^{\beta_{0,j}} \longrightarrow I \longrightarrow 0.$$

The Castelnuovo-Mumford regularity (or simply, regularity) of I, denoted by $\operatorname{reg}(I)$, is defined as

$$\operatorname{reg}(I) = \max\{i \mid \beta_{j,j+i}(I) \neq 0\},$$

and is an important invariant in commutative algebra and algebraic geometry. Computing and finding bounds for the regularity of a monomial ideal have been studied by a number of researchers (see for example [2, 3, 5, 7–11]).

Let G be a graph without isolated vertices. Recall that the edge ideal of G is

$$I(G) = (x_i y_j : \{x_i, y_j\} \text{ is an edge of } G).$$

For any graph G, we write $\operatorname{reg}(G)$ as shorthand for $\operatorname{reg}(I(G))$.

A classical result due to Fröberg says that $\operatorname{reg}(G) = 2$ if and only if the complementary graph G^c is chordal, i.e., has no induced cycle of length at least four (see [4]). In 2011, E. Nevo showed that for a graph G which is both claw-free and gap-free, the Castelnuovo-Munmford regularity of the associated edge ideal $I(G)$ is at most three (see [9, Theorem 2.1]). Later Dao, Huneke and Schwieg gave a simpler proof of this result (see [3, Theorem 3.4]). By

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this motivation, we introduce a class of edge ideals with Castelnuovo-Mumford regularity at most four.

2. Preliminaries

In this section, we provide the definitions and the basic facts which will be used in the next section.

Let G be a finite, simple graph with vertex set $V(G)$ and edge set $E(G)$. For $v, w \in V(G)$, we write $d(v, w)$ for the *distance* between v and w, the minimum number of edges in a path from v to w.

A subgraph $H \subseteq G$ is called *induced* if $\{v, w\}$ is an edge of H whenever v and w are vertices of H and $\{v, w\}$ is an edge of G.

The *complement* of a graph G, denote by G^c, is the graph on the same vertex set as G, in which $\{x, y\}$ is an edge of G^c if and only if it is not an edge of G.

We let C_n denote the cycle on n vertices, K_n denote the complete graph on n vertices and $k_{m,n}$ denote the complete bipartite graph with m vertices on one side, and n on the other. Adding a *whisker* to G at a vertex v means adding a new vertex u and the edge $\{u, v\}$ to G. The graph which is obtained from G by adding a whisker to all of its vertices will be denoted by $W(G)$.

A subset $M \subseteq E(G)$ is a *matching* if $e \cap e' = \emptyset$ for every pair of edges $e, e' \in M$. The cardinality of the largest matching of G is called the *matching number* of G and is denoted by $\text{match}(G)$. The minimum cardinality of the maximal matchings of G is the *minimum matching number* of G and is denoted by $\text{min-match}(G)$. A matching M of G is an *induced matching* of G if for every pair of edges $e, e' \in M$, there is no edge $f \in E(G) \setminus M$ with $f \subset e \cup e'$. The cardinality of the largest induced matching of G is called the *induced matching number* of G and is denoted by $\text{ind-match}(G)$.

Let G be a graph. We say two edges $\{w, x\}$ and $\{y, z\}$ form a gap in G if G does not have an edge with one endpoint in $\{w, x\}$ and the other in $\{y, z\}$. In other words, a gap is an induced matching of size two. A graph which has no gap as an induced subgraph is called *gap-free*. Equivalently, G is gap-free if G^c contains no induced C_4.

Any graph isomorphic to the complete bipartite graph $k_{1,3}$ is called a *claw*. A graph without an induced claw is called *claw-free*.

Recall that the *star* of a vertex x of G for which we write $\text{st}(x)$, is given by

$$\text{st}(x) = \{y \in V(G) : \{x, y\} \text{ is an edge of } G\} \cup \{x\}.$$

The following lemma from [3] has a crucial role in this paper.

Lemma 2.1 ([3, Lemma 3.2])**.** *Let x be a vertex of G. Then*

$$\text{reg}(G) \leq \max\{\text{reg}(G - st(x)) + 1, \text{reg}(G - x)\}.$$

Moreover, $\text{reg}(G)$ *is equal to one of these terms.*

3. Main results

In this section, we prove the main result of this paper. Namely, in Theorem 3.3, we introduce a class of graphs with regularity at most four.

Definition 3.1. Let G be a graph. We say three edges $\{w_1, w_2\}$, $\{w_2, w_3\}$, $\{w_4, w_5\}$ form a 3-*gap* in G if G does not have an edge with one end point in $\{w_1, w_2, w_3\}$ and the other in $\{w_4, w_5\}$. A graph which has no induced 3-gap is called 3-*gap-free.*

In the following proposition, we study a property of 3-gap-free graphs which will be used in the proof of Theorem 3.3.

Proposition 3.2. *Let G be a 3-gap-free graph, and let x be vertex of G of highest degree. Then $d(x, y) \leq 3$ for all vertices y of G.*

Proof. By contradiction, assume that G has a vertex y with $d(x, y) = 4$. Let x have degree m, and list neighbors of x as $w_1, w_2, \ldots, w_m$. Without loss of generality suppose that $\{w_1, z_1\}$, $\{z_1, z_2\}$, $\{z_2, y\}$ are edges of G for some vertices z_1, z_2. For any i with $2 \leq i \leq m$, $\{x, w_i\}$ and $\{z_1, z_2\}$, $\{z_2, y\}$ dose not form a 3-gap in G. Thus, there must be an edge with one end point in $\{x, w_i\}$ and one in $\{z_1, z_2, y\}$. Because $d(x, y)$ and $d(x, z_i)$ both exceed 1, this edge cannot have x as an end point. If $d(x, z_i) = 1$, then $d(x, y) = 3 - (i - 1)$ and this contradict $d(x, y) = 4$. Similarly, $\{w_i, y\}$ can not be an edge, as otherwise, we would have $d(x, y) = 2$, which is again a contradiction. Thus $\{z_1, w_i\}$ is an edge for each i with $1 \leq i \leq m$ (note that we already established that $\{z_1, w_1\}$ is an edge). Since $\{z_1, y\}$ is an edge of G as well, the degree of z_1 exceeds m, which is a contradiction. Similarly if $\{w_i, z_2\}$ is an edge for each i with $1 \leq i \leq m$, then degree z_2 exceeds m, which is impossible. $\square$

We are now ready to prove the main result of this paper.

Theorem 3.3. *Let G be a 3-gap-free graph which does not have the following subgraphs as an induced subgraph.*

(1) (2)

Then $\operatorname{reg}(G) \leq 4$.

Proof. We use induction on $|V(G)|$. There is nothing to prove if $|V(G)| \leq 4$. Thus, assume that $|V(G)| \geq 5$. Let x be a vertex of G of highest degree. By Lemma 2.1, we know that

$$\operatorname{reg}(G) \leq \max\{\operatorname{reg}(G - \operatorname{st}(x)) + 1, \operatorname{reg}(G - x)\}.$$

Note that $G - x$ satisfies the assumptions. Hence, by induction hypothesis, we have $\operatorname{reg}(G - x) \leq 4$. It remains to be show that $\operatorname{reg}(G - \operatorname{st}(x)) \leq 3$. By [4] it is enough to show that $(G - \operatorname{st}(x))^c$ contains no induced cycle of length ≥ 4. Suppose on the contrary that $y_1, y_2, \dots, y_m$ are the vertices of an induced cycle in $(G - \operatorname{st}(x))^c$ with $m \geq 4$. By Proposition 3.2, the distance of each y_i from x in G is at most 3. Note that $d(x, y_1) \neq 1$, as y_1 belongs to vertices $G - \operatorname{st}(x)$. If $d(x, y_1) = 2$, then $\{x, z_1\}$ and $\{z_1, y_1\}$ are edges of G for some vertex z_1. Further $\{y_2, y_m\}$ is an edge of G, since y_2 and y_m are not adjacent in $(G - st(x))^c$. Note that $\{y_1, y_2\}$ and $\{y_1, y_m\}$ are not edges of G. Since G is 3-gap-free it contains either of edges $\{z_1, y_2\}$ or $\{z_1, y_m\}$. Since the graph (1) is not an induced subgraph of G, we conclude that G must have both edges $\{z_1, y_2\}$ and $\{z_1, y_m\}$. On the other hand, in this case G contains the graph (2) as an induced subgraph which is a contradiction. Now assume that $d(x, y_1) = 3$. Then $\{x, w_1\}$, $\{w_1, w_2\}$, $\{w_2, y_1\}$ are edges of G for some vertices w_1 and w_2 in G. Further, y_2 and y_m are not adjacent in $(G - \operatorname{st}(x))^c$. Note that $\{y_1, y_2\}$ and $\{y_1, y_m\}$ are not edges of G. Since G is 3-gap-free, it contains either of edges $\{w_2, y_m\}$ or $\{w_2, y_2\}$. As the graph (1) is not an induced subgraph of G, we conclude that G must have both edges $\{w_2, y_2\}$ and $\{w_2, y_m\}$. On the other hand, in this case, G contains the graph (2) as an induced subgraph which is a contradiction. Similarly, $\{w_1, y_2\}$ and $\{w_1, y_m\}$ are not edges of G. □

The following example shows that the assumptions of Theorem 3.3 can not be dropped.

Example 3.4. (1) For every $n \geq 11$, consider the n-cycle graph C_n. Then G has no induced subgraph isomorphic to graphs (1) and (2) of Theorem 3.3. On the other hand, it is clear that C_n contains a 3-gap. We know form [1, Theorem 1.2] that for every $n \geq 11$, the regularity of C_n is at least 5. Thus, the assumption of being 3-gap-free can not be removed from the hypothesis of Theorem 3.3.

(2) For every integer $m \geq 4$, let G be the union of m triangles which have a common vertex. Then clearly, G is 3-gap-free and has no induced subgraph isomorphic to graph (1) of Theorem 3.3. However, it has induced subgraphs isomorphic to graph (2) of Theorem 3.3. As G is a chordal graph, it follows from [6, Corollary 6.9] that $\operatorname{reg}(G) = \operatorname{ind-match}(G) + 1 = m + 1 \geq 5$. Thus, the assumption of having no induced subgraph isomorphic to graph (2) can not be removed from the hypothesis of Theorem 3.3.

(3) For every $n \geq 4$, let $G = W(K_{1,n})$ be the graph obtained from the complete bipartite graph $K_{1,n}$ by attaching a whisker to all of its vertices. Then clearly, G is 3-gap-free and has no induced subgraph isomorphic to graph (2) of Theorem 3.3. However, it has induced subgraphs isomorphic to graph (1) of Theorem 3.3. As G is a chordal graph, it follows from [6, Corollary 6.9] that $\operatorname{reg}(G) = \text{ind-match}(G) + 1 = n + 1 \geq 5$.

Thus, the assumption of having no induced subgraph isomorphic to graph (1) can not be removed from the hypothesis of Theorem 3.3.

By [12], we know that for every graph G,

$$(*) \qquad \mathrm{reg}(I(G)) \leq \text{min-match}(G) + 1.$$

The following example shows that the conclusion of Theorem 3.3 does not follow from this inequality.

Example 3.5. For every integer $m \geq 4$, consider the graph K_{2m}. Then min-match$(G) = m \geq 4$. Thus, the upper bound given in inequality $(*)$ is at least 5, which is weaker than the bound provided in Theorem 3.3. We recall the well-known fact that the regularity of any complete graph is two.

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