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# ON THE LOCATION OF EIGENVALUES OF REAL CONSTANT ROW-SUM MATRICES

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Abstract. New inclusion sets are obtained for the eigenvalues of real matrices for which the all 1's vector is an eigenvector, i.e., the constant row-sum real matrices. A number of examples are provided. This paper builds upon the work of the authors in [7]. The results of this paper are in terms of Geršgorin discs of the second type. An application of the main theorem to bounding the algebraic connectivity of connected simple graphs is obtained.

## 1. Introduction

S. Geršgorin's 1931 article [2] is an important and often cited work on the location of the eigenvalues of $n \times n$ matrices. A main part of Geršgorin's result is that the eigenvalues of an $n \times n$ complex matrix are contained in the union of the Geršgorin discs in the plane. The concept of Geršgorin disc of the second type has been recently introduced in our paper [7]. It is defined as follows:

**Definition 1.1.** Let $A = [a_{ij}]$ be an $n \times n$ real matrix, and let $x_{i1} \geq \cdots \geq x_{in}$ be a rearrangement in non-increasing order of $a_{i1}, \ldots, a_{i,i-1}, 0, a_{i,i+1}, \ldots, a_{in}$ for $i = 1, \ldots, n$. We call a Geršgorin disc of $A$ of the first type a usual Geršgorin disc, while a Geršgorin disc of the second type $\hat{D}(a_{ii}\, , \hat{r}_i)$ of $A$ satisfies the following conditions:

(1) Its center $a_{ii}$ is the diagonal element from the $i^{th}$ row of $A$.
(2) Its radius:

(a) $\hat{r}_i = \displaystyle\sum_{j=1}^{\frac{n-1}{2}} x_{ij} - \sum_{j=\frac{n+3}{2}}^{n} x_{ij}$, if $n$ is odd.

(b) $\hat{r}_i = \displaystyle\sum_{j=1}^{\frac{n}{2}} x_{ij} - \sum_{j=\frac{n}{2}+1}^{n} x_{ij}$, if $n$ is even.

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The following lemma was also fundamental to our analysis in [7]. The reason for the minimum as stated is a familiar fact from statistics: the median is the statistic that minimizes the sum of absolute deviations; see for example [9].

**Lemma 1.2.** *Consider the real function of the real variable* $f(x) = \sum_{i=1}^{n} |x - \beta_i|$, *with* $\beta_1 \geq \cdots \geq \beta_n$, *not necessarily distinct* $n$ *real numbers.*

(1) *If* $n$ *is odd, then* $\min_{x\in\mathbb{R}} f(x) = \big(\beta_1 + \cdots + \beta_{\frac{n-1}{2}}\big) - \big(\beta_{\frac{n+3}{2}} + \cdots + \beta_n\big)$. *This minimum is reached when* $x = \beta_{\frac{n+1}{2}}$.

(2) *If* $n$ *is even, then* $\min_{x\in\mathbb{R}} f(x) = \big(\beta_1 + \cdots + \beta_{\frac{n}{2}}\big) - \big(\beta_{\frac{n}{2}+1} + \cdots + \beta_n\big)$. *This takes place for every* $x \in [\beta_{\frac{n}{2}}, \beta_{\frac{n}{2}+1}]$ *if* $\beta_{\frac{n}{2}} \neq \beta_{\frac{n}{2}+1}$ *and only for* $x = \beta_{\frac{n}{2}}$ *if* $\beta_{\frac{n}{2}} = \beta_{\frac{n}{2}+1}$.

Based on the above lemma and definition we obtained the following result in [7]. Since we refer to the proof of this theorem later, we re-produce it here. As in our previous work, we denote the all 1's column vector with $n$ components by $e$.

**Theorem 1.3.** *Let* $A = [a_{ij}]$ *be a* $n \times n$ *real matrix and suppose that* $\lambda$ *is an eigenvalue of* $A$ *associated with an eigenvector orthogonal to the all* 1*'s vector* $e$. *Then* $\lambda$ *is in a Geršgorin disc of the second type of* $A$.

*Proof.* Suppose that $\lambda$ is associated with the eigenvector $v = (v_1, \ldots, v_n)^T$ and $v^*e = 0$, that is $\sum v_i = 0$. Without loss of generality, suppose that $v_1$ is the largest, in absolute value, among the elements of $v$. Then $Av = \lambda v$ implies

$$(\lambda - a_{11})v_1 = \sum_{j=2}^{n} a_{1j}\, v_j \ = (0 - x)v_1 + \sum_{j=2}^{n}(a_{1j} - x)v_j, \text{ since } \sum_{j=1}^{n} xv_j = 0.$$

Hence,

$$|\lambda - a_{11}||v_1| = |(0 - x)v_1 + \sum_{j=2}^{n}(a_{1j} - x)v_j| \leq \ |0 - x|\,|v_1| + \sum_{j=2}^{n} |(a_{1j} - x)||v_j|$$
$$\leq \big(\,|0 - x| + \sum_{j=2}^{n} |(a_{1j} - x\,)|\big)\,|v_1|.$$

That is,

$$|\lambda - a_{11}| \leq \min_{x\in\mathbb{R}} \big(\sum_{j=2}^{n} |a_{1j} - x| + |0 - x|\big).$$

Therefore, the theorem follows by the use of Definition 1.1 and Lemma 1.2. □

In our current work, we apply this result to stochastic matrices in particular, and more generally to all real matrices for which the all 1's vector is an eigenvector, i.e., the constant row-sum real matrices. As such we make some further definitions.

**Definition 1.4.** A real $n \times n$ matrix for which $e$ is an eigenvector is called an e-matrix. If $A$ is an e-matrix, and $\lambda_t$ is the eigenvalue of $A$ associated with $e$, then we call $\lambda_t$ the trivial eigenvalue of $A$. If $\lambda$ is an eigenvalue of $A$ such that $\lambda \neq \lambda_t$, then we call $\lambda$ a non-trivial eigenvalue of $A$.

Some well-known types of matrices are e-matrices, such as stochastic and Laplacian matrices.

**Definition 1.5.** A Geršgorin region of the second type of a real matrix $A$ is the union of the Geršgorin discs of the second type of $A$. We denote this region by $G_2(A)$. We call the usual Geršgorin region, the Geršgorin region of the first type.

In [8] we obtained a new upper bound for eigenvalues of real constant row-sum matrices (which we are now calling e-matrices); we also made comparisons to some previously known bounds. In this present paper, the emphasis is on the localization of eigenvalues of e-matrices; the location is done with Geršgorin discs of the second type. We mention two interesting recent articles on localization of stochastic matrices in particular, namely [1] and [4].

## 2. Main results

The following result follows from the "principle of biorthogonality", see [3, Theorem 1.4.7(a)].

**Lemma 2.1.** *Suppose $A$ is an $n \times n$ e-matrix, $\lambda$ is a non-trivial eigenvalue of $A$, and $v$ is any left eigenvector of $A$ associated with $\lambda$. Then $v$ is orthogonal to $e$.*

Now we state our first main theorem.

**Theorem 2.2.** *If $A$ is an $n \times n$ e-matrix, then all the non-trivial eigenvalues of $A$ are in the Geršgorin region of the second type of $A^T$. This applies in particular to stochastic matrices. Also, if $A$ is a doubly stochastic matrix, then all the non-trivial eigenvalues of $A$ are in the intersection of the Geršgorin regions of the second type of $A$ and $A^T$.*

*Proof.* Let $\lambda$ be a non-trivial eigenvalue of $A$ and $v$ be a left eigenvector of $A$ associated with $\lambda$. Then $v^*A = \lambda v^*$, so that $A^T v = \bar{\lambda} v$. By Lemma 2.1, $v$ is orthogonal to $e$. Hence, by Theorem 1.3, $\bar{\lambda}$ is in a Geršgorin disc of the second type of $A^T$. Since $A$ is a real matrix, the center of each such disc is on the real-axis. Thus, it is also the case that $\lambda$ is in that same Geršgorin disc, so that $\lambda$ is in the Geršgorin region of the second type of $A^T$. The statements about stochastic matrices should be clear. □

**Example 2.3.** Let

$$A = \begin{bmatrix} 3 & 2 & 5 \\ 5 & 2 & 3 \\ 4 & 2 & 4 \end{bmatrix}.$$

The Geršgorin region of the first type of $A^T$, $G_1(A^T)$, is the union of the 3 discs $D(3,9) \cup D(2,4) \cup D(4,8)$. The Geršgorin region of the second type of $A^T$, $G_2(A^T)$, is the union $\hat{D}(3,5) \cup \hat{D}(2,2) \cup \hat{D}(4,5)$. Observe that for this matrix, $G_2(A^T)$ is significantly smaller than $G_1(A^T)$. By definition, $G_2(A^T) \subseteq G_1(A^T)$.

**Corollary 2.4.** *Let $A$ be an $n \times n$ e-matrix, with trivial eigenvalue different than $0$. If the number $0$ is outside of the Geršgorin region of the second type of $A^T$, then $A$ is nonsingular.*

**Example 2.5.** Let

$$A = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}.$$

Then it can be seen that $A$ is such a nonsingular matrix.

Recently we have proved in [5] that if an eigenvalue has geometric multiplicity $k$, then it is in the intersection of at least $k$ Geršgorin discs of the first type. We also obtained a result concerning the rank of matrices in [6]: if the number 0 is outside of $k$ Geršgorin discs of the first type of an $n \times n$ matrix $A$, then the rank of this matrix is greater than or equal to $n - k$. The key idea behind these results is the following lemma which can be found in [5].

**Lemma 2.6.** *Let $S$ be a $k$-dimensional subspace of $\mathbb{C}^n$. There is a basis $\{v_1, v_2, \ldots, v_k\}$ of $S$ with the following property: for each $i = 1, 2, \ldots, k$, there is a distinct integer $p_i$, with $1 \leq p_i \leq n$ and $p_i \neq p_j$ for $i \neq j$, such that a largest modulus entry of each $v_i$ is in position $p_i$.*

**Theorem 2.7.** *Let $A$ be an e-matrix. If $\lambda$ is a non-trivial eigenvalue of $A$ with geometric multiplicity $k$, then $\lambda$ is in at least $k$ Geršgorin discs of the second type of $A^T$.*

*Proof.* If $\lambda$ has geometric multiplicity $k$, then the left eigenspace of $\lambda$ has dimension $k$, and by Lemma 2.1 it is orthogonal to $e$. According to Lemma 2.6, this subspace is spanned by $k$ linearly independent left eigenvectors associated with $\lambda$ such that these eigenvectors have largest moduli entries in different positions. Denote these left eigenvectors of $\lambda$ by $v_i = (v_{i1}, \ldots, v_{in})^T$ for $i = 1, \ldots, k$, and without loss of generality, suppose that the largest modulus of $v_i$ is $v_{ii}$. Taking into account that the left eigenvectors of $A$ are right eigenvectors of $A^T$ and using the same reasoning as in the proof of Theorem 2.2, we construct from the $i^{th}$ row of $A^T$, a Geršgorin disc of the second type that contains $\lambda$ (see also the proof of Theorem 1.3). Hence $\lambda$ belongs to the intersection of at least $k$ Geršgorin discs of the second type constructed from $k$ different rows of $A^T$. $\square$

**Corollary 2.8.** *Let $A$ be an e-matrix. If no more than $k$ Geršgorin discs of the second type of $A^T$ are connected, then no non-trivial eigenvalue of $A$ has geometric multiplicity larger than $k$.*

**Corollary 2.9.** *Let $A$ be an e-matrix with trivial eigenvalue different than $0$. If $0$ is outside $k$ Geršgorin discs of the second type of $A^T$, then the rank of $A$ is greater than or equal to $k$.*

*Proof.* If 0 is outside $k$ Geršgorin discs of the second type of $A^T$, then it belongs to at most $n-k$ Geršgorin discs of the second type of $A^T$. Hence, if 0 is an eigenvalue of $A$, then according to Theorem 2.7, it cannot have geometric multiplicity greater than $n-k$. In other words, the rank of $A$ cannot be less than $k$. □

If $A$ is an e-matrix, and $\lambda_t$ is its trivial eigenvalue, then clearly $\lambda_t$ is on the boundary of every Geršgorin disc of the first type of $A$. This fact implies that the Geršgorin region of the first type of $A$ is connected. The question that arises now is the following: what about the Geršgorin region of the second type of $A^T$, is it also connected? We will discuss this question in the next section. Now let's emphasize the geometric multiplicity of $\lambda_t$ and its relationship with the Geršgorin region of the second type of $A$. If $\lambda_t$ has geometric multiplicity $k \geq 2$, we can use the following result that can be found in [7].

**Lemma 2.10.** *Let $A$ be a real matrix and let $\lambda$ be an eigenvalue of $A$ with geometric multiplicity $k \geq 2$. Then $\lambda$ is in the intersection of at least $k-1$ Geršgorin discs of the second type of $A$ and at least one Geršgorin disc of the first type of the matrix $C_k(A)$, all the discs being constructed from different rows.*

The matrix $C_k(A)$ is constructed from $A$ by replacing in every row of $A$, $k-1$ smallest off-diagonal entries in absolute value by 0.

**Corollary 2.11.** *Let $A$ be an e-matrix and suppose that $\lambda_t$, its trivial eigenvalue, has geometric multiplicity $k \geq 2$. Then $\lambda_t$ is in the intersection of at least $k-1$ discs of the second type of $A$.*

The above corollary has a strong connection with the number of zeros in some rows of $A$. This is because $\lambda_t$ as we have mentioned before is on the boundary of every Geršgorin discs of the first type of $A$, and since it has geometric multiplicity $k$ then it must be in at least $k-1$ discs of second type of $A$ according to Corollary 2.11. We know also that every Geršgorin disc of the second type is a subset of a Geršgorin disc of the first type constructed from the same row of $A$. All these facts together imply that $k-1$ discs of the second type of $A$ must be identical, respectively, to the Geršgorin disc of the first type of $A$ constructed from the same row. When we look at how these $k-1$ discs are constructed, we will deduce for the case of stochastic matrices that the corresponding rows must have the following properties.

**Theorem 2.12.** *Let $A$ be an $n \times n$ stochastic matrix. Suppose that $\lambda_t = 1$, its trivial eigenvalue, has geometric multiplicity $k \geq 2$.*

*- If $n$ is odd, then there are at least $k-1$ rows of $A$, each one of which has*

*at least* $\frac{n-1}{2}$ *off-diagonal entries equal to* 0.

*- If* $n$ *is even, then there are at least* $k-1$ *rows of* $A$*, each one of which has at least* $\frac{n}{2}-1$ *off-diagonal entries equal to* 0.

**Example 2.13.** Suppose that $A$ is a $100 \times 100$ stochastic matrix. For the eigenvalue 1 to have geometric multiplicity 3, there must be at least two rows of $A$, each of which has at least 49 zero off-diagonal entries.

**Application of Theorem 2.2 to algebraic connectivity of connected graphs**

Let $A = [a_{ij}]$ be an e-matrix and let $\lambda$ be a non-trivial eigenvalue of $A$. Theorem 2.2 implies that $\lambda \in \bigcup_{i=1}^{n} \hat{D}_i(a_{ii}, r_i)$, where $\hat{D}_i(a_{ii}, r_i)$ is the Geršgorin disc of the second type having center $a_{ii}$, radius $r_i$ and constructed from the $i^{th}$ column of $A$. If in addition, $A$ is symmetric, then this implies that there exists some $j \in \{1, \dots, n\}$ such that

$$a_{jj} - r_j \leq \lambda \leq a_{jj} + r_j$$

which leads to:

**Corollary 2.14.** *Let* $A = [a_{ij}]$ *be a symmetric e-matrix and let* $\lambda$ *be a non-trivial eigenvalue of* $A$*. Then*

$$\min_{1 \leq i \leq n} \{a_{ii} - r_i\} \leq \lambda \leq \max_{1 \leq i \leq n} \{a_{ii} + r_i\},$$

*where* $r_i$ *is the radius of the Geršgorin disc of the second type constructed from the* $i^{th}$ *column of* $A$*.*

Now, let $L = [l_{ij}]$ be the Laplacian of a connected simple graph $G$. Then $L$ is a symmetric e-matrix. The trivial eigenvalue of $L$ is 0 and its algebraic connectivity $\mu$ is strictly greater than 0 since $G$ is connected. The algebraic connectivity $\mu$ is also the smallest non-trivial eigenvalue of $L$, so it is the closest non-trivial eigenvalue of $L$ to the lower bound given in Corollary 2.14. That is,

$$\mu \geq \min_{1 \leq i \leq n} \{l_{ii} - r_i\},$$

where $r_i$ is the radius of the Geršgorin disc of the second type constructed from the $i^{th}$ column of $L$. Next we try to find some explicit forms of the above formula. Suppose that the connected simple graph $G$ has vertices $v_1, \dots, v_n$ with degrees $d_1, \dots, d_n$, respectively. Then the entries of $L$ are

$$l_{ij} = \begin{cases} d_i, & \text{if } i = j \\ -1, & \text{if } v_i v_j \in E \\ 0, & \text{if } v_i v_j \notin E. \end{cases}$$

If $n$ is even and $d_i > \frac{n}{2}$, then

$$r_i = \big[0 + \cdots + 0\big] + \overbrace{\big[(-1) + \cdots + (-1)\big]}^{(d_i - \frac{n}{2}) \text{ terms}} - \overbrace{\big[(-1) + \cdots + (-1)\big]}^{\frac{n}{2} \text{ terms}}.$$

That is $r_i = n - d_i$. It follows that, if $\lambda$ is a non-trivial eigenvalue of $L$ that lies within the Geršgorin disc of the second type constructed from the $i^{th}$ column, then $\lambda \geq 2d_i - n$.

If $n$ is odd and $d_i \geq \frac{n+1}{2}$, then

$$r_i = \big[0 + \cdots + 0\big] + \overbrace{\big[(-1) + \cdots + (-1)\big]}^{(d_i - \frac{n+1}{2})\text{ terms}} - \overbrace{\big[(-1) + \cdots + (-1)\big]}^{\frac{n-1}{2}\text{ terms}}.$$

That is, again $r_i = n - d_i$. Therefore, if $\lambda$ is a non-trivial eigenvalue of $L$ that lies within the Geršgorin disc of the second type constructed from the $i^{th}$ column, then $\lambda \geq 2d_i - n$.

In the case where $n$ is even and $d_i \leq \frac{n}{2}$ and in the case where $n$ is odd and $d_i \leq \frac{n-1}{2}$, the lower bound given by Corollary 2.14 is equal to 0, therefore trivial.

In view of the above discussion we have the following result.

**Theorem 2.15.** *Let $G$ be a simple connected graph of order $n$ such that every vertex is connected to a number of vertices no less than $\frac{n}{2} + 1$ if $n$ is even, or no less than $\frac{n+1}{2}$ if $n$ is odd. Let $\mu$ be the algebraic connectivity of $G$. Then*

$$\mu \geq 2d - n,$$

*where $d$ is the minimum among the degrees of the vertices of $G$.*

**Example 2.16.** Consider the $10 \times 10$ Laplacian matrix

$$L = \begin{bmatrix} 7 & -1 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & -1 \\ -1 & 8 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 7 & -1 & -1 & -1 & 0 & -1 & -1 & -1 \\ 0 & -1 & -1 & 8 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 9 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 & 8 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 & -1 & 7 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 9 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & 7 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 8 \end{bmatrix}.$$

The algebraic connectivity of $L$ is $\mu = 6.382$ and the lower bound given by Theorem 2.15 is 4.

*Remark* 2.17. This paper is more about the improvement of the location of eigenvalues of e-matrices in general and stochastic matrices in particular, using the new concept of Geršgorin discs of the second type introduced in [7]. Therefore, it does not study in depth the algebraic connectivity, and neither the applications of Theorem 2.15 nor its comparison to other existing results about the bounds of algebraic connectivity.

## 3. Some properties of the Geršgorin region of the second type of stochastic matrices

**Theorem 3.1.** *The Geršgorin region of the second type of a* $2 \times 2$ *stochastic matrix is connected.*

*Proof.* Every $2 \times 2$ stochastic matrix $S$ can be written in the form

$$S = \begin{bmatrix} a & 1-a \\ 1-b & b \end{bmatrix},$$

where $a$ and $b$ are real numbers such that $0 \leq a \leq 1$ and $0 \leq b \leq 1$. The Geršgorin discs of the second type of $S$ are $D_1(a, 1-a)$ and $D_2(b, 1-b)$. We have $\Big[(1-a)+(1-b)\Big]^2 - (|a-b|)^2 = 2-2(a+b)+2ab = 2(1-a)(1-b) \geq 0$. Hence $(1-a)+(1-b) \geq |a-b|$, which means that the discs $D_1$ and $D_2$ are connected. □

For $n \times n$ stochastic matrices with $n \geq 3$, it may happen that the Geršgorin region of the second type of the transpose consists of disjoint parts as in the following examples.

**Example 3.2.** Let

$$S_1 = \frac{1}{20} \begin{bmatrix} 14 & 4 & 2 \\ 4 & 14 & 2 \\ 9 & 9 & 2 \end{bmatrix}$$

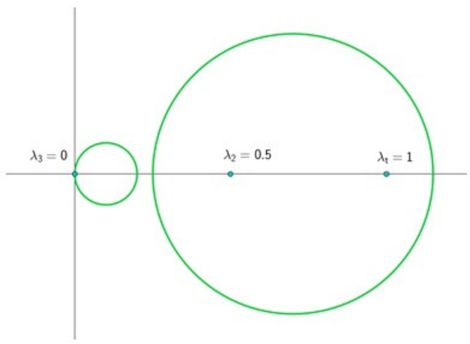


FIGURE 1. $G_2(S_1^T)$

and $S_2 = \frac{1}{100} \begin{bmatrix} 65 & 14 & 13 & 8 \\ 10 & 72 & 10 & 8 \\ 10 & 10 & 70 & 10 \\ 25 & 30 & 29 & 16 \end{bmatrix}$.

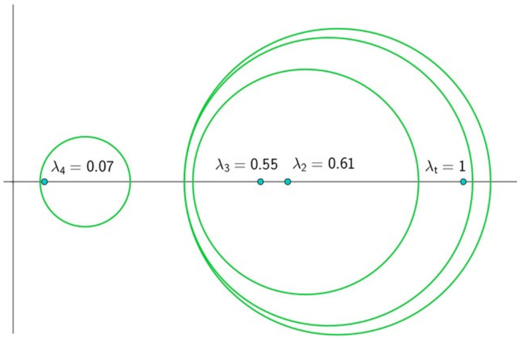


FIGURE 2. $G_2(S_2^T)$

The eigenvalues of $S_1$ are $\lambda_t = 1, \lambda_2 = 0.5$ and $\lambda_2 = 0$. The eigenvalues of $S_2$ are $\lambda_t = 1, \lambda_2 \approx 0.61, \lambda_3 \approx 0.55$ and $\lambda_4 \approx 0.07$. The non-trivial eigenvalues of each of these two matrices are contained in the Geršgorin region of the second type of its transpose. Each one of these two regions is made up of two disjoint parts, as it can be seen in Figure 1 and Figure 2.

One can see also that for both $S_1$ and $S_2$, the trivial eigenvalue $\lambda_t = 1$ is in some discs of the second type of $S_1^T$ and $S_2^T$, but this is not always true in the general case, as it is shown by the example below.

**Example 3.3.** Let

$$S_3 = \frac{1}{10}\begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 4 \\ 4 & 3 & 3 \end{bmatrix}.$$

The trivial eigenvalue $\lambda_t = 1 \notin G_2(S_3) = \hat{D}(0.1, 0.4) \cup \hat{D}(0.2, 0.4) \cup \hat{D}(0.3, 0.5)$.

There is a connection between the Geršgorin region of the second type and the primitivity of stochastic matrices, as stated by the following result.

**Theorem 3.4.** *Let $S$ be an $n \times n$ irreducible stochastic matrix. If 1 is outside the Geršgorin region of the second type of $S^T$, then $S$ is primitive.*

*Proof.* If 1 is outside the Geršgorin region of the second type of $S^T$, then all the Geršgorin discs of the second type of $S^T$ are strictly included inside the circle having center the origin and radius equal to 1. Therefore, by Theorem 2.2, every non-trivial eigenvalue $\lambda$ of $S$ is such that $|\lambda| < 1$. Since $S$ is irreducible, it follows that $S$ is primitive. □

**Example 3.5.** Let

$$S_4 = \frac{1}{10}\begin{bmatrix} 0 & 5 & 5 & 0 \\ 7 & 0 & 0 & 3 \\ 4 & 3 & 0 & 3 \\ 6 & 0 & 4 & 0 \end{bmatrix}.$$

Despite the fact that a significant number of entries are zero, this matrix is primitive by Theorem 3.4.

We can also remark that when looking at the Geršgorin region of the second type of the matrices $S_1^T$ and $S_2^T$ from Example 3.2, there is an eigenvalue in the isolated disk. Let us recall the second part of the original Geršgorin theorem [2], which states that if $R_1$, the union of $k$ of the $n$ Geršgorin discs of the first type of a matrix $A$, is disjoint from the union of the remaining $n-k$ discs, then $R_1$ contains exactly $k$ eigenvalues of $A$ counting the algebraic multiplicities. In an analogous way, we state the following open question.

**Open Question 3.6.** Prove or disapprove the following proposition. Let $S$ be an $n \times n$ stochastic matrix. If $k$ Geršgorin discs of the second type constructed from k different columns, form a region $R_1$ that is disjoint from the remaining $(n-k)$ discs, then either $k$ or $(k-1)$ eigenvalues, counting the algebraic multiplicities, are included in the region $R_1$.

So much has been written on Geršgorin theorem and its application. However, as we know, there is no algebraic proof of the second part of this theorem. Instead, there is a clever analytical proof based on the well known fact that the eigenvalues of every complex matrix are continuous functions of its entries. Unfortunately, this analytical proof seems not to work directly for solving the above open question, in case it has an affirmative answer.

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