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ON COMMUTATIVITY OF REGULAR PRODUCTS

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Abstract. We study the one-sided regularity of matrices in upper triangular matrix rings in relation with the structure of diagonal entries. We next consider a ring theoretic condition that ab being regular implies ba being also regular for elements a, b in a given ring. Rings with such a condition are said to be *commutative at regular product* (simply, *CRP* rings). CRP rings are shown to be contained in the class of directly finite rings, and we prove that if R is a directly finite ring that satisfies the descending chain condition for principal right ideals or principal left ideals, then R is CRP. We obtain in particular that the upper triangular matrix rings over commutative rings are CRP.

This article concerns a ring property related to directly finite (or Dedekind finite) condition, which extends the study of one-sided inverses (e.g., Baer [3] and Jacobson [9]) to one of one-sided regularity. In Section 1, the structure of diagonal entries of one-sided regular upper triangular matrices are investigated, and this gives useful information to observe the commutativity of regular products in upper triangular matrix rings. In Section 2, we study the structure of rings which satisfy the commutativity of regular products. In the procedure we observe directly finite rings which do not satisfy the commutativity of regular products, which provides interesting information to our study.

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. Use $U(R)$, $N^*(R)$, and $N(R)$ to denote the group of units, the upper nilradical (i.e., sum of all nil ideals), and the set of all nilpotent elements in R, respectively. Clearly $N^*(R) \subseteq N(R)$. The polynomial (resp., power series) ring with an indeterminate x over R is denoted by $R[x]$ (resp., $R[[x]]$). $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). Denote the n by n full (resp., upper triangular) matrix ring over R by $\mathrm{Mat}_n(R)$ (resp., $T_n(R)$), and $D_n(R)$ denotes the subring $\{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ of $T_n(R)$. Use E_{ij} for the matrix with (i, j)-entry 1 and elsewhere 0. These notations are usually used in the literature.

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1. One-sided regularity of upper triangular matrices

In this section we investigate the right (left) regularity of matrices in upper triangular matrix rings. We follow the literature in using the next definitions. An element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. The *left regular* can be defined similarly. An element is *regular* if it is both left and right regular.

Theorem 1.1. *Let R be a ring, $n \geq 2$, and $(a_{ij}) \in T_n(R)$.*

(1) *If a_{ii} is right (resp. left) regular in R for all $i \in \{1, \ldots, n\}$, then (a_{ij}) is right (resp. left) regular in $T_n(R)$.*
(2) *If (a_{ij}) is right (resp., left) regular in $T_n(R)$, then a_{11} is right regular (resp., a_{nn} is left regular) in R.*
(3) *Let R be a commutative ring. Then (a_{ij}) is right or left regular in $T_n(R)$ if and only if a_{ii} is regular in R for all $i \in \{1, \ldots, n\}$ if and only if (a_{ij}) is regular in $T_n(R)$.*

Proof. (1) Let $A = (a_{ij})$. Suppose that every a_{ii} is right regular for $1 \leq i \leq n$, and let $AB = 0$ for $B = (b_{st}) \in T_n(R)$. Clearly $b_{ss} = 0$ for all s. Assume $B \neq 0$ on the contrary. Take $b_{ef} \neq 0$ so that e and f are largest respectively. Then $e < f$ and the (e, f)-entry of AB is $a_{ee}b_{ef} \neq 0$, hence $AB \neq 0$, contrary to $AB = 0$. Thus $B = 0$. The proof for the case of left regular is similar.

(2) Suppose that $A = (a_{ij})$ is right regular in $T_n(R)$. Assume on the contrary that $a_{11}b = 0$ for some nonzero $b \in R$, and set $B = (b_{st})$ with $b_{11} = b$ and elsewhere zeros. Then $AB = 0$. But since A is right regular, we get $B = 0$, contrary to $b \neq 0$. Thus a_{11} is right regular. The proof for the case of left regular is similar.

(3) By (1) and the commutativity of R, it is enough to show that if $A = (a_{ij})$ is right or left regular in $T_n(R)$, then a_{ii} is regular in R for all $i \in \{1, \ldots, n\}$.

First, let $A = (a_{ij}) \in T_n(R)$ and suppose that A is right (resp., left) regular in $T_n(R)$. Then a_{11} (resp., a_{nn}) is regular by (2). We use this fact freely hereafter. We will proceed the proof by induction on n.

Consider first the case of $n = 2$. Assume on the contrary that a_{22} is not regular. Say that $a_{22}b = 0$ for some $0 \neq b \in R$. Consider $B = \begin{pmatrix} 0 & ba_{12} \\ 0 & -ba_{11} \end{pmatrix}$ in $T_2(R)$. Then

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} 0 & ba_{12} \\ 0 & -ba_{11} \end{pmatrix} = \begin{pmatrix} 0 & b(a_{11}a_{12} - a_{12}a_{11}) \\ 0 & -a_{22}ba_{11} \end{pmatrix} = 0.$$

But since a_{11} is regular, we have $ba_{11} \neq 0$, implying $B \neq 0$. Thus A is not right regular, contrary to A being right regular. Therefore a_{22} is also regular.

We next prove the case of $T_n(R)$ for $n \geq 3$. Assume that we use the matrix

$$E_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & e_0\beta'_1 \\ 0 & 0 & \cdots & 0 & e_0\beta'_2 \\ \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 & \cdots & 0 & e_0\beta'_{n-2} \\ 0 & 0 & \cdots & 0 & e_0\beta'_{n-1} \end{pmatrix}$$

with $\beta'_{n-1} = -a_{(n-2)(n-2)} \cdots a_{22}a_{11}, \beta'_{n-2}, \ldots, \beta'_1 \in R$, in the procedure to obtain that a_{ii} is regular for all $i = 1, \ldots, n-1$ in $T_{n-1}(R)$, where $a_{(n-1)(n-1)}e_0 = 0$ for some $0 \neq e_0 \in R$.

Suppose that a_{ii} is regular in R for all $i \in \{1, 2, \ldots, n-1\}$. Assume on the contrary that a_{nn} is not regular. Say that $a_{nn}e = 0$ for some $0 \neq e \in R$. Then, by help of the preceding matrix E_0 in $T_{n-1}(R)$, we can find

$$E = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -e[a_{12}\beta_1 + e_{13}\beta_2 + \cdots + a_{1(n-1)}\beta_{n-2} + a_{1n}\beta_{n-1}] \\ 0 & 0 & 0 & \cdots & 0 & e\beta_1 a_{11} \\ 0 & 0 & 0 & \cdots & 0 & e\beta_2 a_{11} \\ \vdots & \vdots & \vdots & \cdots & 0 & \vdots \\ 0 & 0 & 0 & \cdots & 0 & e\beta_{n-2}a_{11} \\ 0 & 0 & 0 & \cdots & 0 & e\beta_{n-1}a_{11} \end{pmatrix} \in T_n(R)$$

such that $AE = 0$, where β_k is obtained from β'_k by replacing a_{st} by $a_{(s+1)(t+1)}$ for all $k = 1, \ldots, n-1$. But every a_{ii} is regular for $i = 1, 2, \ldots, n-1$, so $a_{11} \cdots a_{(n-1)(n-1)}$ is also regular. This implies $ea_{(n-1)(n-1)} \cdots a_{11} \neq 0$ because $e \neq 0$, entailing $E \neq 0$. Thus A is not right regular, contrary to A being right regular. Therefore a_{nn} is also regular.

Next we also claim that if A is left regular, then a_{ii} is regular in R for all i.

Consider the case of $n = 2$. Assume on the contrary that a_{11} is not regular. Say that $a_{11}b = 0$ for some $0 \neq b \in R$. Consider $B = \left(\begin{smallmatrix} -a_{22}b & a_{12}b \\ 0 & 0 \end{smallmatrix}\right)$ in $T_2(R)$. Then $B \neq 0$ since $a_{22}b \neq 0$, and

$$BA = \begin{pmatrix} -a_{22}b & a_{12}b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \begin{pmatrix} -a_{22}ba_{11} & -a_{22}ba_{12} + a_{12}ba_{22} \\ 0 & 0 \end{pmatrix} = 0,$$

contradicting A being left regular. Therefore a_{11} is also regular. We use this fact freely hereafter. We will proceed the proof by induction on n.

We next prove the case of $T_n(R)$ for $n \geq 3$. Assume that we use the matrix

$$E_0 = \begin{pmatrix} e_0\beta_1 & e_0\beta_2 & \cdots & e_0\beta_{n-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

with $\beta_1 = -a_{22}a_{33} \cdots a_{(n-1)(n-1)}, \beta_2, \ldots, \beta_{n-1} \in R$, in the procedure to obtain that a_{ii} is regular for all $i = 1, \ldots, n-1$ in $T_{n-1}(R)$, where $a_{11}e_0 = 0$ for some $0 \neq e_0 \in R$.

Suppose that a_{ii} is regular in R for all $i \in \{2, \ldots, n\}$. Assume on the contrary that a_{11} is not regular. Say that $a_{11}e = 0$ for some $0 \neq e \in R$. Then, by help of the preceding matrix E_0 in $T_{n-1}(R)$, we can find

$$E = \begin{pmatrix} e\beta_1 a_{nn} & e\beta_2 a_{nn} & e\beta_3 a_{nn} & \cdots & e\beta_{n-1} a_{nn} & -e[\beta_1 a_{1n} + \beta_2 a_{2n} + \beta_3 a_{3n} + \cdots + \beta_{n-1} a_{(n-1)n}] \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

in $T_n(R)$ such that $EA = 0$. Note $e\beta_1 a_{nn} = -ea_{22}a_{33} \cdots a_{(n-1)(n-1)}a_{nn}$. But every a_{ii} is regular for $i = 2, \ldots, n$, so $a_{22} \cdots a_{nn}$ is also regular. This implies $ea_{22} \cdots a_{nn} \neq 0$ because $e \neq 0$, entailing $E \neq 0$. Thus A is not left regular, contradicting A being left regular. Therefore a_{11} is also regular. □

The converse of Theorem 1.1(1) need not be true by the following.

Example 1.2. Let R be a ring which has right regular elements a and b satisfying $aR \cap bR = 0$. Consider

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & e \end{pmatrix} = \begin{pmatrix} ac & ad + be \\ 0 & 0 \end{pmatrix} = 0$$

for $\left(\begin{smallmatrix} c & d \\ 0 & e \end{smallmatrix}\right) \in T_2(R)$. Since a is right regular, we get $c = 0$. From $ad + be = 0$, we obtain $d = 0 = e$ by the condition that $ad = -be \in aR \cap bR = 0$, because a and b are right regular. Thus $\left(\begin{smallmatrix} c & d \\ 0 & e \end{smallmatrix}\right) = 0$, and so $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ is right regular in $T_2(R)$.

For example, let K be a field and $R = K\langle x, y \rangle$ be the free algebra generated by the non-commuting indeterminates x, y over K. Then $\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right)$ is right regular in $T_2(R)$ by the argument above, because $xR \cap yR = 0$.

Considering Theorem 1.1(3), it is natural to ask whether if (a_{ij}) is regular in $T_n(R)$, then every a_{ii} is regular in R when R is a noncommutative ring. The answer is negative by the following.

Example 1.3. (I) Then case of $T_2(R)$:

Let $R_0 = K\langle x, y \rangle$ be the free algebra generated by the non-commuting indeterminate x, y over a field K.

We use the matrix $\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right)$ in Example 1.2 that is right regular in $T_2(R_0)$. But this matrix is not left regular in $T_2(R_0)$ by Theorem 1.1(2), since its (2,2)-entry is not left regular. Write $R = T_2(R_0)$. Consider a matrix

$$A = \begin{pmatrix} \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \end{pmatrix} \in T_2(R).$$

Then $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ is regular in R by Theorem 1.1(1), and so A is right regular in $T_2(R)$ also by Theorem 1.1(1). Suppose that $BA = 0$ for $B = (b_{ij}) \in T_2(R)$ with $b_{11} = \left(\begin{smallmatrix} a_1 & b_1 \\ 0 & c_1 \end{smallmatrix}\right)$, $b_{12} = \left(\begin{smallmatrix} a_2 & b_2 \\ 0 & c_2 \end{smallmatrix}\right)$. Since $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ is regular, we have $b_{22} = 0$.

From $\left(\begin{smallmatrix} a_1 & b_1 \\ 0 & c_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right) = 0$, we get $a_1 = 0$. So, from

$$0 = \begin{pmatrix} 0 & b_1 \\ 0 & c_1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} a_2x & b_1x + b_2y \\ 0 & c_1x + c_2y \end{pmatrix},$$

we first get $a_2 = 0$, and next obtain $b_1 = 0 = b_2$ and $c_1 = 0 = c_2$ because $R_0x \cap R_0y = 0$. Thus $B = 0$ and so A is left regular in $T_2(R)$. Therefore A is regular in $T_2(R)$, but the (1,1)-entry of A is not left regular in R.

(II) Then case of $T_n(R)$ for $n \geq 3$:

We first argue about the case of $T_3(R)$. Let $R = K\langle x, y\rangle$ as in (I). Consider $C = \left(\begin{smallmatrix} x & y & 0 \\ 0 & 0 & x \\ 0 & 0 & y \end{smallmatrix}\right)$ in $T_3(R)$, and let $CD = 0$ for $D = (d_{ij}) \in T_3(R)$. Then $d_{11} = 0 = d_{33}$, and we obtain $d_{12} = d_{13} = d_{22} = d_{23} = 0$ from $xd_{12} + yd_{22} = 0$ and $xd_{13} + yd_{23} = 0$ because $xR_0 \cap yR_0 = 0$. Let $EC = 0$ for $(e_{ij}) \in T_3(R)$. Then $e_{11} = 0 = e_{33}$, and we obtain $e_{12} = e_{13} = e_{22} = e_{23} = 0$ from $e_{12}x + e_{13}y = 0$ and $e_{22}x + e_{23}y = 0$ because $R_0x \cap R_0y = 0$. These imply that C is regular in $T_3(R)$, but the (2,2)-entry of C is zero.

We extend the preceding result to the general case. Let R be the free algebra $K\langle x_1, x_2, \ldots, x_{n-1}\rangle$ generated by the non-commuting indeterminates $x_1, x_2, \ldots, x_{n-1}$ over K, where $n \geq 3$. Consider a matrix

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & \cdots & x_{n-2} & x_{n-1} & 0 \\ 0 & x_2 & 0 & 0 & \cdots & 0 & 0 & x_1 \\ 0 & 0 & x_3 & 0 & \cdots & 0 & 0 & x_2 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x_{n-2} & 0 & x_{n-3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x_{n-1} \end{pmatrix}$$

in $T_n(R)$. Let $MF = 0$ for $F = (f_{ij}) \in T_n(R)$. Then $f_{11} = f_{22} = \cdots = f_{(n-2)(n-2)} = f_{nn} = 0$, and we obtain $f_{(n-1)(n-1)} = 0$ and $f_{ij} = 0$ for all i, j with $i \neq j$ from the equalities

$$x_1f_{1k} + x_2f_{2k} + \cdots + x_{k-1}f_{(k-1)k} + x_kf_{kk} = 0 \text{ for } k = 2, \ldots, n-1$$

and

$$x_1f_{1n} + x_2f_{2n} + \cdots + x_{n-2}f_{(n-2)n} + x_{n-1}f_{(n-1)n} = 0,$$

because $x_sR \cap (\sum_{t \neq s} x_tR) = 0$ for all $s = 1, \ldots, n-1$. Thus $F = 0$. Let $GM = 0$ for $G = (g_{ij}) \in T_n(R)$. Then $g_{11} = g_{22} = \cdots = g_{(n-2)(n-2)} = g_{nn} = 0$, and we obtain $g_{(n-1)(n-1)} = 0$ and $g_{ij} = 0$ for all i, j with $i \neq j$ from the equality

$$g_{h2}x_1 + g_{h3}x_2 + \cdots + g_{h(n-1)}x_{n-2} + g_{hn}x_{n-1} = 0 \text{ for } h = 1, \ldots, n-1,$$

because $Rx_s \cap (\sum_{t \neq s} Rx_t) = 0$ for all $s = 1, \ldots, n-1$. Thus $G = 0$, entailing that M is regular in $T_n(R)$. But the $(n-1, n-1)$-entry of M is zero.

2. Commutative property at regular products

In this section we study the structure of rings with a kind of commutative property at regular products. Following the literature, a ring R is said to be *directly finite* (or *Dedekind finite*) if $ab = 1$ implies $ba = 1$ for $a, b \in R$. A ring is usually called *Abelian* if every idempotent is central. A ring is usually called *reduced* if it has no nonzero nilpotent elements. It is easily checked that Abelian rings are directly finite and reduced rings are Abelian. The class of directly finite rings is obviously closed under subrings. Left or right Noetherian rings are directly finite by [9, Theorem 1]. It is well-known that the class of directly finite rings contains algebraic algebras over fields, PI-algebras, and rings with finite number of nilpotent elements. Note that $\mathrm{Mat}_n(R)$ over a commutative domain R is directly finite by the preceding facts. But there exists a domain R such that $\mathrm{Mat}_2(R)$ is not directly finite by [12, Theorem 1.0].

We first observe several useful equivalent conditions to the direct finiteness in the following.

Lemma 2.1. *Given a ring R, the following conditions are equivalent*:

(1) *R is directly finite.*

(2) *If $a_1a_2\cdots a_n \in U(R)$ for $a_1, a_2, \ldots, a_n \in R$, then*

$$a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)} \in U(R)$$

for any permutation σ of the set $\{1, 2, \ldots, n\}$, where $n \geq 2$.

(3) *$ab \in U(R)$ implies $ba \in U(R)$ for all $a, b \in R$.*

(4) *If $a_1a_2\cdots a_n = 1$ for $a_1, a_2, \ldots, a_n \in R$, then*

$$a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)} \in U(R)$$

for any permutation σ of the set $\{1, 2, \ldots, n\}$, where $n \geq 2$.

(5) *$ab = 1$ implies $ba \in U(R)$ for all $a, b \in R$.*

(6) *$ab = 1$ implies that both a and b are regular for all $a, b \in R$.*

Proof. (1) $\Rightarrow$ (2): Let R be directly finite, and suppose that $a_1a_2\cdots a_n \in U(R)$ for $a_1, a_2, \ldots, a_n \in R$, where $n \geq 2$. Say that $a_1a_2\cdots a_nb = ba_1a_2\cdots a_n = 1$ for $b \in R$. Since R is directly finite and $a_1(a_2\cdots a_nb) = 1$, we have $(a_2\cdots a_nb)a_1 = 1$, entailing $a_1 \in U(R)$. Next since R is directly finite and $a_2(a_3\cdots a_nba_1) = 1$, we have $(a_3\cdots a_nba_1)a_2 = 1$, entailing $a_2 \in U(R)$. Inductively we have $a_i \in U(R)$ for all i. This yields that $a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)} \in U(R)$ for any permutation σ of the set $\{1, 2, \ldots, n\}$.

(2) $\Rightarrow$ (3), (2) $\Rightarrow$ (4), (3) $\Rightarrow$ (5), and (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (6): Let $ab = 1$ for $a, b \in R$, and suppose that the condition (5) holds. If $ar = 0$ for $r \in R$, then $(ba)r = 0$ and so $r = 0$ since $ba \in U(R)$. If $r'a = 0$ for $r' \in R$, then $0 = r'ab = r'$. These show that a is regular. Similarly, it can be obtained that b is also regular.

(6) $\Rightarrow$ (1): Let $ab = 1$ for $a, b \in R$, and suppose that the condition (6) holds. Then $ab = 1$ implies $(ba)^2 = ba$, and $ba(ba - 1) = 0$. So $ba = 1$ since ba is regular by (6). □

We can see a basic extension preserving the direct finiteness in the following. Recall the subring

$$V_n(R) = \{m = (m_{ij}) \in D_n(R) \mid m_{st} = m_{(s+1)(t+1)} \text{ for } s = 1, \ldots, n-2 \text{ and } t = 2, \ldots, n-1\}$$

of $D_n(R)$, where R is given a ring and $n \geq 2$. Note that $R[x]/x^nR[x]$ is isomorphic to $V_n(R)$, via $(a_0 + a_1x + \cdots + a_{n-1}x^{n-1}) + x^nR[x] \mapsto (a_0, a_1, \ldots, a_{n-1})$, where $(a_{ij}) \in V_n(R)$ is expressed by $(a_0, a_1, \ldots, a_{n-1})$ with $a_{1j} = a_{j-1}$. Consider a power series $f(x) = \sum_{i=0}^{\infty} a_ix^i$ over given a ring R. Note that $a_0 \in U(R)$ if and only if $f(x) \in U(R[[x]])$.

Corollary 2.2. *Let R be a ring and $n \geq 2$.*

(1) *A ring R is directly finite if and only if $T_n(R)$ is directly finite if and only if $D_n(R)$ is directly finite if and only if $V_n(R)$ is directly finite.*
(2) *A ring R is directly finite if and only if so is $R[[x]]$.*

Proof. (1) It suffices to show that $E = T_n(R)$ is directly finite when R is a directly finite ring, because the class of directly finite rings is closed under subrings. Let $(a_{ij})(b_{ij}) = 1$ for $(a_{ij}), (b_{ij}) \in E$. Then $a_{ii}b_{ii} = 1$ for all $i = 1, \ldots, n$. Since R is directly finite, $b_{ii}a_{ii} = 1$ for all i. This implies that $(b_{ij})(a_{ij}) \in U(E)$. Then E is directly finite by Lemma 2.1.

(2) It also suffices to show the necessity. Let $f(x)g(x) = 1$ for $f(x), g(x) \in R[[x]]$, where $f(x) = \sum_{i=0}^{\infty} a_ix^i$ and $g(x) = \sum_{j=0}^{\infty} b_jx^j$. Then $a_0b_0 = 1$. Since R is directly finite, we have $b_0a_0 = 1$, entailing $a_0, b_0 \in U(R)$. This implies that $f(x), g(x) \in U(R[[x]])$. Then $g(x)f(x) = 1$ by Lemma 2.1. □

From the condition (3) in Lemma 2.1, we next introduce a new concept through the argument to follow. First for a given ring R consider the following:

$(*)$ If ab is regular for $a, b \in R$, then ba is also regular.

A ring R satisfying the condition $(*)$ is directly finite. For, letting $ab = 1$ for $a, b \in R$, ba is regular because R satisfies $(*)$. Then $(ba)^2 = ba$ and so $ba(ba - 1) = 0$, implying $ba = 1$. So it is natural to ask whether directly finite rings satisfy the condition $(*)$. But the answer is negative by the following.

Example 2.3. We use the ring in [2, Example 4.8] and the argument in [11, Proof of Theorem 1]. Let K be a field and $A = K\langle a, b\rangle$ be the free algebra generated by the noncommuting indeterminates a, b over K. Let I be the ideal of A generated by b^n and set $R = A/I$, where $n \geq 2$. Write $\bar{r} = r + I$ for $r \in A$. Then R is Abelian (hence directly finite) by [2, Theorem 4.7] and [7, Corollary 8]. We next claim that R does not satisfy the condition $(*)$. Let $\alpha, \beta \in R\backslash\{0\}$ satisfy $\alpha\beta = 0$. Then $\alpha = \alpha'\bar{b}^s$ and $\beta = \bar{b}^t\beta'$ for some $\alpha', \beta' \in R$, where $1 \leq s, t \leq n-1$ and $s + t \geq n$, by help of the argument in [11, Proof of Theorem 1]. Consider the element $\bar{a}\bar{b}\bar{a}$ in R. Then $\bar{a}\bar{b}\bar{a} = \bar{a}(\bar{b}\bar{a})$ is regular by the preceding argument. But $(\bar{b}\bar{a})\bar{a}$ is not (left) regular as can be seen by $\bar{b}^{n-1}(\bar{b}\bar{a})\bar{a} = 0$, noting $\bar{b}^{n-1} \neq 0$. So R does not satisfy the condition $(*)$.

A ring R shall be said to be *commutative at regular-product* (simply, *CRP*) if R satisfies the condition $(*)$, based on Lemma 2.1 and Example 2.3. Then CRP rings are directly finite by an argument above. Due to Bell [4], a ring R is said to satisfy *Insertion-of-Factors-Property* (or simply, be an *IFP* ring) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. It is easily checked that reduced rings are IFP, and the converse need not hold because any commutative ring is clearly IFP (e.g., $\mathbb{Z}_{m^n}$ for $m, n \geq 2$). The proof for IFP rings to be Abelian is simple. Following Huh et al. [7], a ring is called *locally finite* if every finite subset generates a finite multiplicative semigroup. It is shown that a ring is locally finite if and only if every finite subset generates a finite subring by [6, Theorem 2.2(1)]. Finite rings, infinite direct sums of finite rings, and algebraic closures of finite fields are basic examples of locally finite rings.

Theorem 2.4. (1) *Left or right Artinian rings are CRP.*

(2) *IFP rings are CRP.*

(3) *Locally finite rings are CRP.*

(4) *A ring R is CRP if and only if ab being regular for $a, b \in R$ implies that both a and b are regular.*

Proof. (1) Let R be a right Artinian ring and suppose that ab is regular for $a, b \in R$. Then $(ba)^k = (ba)^{k+1}c$ for some $k \geq 1$ and $c \in R$. This yields

$$0 = a(ba)^k - a(ba)^{k+1}c = (ab)^k(a - abac) \text{ and so } a - abac = 0$$

because ab is regular. Multiplying $a - abac = 0$ by b on the right-hand side, we get $ab(1 - acb) = 0$ and moreover $acb = 1$ because ab is regular. Since right Artinian rings are directly finite, we obtain $b(ac) = 1$ and $(cb)a = 1$ from $acb = 1$. So both a and b are regular, implying that ba is regular. Therefore R is CRP. The proof for the case of a left Artinian ring is similar.

(2) Let R be an IFP ring, and assume on the contrary that there exist $a, b \in R$ such that ab is regular but ba not regular. Then $(ba)\alpha = 0$ for some $\alpha \in R\backslash\{0\}$, or $\beta(ba) = 0$ for some $\beta \in R\backslash\{0\}$. Since R is IFP, $a(ba)b\alpha = 0$ (resp., $\beta a(ba)b = 0$) implies $\alpha = 0$ (resp., $\beta = 0$) because ab is regular. This result induces a contradiction to $\alpha, \beta \in R\backslash\{0\}$. Therefore R is CRP.

(3) Let R be a locally finite ring. Suppose that ab is regular for $a, b \in R$. Consider the subring S of R generated by $a, b, 1$. Then S is finite because R is locally finite; hence S is CRP by (1). But ab is also regular in S, implying that ba is regular in S. Since R is locally finite, both $(ab)^m$ and $(ba)^n$ are idempotents for some $m, n \geq 1$ by the proof of [7, Proposition 16]. This forces $(ab)^m = 1$ and $(ba)^n = 1$ because both ab and ba are regular in S. So $a, b \in U(R)$, entailing that ba is regular in R. Therefore R is CRP.

(4) It suffices to show the necessity. Let R be a CRP ring and suppose that ab is regular for $a, b \in R$. Then a is left regular and b is right regular. Since R is CRP, ba is regular and so this implies that a is right regular and b is left regular. $\square$

By Theorem 2.4(1), a CRP ring need not be Abelian. In fact, $\mathrm{Mat}_n(R)$ is CRP but non-Abelian over a right Artinian ring R for $n \geq 2$. So the concepts of regular-commutativity and Abelian are independent of each other, considering Example 2.3. Thus direct finiteness is a ring theoretic property which unifies the concepts of regular-commutativity and Abelian.

Proposition 2.5. *For a ring R, the following are equivalent*:

(1) *R is CRP.*

(2) *abc being regular for $a, b, c \in R$ implies that acb is regular.*

(3) *abc being regular for $a, b, c \in R$ implies that bac is regular.*

(4) *$a_1a_2 \cdots a_n$ being regular for $a_1, a_2, \dots a_n \in R$ implies that $a_{\sigma(1)}a_{\sigma(2)} \cdots a_{\sigma(n)}$ is regular for any permutation σ of the set $\{1, 2, \dots, n\}$.*

Proof. (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3): Let R be a CRP ring. Suppose that abc is regular for $a, b, c \in R$. Then both $(bc)a$ and $c(ab)$ are regular and so a, b and c are regular by Theorem 2.4(4). This implies that both acb and bac are regular.

Each converse is clear since $ab = 1 \cdot ab = ab \cdot 1$.

(2) $\Rightarrow$ (4): This follows from the same argument as the proof of [1, Theorem I.1], using regular in place of 0.

(4) $\Rightarrow$ (1): It is obvious. □

If a ring is not directly finite, then it has infinitely many matrix units by [9, page 1]. We see a similar result for non-regular elements as follows.

Theorem 2.6. *Let R be a directly finite ring. If R is not CRP, then we have the following results*:

(1) *R has an infinite set of nonzero non-regular elements which intersects with $N(R)$ at emptiness.*

(2) *R has both an infinite properly descending chain of principal right ideals and an infinite properly descending chain of principal left ideals.*

Proof. (1) Since R is not CRP, there exist $a, b \in R$ such that ab is regular but ba is not regular. Since ab is regular, a is left regular and b is right regular. Since ba is not regular, $(ba)\alpha = 0$ (hence $a\alpha = 0$) for some $\alpha \in R\backslash\{0\}$, or $\beta(ba) = 0$ (hence $\beta b = 0$) for some $\beta \in R\backslash\{0\}$.

We first note that $(ba)^k$ is nonzero for any $k \geq 1$. For, the regularity of $a(ba)^k b = (ab)^{k+1}$ implies $(ba)^k \neq 0$. We next claim that ab is not a unit. If $(ca)b = a(bc) = 1$ for some $c \in R$, then a and b are regular by Lemma 2.1(6). This is contrary to that ba is not regular. Thus ab is not a unit.

Assume that $(ba)^m = (ba)^n$ for some $1 \leq m < n$. Then

$$(ab)^{m+1} = a[(ba)^m]b = a[(ba)^n]b = (ab)^{n+1} \text{ and } (ab)^{m+1}[1 - (ab)^{n-m}] = 0.$$

This implies $(ab)^{n-m} = 1$ because $(ab)^{m+1}$ is regular. So ab is a unit, contrary to the result above. Thus $(ba)^m \neq (ba)^n$ for all $1 \leq m < n$. Therefore we now obtain an infinite set

$$\{(ba)^n \mid n = 1, 2, \dots\}$$

of nonzero non-regular elements in R. Since every $(ba)^n$ is nonzero, the intersection of $\{(ba)^n \mid n = 1, 2, \ldots\}$ and $N(R)$ is an empty set.

(2) Recall the elements a and b in R such that ab is regular but ba is not regular, in the proof of (1). Consider the descending chain

$$(\dagger) \qquad baR \supseteq (ba)^2R \supseteq \cdots \supseteq (ba)^nR \supseteq (ba)^{n+1}R \supseteq \cdots$$

of principal right ideals in R, where $n = 1, 2, \ldots$. Assume that $(ba)^kR = (ba)^{k+1}R$ for some $k \geq 1$. Then $(ba)^k = (ba)^{k+1}c$ for some $c \in R$. Multiplying this equality by a on the left-hand side, we get

$$(ab)^ka = a(ba)^k = a(ba)^{k+1}c = (ab)^kabac \text{ and}$$

$$0 = a(ba)^k - a(ba)^{k+1}c = (ab)^k(a - abac).$$

This yields $a - abac = 0$ because ab is regular. Multiplying $a - abac = 0$ by b on the right-hand side, we get

$$ab - abacb = 0 \text{ and } ab(1 - acb) = 0.$$

This yields $acb = 1$ because ab is regular. Then $a(cb) = 1 = (ac)b$ imply that both a and b are regular by Lemma 2.1(6), contrary to ba being not regular. So the descending chain (†) is not stationary.

The argument for the chain principal left ideals is similar to the preceding one. □

Recall the ring R in Example 2.3 which is Abelian but not CRP. The element aba of R is regular but ba^2 is not left regular (in fact, $b^{n-1}ba^2 = 0$). So, by the proof of Theorem 2.6(1), we get an infinite set $\{(ba^2)^k \mid k = 1, 2, \ldots\}$ of nonzero non-regular elements. We see an application of Theorem 2.6(1) in the following.

Example 2.7. We use the construction in [10, Theorem 2.2(2)]. Let F be a domain. Define a map $\sigma : D_{2^n}(F) \to D_{2^{n+1}}(F)$ by $B \mapsto \left(\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right)$. Then $D_{2^n}(F)$ can be considered as a subring of $D_{2^{n+1}}(F)$ via σ (i.e., $B = \sigma(B)$ for $B \in D_{2^n}(F)$). Set $R = \bigcup_{n=1}^{\infty} D_{2^n}(F)$. Then R is Abelian (hence directly finite) by [5, Lemma 2].

Since F is a domain, we have $N(R) = \{A \in R \mid \text{the diagonal entries of } A \text{ are all zero}\} = N^*(R)$. So every matrix in $R \backslash N(R)$ is regular by [8, Lemma 2.1] because the diagonal entries are nonzero and F is a domain. Consequently the set of non-nilpotent non-regular elements is empty. Thus R is CRP by Theorem 2.6(1).

In the following we see an application of Theorem 2.6(2).

Proposition 2.8. *If R is a directly finite ring that satisfies the descending chain condition for principal right ideals or principal left ideals, then R is CRP.*

Proof. Let R be a directly finite ring that satisfies the descending chain condition for principal right ideals or principal left ideals. Assume on the contrary that R is not CRP. Then R has both an infinite properly descending chain of principal right ideals and an infinite properly descending chain of principal left ideals by Theorem 2.6(2), a contradiction to the condition. Therefore R is CRP. $\square$

We can also obtain Theorem 2.4(1) as a corollary of Proposition 2.8. In what follows, we elaborate upon the structure of descending chains in the proof of Theorem 2.6(2).

Note. Let R be a directly finite ring that is not CRP. Then there exist $a, b \in R$ such that ab is regular but ba is not regular. So a is not right regular or b is not left regular. Suppose that b is not left regular. Consider a descending chain

$$(\delta) \qquad bR \supseteq b^2R \supseteq \cdots \supseteq b^nR \supseteq b^{n+1}R \supseteq \cdots$$

of principal right ideals in R, where $n = 1, 2, \ldots$. Assume that $b^kR = b^{k+1}R$ for some $k \geq 1$. Then $b^k = b^{k+1}d$ for some $d \in R$. Thus $b^k(1 - bd) = 0$ implies $1 = bd$ because b is right regular. Then b is regular by Lemma 2.1(6), a contradiction. Therefore the chain (δ) is non-stationary.

Next suppose that a is not right regular. Consider a descending chain

$$(\gamma) \qquad Ra \supseteq Ra^2 \supseteq \cdots \supseteq Ra^n \supseteq Ra^{n+1} \supseteq \cdots$$

of principal left ideals in R, where $n = 1, 2, \ldots$. Assume that $Ra^h = Ra^{h+1}$ for some $h \geq 1$. Then $a^h = ea^{h+1}$ for some $e \in R$. Thus $(1 - ea)a^h = 0$ implies $1 = ea$ because a is left regular. Then a is regular by Lemma 2.1(6), a contradiction. Therefore the chain (γ) is non-stationary.

We see in the following some sorts of matrix rings through which the CRP property passes.

Theorem 2.9. (1) *For a ring R the following conditions are equivalent*:

- (a) *R is CRP.*
- (b) *$D_n(R)$ is CRP for $n \geq 2$.*
- (c) *$V_n(R)$ is CRP for $n \geq 2$.*

(2) *If R is a commutative ring, then $T_n(R)$ is a CRP ring for all $n \geq 2$.*
(3) *If R is a domain, then $T_2(R)$ is a CRP ring.*
(4) *Let R be a ring and $n \geq 2$. If $T_n(R)$ is CRP, then so is R.*

Proof. (1) (a) $\Leftrightarrow$ (b): Let R be CRP and suppose that AB is regular for $A = (a_{ij}), B = (b_{st}) \in D_n(R)$. Then $a_{uu}b_{uu}$ is regular in R for all $u = 1, \ldots, n$ by [8, Lemma 2.1]. Since R is CRP, a_{uu} and b_{uu} are regular for any u by Theorem 2.4(4). Then both A and B are regular by [8, Lemma 2.1], and thus $D_n(R)$ is CRP by Theorem 2.4(4).

Conversely, suppose that $D_n(R)$ is CRP and let ab be regular for $a, b \in R$. Consider two matrices $D = (d_{ij})$ and $E = (e_{st})$ in $D_n(R)$ such that $d_{ii} = a$, $e_{ss} = b$, and $d_{ij} = 0 = e_{st} = 0$ for $i \neq j$ and $s \neq t$. Then DE is regular in $D_n(R)$ by [8, Lemma 2.1]. Since $D_n(R)$ is CRP, both D and E are regular by Theorem 2.4(4). Hence both a and b are regular by [8, Lemma 2.1]. Therefore R is CRP.

The proof of (a) ⇔ (c) is almost the same as one of (a) ⇔ (b).

(2) Let R be a commutative ring and suppose that AB is regular for $A = (a_{ij}), B = (b_{st}) \in T_n(R)$. Then $a_{uu}b_{uu}$ is regular in R for all $u = 1, \ldots, n$, implying that a_{uu} and b_{uu} are regular in R because R is commutative. Hence both A and B are regular in $T_n(R)$ by Theorem 1.1(3). Therefore $T_n(R)$ is CRP.

(3) Let R be a domain and suppose that $(a_{ij})(b_{st})$ is regular in $T_2(R)$ for $(a_{ij}), (b_{st}) \in T_2(R)$. Then $a_{11}b_{11}$ is right regular and $a_{22}b_{22}$ is left regular in R by Theorem 1.1(2). This implies that a_{11}, a_{22}, b_{11} and b_{22} are regular because R is a domain. Thus both (a_{ij}) and (b_{st}) are regular by Theorem 1.1(1), entailing that $T_2(R)$ is CRP.

(4) Suppose that ab is regular for $a, b \in R$. Consider two matrices $D = (d_{ij})$ and $E = (e_{st})$ in $T_n(R)$ such that $d_{ii} = a$, $e_{ss} = b$, and $d_{ij} = 0 = e_{st}$ for $i \neq j$ and $s \neq t$. Then DE is regular by Theorem 1.1(1). If $T_n(R)$ is CRP, then both D and E are regular in $T_n(R)$. Hence both a and b are regular in R by Theorem 1.1(2). Thus R is CRP. □

Considering Theorem 2.9, one may naturally ask whether $Mat_n(R)$ is CRP when R is a CRP ring. But the answer is negative by the following.

Example 2.10. There exists a domain (hence CRP by Theorem 2.4(2)) R such that $\mathrm{Mat}_2(R)$ is not directly finite (hence not CRP) by [12, Theorem 1.0]. We extend this argument to the general case of $n \geq 2$. $\mathrm{Mat}_2(R)$ is isomorphic to a subring of $\mathrm{Mat}_n(R)$ via the corresponding $(a_{ij}) \mapsto (a'_{ij})$ where $a'_{ij} = a_{ij}$ for all $1 \leq i, j \leq 2$, $a'_{ii} = 1$ for all $3 \leq i \leq n$, and $a'_{ij} = 0$ for all $3 \leq i, j \leq n$ with $i \neq j$. Thus $\mathrm{Mat}_n(R)$ is not directly finite (hence not CRP) because the class of directly finite rings is closed under subrings.

Considering Theorem 2.9(3), one may ask whether $T_n(R)$ is also CRP over a domain R for $n \geq 3$. But we see a negative answer in what follows.

Example 2.11. Let $R = K\langle x, y\rangle$ be the free algebra generated by the non-commuting indeterminate x, y over a field K. Note that $\left(\begin{smallmatrix} x & y & 0 \\ 0 & 0 & x \\ 0 & 0 & y \end{smallmatrix}\right)$ is regular in $T_3(R)$ by (II) in Example 1.3.

$$\begin{pmatrix} x & y & 0 \\ 0 & 0 & x \\ 0 & 0 & y \end{pmatrix} = \begin{pmatrix} x & 1 & 0 \\ 0 & 0 & x \\ 0 & 0 & y \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & 1 & 0 \\ 0 & 0 & x \\ 0 & 0 & y \end{pmatrix} = \begin{pmatrix} x & 1 & 0 \\ 0 & 0 & yx \\ 0 & 0 & y \end{pmatrix}.$$

The former is regular but the latter is not regular in $T_3(R)$ as can be seen by

$$\begin{pmatrix} x & 1 & 0 \\ 0 & 0 & yx \\ 0 & 0 & y \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -x \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

So $T_3(R)$ is not CRP, in spite of R being a domain (hence CRP). This result is also shown by Theorem 2.4(4) because $\left(\begin{smallmatrix} x & 1 & 0 \\ 0 & 0 & x \\ 0 & 0 & y \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & -x \\ 0 & 0 & 0 \end{smallmatrix}\right) = 0$. This result can be extended to the case of $n \geq 4$.

A ring R is usually called *right Ore* if given $a, b \in R$ with b regular there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is well-known that right Noetherian domains are right Ore. The ring R in Example 2.11 is not right Ore.

Proposition 2.12. *Let R be a domain. If R is right Ore, then $T_3(R)$ is CRP.*

Proof. Let R be right Ore and suppose that AB is regular in $T_3(R)$ for $A = (a_{ij}), B = (b_{st}) \in T_3(R)$. Write $C = AB = (c_{pq})$. Then both c_{11} and c_{33} are nonzero by Theorem 1.1(2), entailing that $a_{11}, a_{33}, b_{11}, b_{33} \in R\backslash\{0\}$, since R is a domain.

Assume $c_{22} = 0$ (i.e., $a_{22} = 0$ or $b_{22} = 0$). Since R is right Ore and $c_{11} \neq 0$, there exist $\alpha \neq 0$ and β in R such that $c_{11}\beta + c_{12}\alpha = 0$. This implies

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ 0 & 0 & c_{23} \\ 0 & 0 & c_{33} \end{pmatrix}\begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_{11}\beta + c_{12}\alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

Thus C is not right regular because $\left(\begin{smallmatrix} 0 & 0 & \beta \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{smallmatrix}\right)$ is nonzero, contrary to C being regular. Hence we have $c_{22} \neq 0$, and this implies that both a_{22} and b_{22} are nonzero. Therefore both A and B are regular by Theorem 1.1(1), and hence $T_3(R)$ is CRP by Theorem 2.4(4). □

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