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NONLOCAL BOUNDARY VALUE PROBLEMS FOR HILFER FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we initiate the study of boundary value problems involving Hilfer fractional derivatives. Several new existence and uniqueness results are obtained by using a variety of fixed point theorems. Examples illustrating our results are also presented.

1. Introduction

The theory of fractional differential equations received in recent years considerable interest both in pure mathematics and applications. In the literature, there exist several different definitions of fractional integrals and derivatives, for example, the most popular of them are fractional derivatives in the sense of Riemann-Liouville and Caputo. Other less-known definitions are the Hadamard fractional derivative, the Erdeyl-Kober fractional derivative and so on. We refer the interested in fractional calculus reader to the classical reference texts such as [2, 9, 11–14, 16]. A generalization of derivatives of both Riemann-Liouville and Caputo was given by R. Hilfer in [6] when he studied fractional time evolution in physical phenomena. He named it as *generalized fractional derivative of order* $\alpha \in (0,1)$ *and a type* $\beta \in [0,1]$ which can be reduced to the Riemann-Liouville and Caputo fractional derivatives when $\beta = 0$ and $\beta = 1$, respectively (See Definition 2.4). Many authors call it *the Hilfer fractional derivative.* Such derivative interpolates between the Riemann-Liouville and Caputo derivative in some sense (cf. Remark 2.5). Some properties and applications of the Hilfer derivative are given in [7], [8] and references cited therein.

Initial value problems involving Hilfer fractional derivatives were studied by several authors. In year 2012 Furati, Kassim and Tatar [3] considered the initial

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value problem involving Hilfer fractional derivative

$$\begin{cases} {}^{H}D^{\alpha,\beta}x(t) = f(t,x(t)), \quad t \in (a,\infty),\ 0<\alpha<1,\ 0\le\beta\le 1, \\ I^{1-\gamma}x(a^{+}) = x_a, \quad \gamma = \alpha+\beta-\alpha\beta, \end{cases} \tag{1.1}$$

where ${}^{H}D^{\alpha,\beta}$ is the generalized Riemann-Liouville fractional derivative operator introduced by Hilfer and $I^{1-\gamma}$ is the Riemann-Liouville fractional integral of order $1-\gamma$. They proved existence and uniqueness of global solutions in the space of weighted continuous functions and also analyzed stability of the solution for a weighted Cauchy-type problem. In year 2015, Gu and Trujillo [5] investigated existence of mild solution for evolution equation with Hilfer fractional derivative of the form

$$\begin{cases} {}^{H}D^{\alpha,\beta}x(t) = Ax(t) + f(t,x(t)), \quad t \in (0,b], \\ I^{(1-\alpha)(1-\beta)}x(0) = x_0, \end{cases} \tag{1.2}$$

where $0<\alpha<1$, $0\le\beta\le 1$, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators in Banach space. In the same year Wang and Zhang [15] discussed the existence of solutions to nonlocal initial value problem for differential equations with Hilfer fractional derivative

$$\begin{cases} {}^{H}D^{\alpha,\beta}x(t) = f(t,x(t)), \quad t \in (a,b],\ 0<\alpha<1,\ 0\le\beta\le 1, \\ I^{1-\gamma}x(a^{+}) = \displaystyle\sum_{i=1}^{m}\lambda_i x(\tau_i), \quad \gamma = \alpha+\beta-\alpha\beta,\ \tau_i \in (a,b]. \end{cases} \tag{1.3}$$

Using Krasnoselskii and Schauder fixed point theorems, they proved the existence of problem (1.3). However, to the best of our knowledge, there is no work on boundary value problems with Hilfer fractional derivatives in the literature.

The objective of the present work is to introduce a new class of boundary value problems of Hilfer-type fractional differential equations with nonlocal integral boundary conditions, and develop the existence and uniqueness criteria for the solutions of such problems. In precise terms, we consider the nonlocal boundary value problem

$${}^{H}D^{\alpha,\beta}x(t) = f(t,x(t)), \quad t \in [a,b], \ \ 1<\alpha<2, \ \ 0\le\beta\le 1, \tag{1.4}$$

$$x(a)=0,\ x(b) = \sum_{i=1}^{m}\delta_i I^{\varphi_i}x(\xi_i), \quad \varphi_i>0,\ \delta_i\in\mathbb{R},\ \xi_i\in[a,b], \tag{1.5}$$

where ${}^{H}D^{\alpha,\beta}$ is the Hilfer fractional derivative of order α, $1<\alpha<2$ and parameter β, $0\le\beta\le 1$, I^{φ_i} is the Riemann-Liouville fractional integral of order $\varphi_i>0$, $\xi_i\in[a,b]$, $a\ge 0$ and $\delta_i\in\mathbb{R}$, $i=1,\dots,m$.

Several existence and uniqueness results are proved by using a variety of fixed point theorems. We make use of Banach's fixed point theorem, Hölder's inequality and Boyd and Wong fixed point theorem for nonlinear contractions [1] to obtain the uniqueness results, while nonlinear alternative of Leray-Schauder type [4] and Krasnoselskii's fixed point theorem [10] are applied to obtain the existence results for the problem (1.4)-(1.5).

The paper is organized as follows: We present our main work in Section 3, while Section 2 contains some preliminary concepts related to our problem. Examples are constructed in every section to illustrate the main results.

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [9, 13].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function is defined by

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)ds, \qquad n-1<\alpha<n,$$

where $n = [\alpha] + 1$ denotes the integer part of real number α, provided the right-hand side is point-wise defined on (a, ∞).

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function is defined by

$$\begin{aligned} {}^{RL}D^{\alpha}u(t) &:= D^n I^{n-\alpha}u(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1}u(s)ds, \qquad n-1<\alpha<n, \end{aligned}$$

where $n = [\alpha] + 1$ denotes the integer part of real number α, provided the right-hand side is point-wise defined on (a, ∞).

Definition 2.3. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function is defined by

$$\begin{aligned} {}^{C}D^{\alpha}u(t) &:= I^{n-\alpha}D^n u(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}\left(\frac{d}{ds}\right)^n u(s)ds, \qquad n-1<\alpha<n, \end{aligned}$$

where $n = [\alpha] + 1$ denotes the integer part of real number α, provided the right-hand side is point-wise defined on (a, ∞).

In [6] (see also [8]) another new definition of the fractional derivative was suggested. *The generalized Riemann-Liouville fractional derivative* defined as:

Definition 2.4. The generalized Riemann-Liouville fractional derivative or Hilfer fractional derivative of order α and parameter β of a function is defined by

$${}^{H}D^{\alpha,\beta}u(t) = I^{\beta(n-\alpha)}D^n I^{(1-\beta)(n-\alpha)}u(t),$$

where $n-1<\alpha<n$, $0 \le \beta \le 1$, $t > a$, $D = \frac{d}{dt}$.

Remark 2.5. Observe that if $\beta = 0$, (1.4) is reduced to the Riemann-Liouville fractional differential equation of the form:

$$ {}^{RL}D^{\alpha}x(t) = f(t, x(t)), \quad t \in [a, b], \tag{2.1} $$

and if $\beta = 1$, (1.4) is also reduced to the Caputo fractional differential equation

$$ {}^{C}D^{\alpha}x(t) = f(t, x(t)), \quad t \in [a, b], \tag{2.2} $$

where ${}^{RL}D^{\alpha}$ and ${}^{C}D^{\alpha}$ are respectively the Riemann-Liouville and Caputo fractional differential operators of order α.

Lemma 2.6 ([9, Lemma 2.5])**.** *Let* $1 < \alpha \leq 2$. *Then*

$$ I^{\alpha}\left({}^{RL}D^{\alpha}f\right)(t) = f(t) - \frac{(I^{1-\alpha}f)(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1} - \frac{(I^{2-\alpha}f)(a)}{\Gamma(\alpha-1)}(t-a)^{\alpha-2}. $$

3. Main results

The following lemma deals with a linear variant of the boundary value problem (1.4)-(1.5).

Lemma 3.1. *Let*

$$ \Lambda = \sum_{i=1}^{m} \frac{\delta_i(\xi_i - a)^{\gamma+\varphi_i-1}}{\Gamma(\gamma+\varphi_i)} - \frac{(b-a)^{\gamma-1}}{\Gamma(\gamma)} \neq 0, \tag{3.1} $$

$\varphi_i > 0$, $\xi_i \in [a, b]$, $a \geq 0$, $\delta_i \in \mathbb{R}$, $i = 1, \ldots, m$, $1 < \alpha < 2$, $\gamma = \alpha + 2\beta - \alpha\beta$ *and* $h \in C([a, b], \mathbb{R})$. *Then the function* x *is a solution of the boundary value*

$$ {}^{H}D^{\alpha,\beta}x(t) = h(t), \quad t \in [a, b], \ \ 1 < \alpha < 2, \ \ 0 \leq \beta \leq 1, \tag{3.2} $$

$$ x(a) = 0, \ \ x(b) = \sum_{i=1}^{m} \delta_i I^{\varphi_i} x(\xi_i), \ \ \varphi_i > 0, \ \ \delta_i \in \mathbb{R}, \ \ \xi_i \in [a, b], \tag{3.3} $$

if and only if

$$ x(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\left(I^{\alpha}h(b) - \sum_{i=1}^{m}\delta_i I^{\alpha+\varphi_i}h(\xi_i)\right) + I^{\alpha}h(t). \tag{3.4} $$

Proof. The equation (3.2) can be written as

$$ I^{\beta(2-\alpha)}D^2 I^{(1-\beta)(2-\alpha)}x(t) = h(t). \tag{3.5} $$

Applying the Riemann-Liouville fractional integral of order α to the both sides of the equation (3.5), we obtain

$$ I^{\alpha}I^{\beta(2-\alpha)}D^2 I^{(1-\beta)(2-\alpha)}x(t) = I^{\alpha}h(t). $$

Indeed

$$ I^{\alpha}I^{\beta(2-\alpha)}D^2 I^{(1-\beta)(2-\alpha)}x(t) = I^{\gamma}D^2 I^{2-\gamma}x(t) = I^{\gamma}\left({}^{RL}D^{\gamma}x\right)(t), $$

and therefore

$$ I^{\gamma}\left({}^{RL}D^{\gamma}x\right)(t) = I^{\alpha}h(t). $$

By using Lemma 2.6 and setting $\left(I^{2-\alpha}x\right)(a) = c_1$, $\left(I^{1-\alpha}x\right)(a) = c_2$, we obtain

$$x(t) = \frac{c_2}{\Gamma(\gamma)}(t-a)^{\gamma-1} + \frac{c_1}{\Gamma(\gamma-1)}(t-a)^{\gamma-2} + I^{\alpha}h(t). \tag{3.6}$$

From the first boundary condition $x(a) = 0$, we obtain $c_1 = 0$. Then we get

$$x(t) = \frac{c_2}{\Gamma(\gamma)}(t-a)^{\gamma-1} + I^{\alpha}h(t), \tag{3.7}$$

and

$$\sum_{i=1}^{m} \delta_i I^{\varphi_i} x(\xi_i) = c_2 \sum_{i=1}^{m} \frac{\delta_i(\xi_i-a)^{\gamma+\varphi_i-1}}{\Gamma(\gamma+\varphi_i)} + \sum_{i=1}^{m} \delta_i I^{\alpha+\varphi_i} h(\xi_i). \tag{3.8}$$

From $x(b) = \sum_{i=1}^{m} \delta_i I^{\varphi_i} x(\xi_i)$, by using (3.8), we have

$$c_2 \left(\sum_{i=1}^{m} \frac{\delta_i(\xi_i-a)^{\gamma+\varphi_i-1}}{\Gamma(\gamma+\varphi_i)} - \frac{(b-a)^{\gamma-1}}{\Gamma(\gamma)} \right) = I^{\alpha}h(b) - \sum_{i=1}^{m} \delta_i I^{\alpha+\varphi_i} h(\xi_i),$$

from which we get

$$c_2 = \frac{1}{\Lambda}\left(I^{\alpha}h(b) - \sum_{i=1}^{m} \delta_i I^{\alpha+\varphi_i} h(\xi_i) \right).$$

Substituting the value of c_1 and c_2 in (3.6), we obtain the solution (3.4). The converse follows by direct computation. This completes the proof. □

Let $\mathcal{C} = C([a,b],\mathbb{R})$ denotes the Banach space of all continuous functions from $[a,b]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t\in[a,b]} |x(t)|$. In view of Lemma 3.1, we define an operator $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}(\mathcal{A}x)(t) = {} & \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)} \left(I^{\alpha} f(s,x(s))(b) - \sum_{i=1}^{m} \delta_i I^{\alpha+\varphi_i} f(s,x(s))(\xi_i) \right) \\ & + I^{\alpha} f(s,x(s))(t),\end{aligned} \tag{3.9}$$

where the notation $I^{\phi} f(s,x(s))(y)$ means

$$I^{\phi} f(s,x(s))(y) = \frac{1}{\Gamma(\phi)} \int_a^y (y-s)^{\phi-1} f(s,x(s))ds,$$

with $\phi \in \{\alpha, \alpha+\varphi_i\}$, $y \in \{b, \xi_i, t\}$. It should be noticed that problem (1.4)-(1.5) has a solution if and only if the operator $\mathcal{A}$ has fixed points. In the following, for the sake of convenience, we set a positive constant

$$\Omega = \frac{(b-a)^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} \frac{|\delta_i|(\xi_i-a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.10}$$

In the following subsections we prove existence, as well as existence and uniqueness results, for the boundary value problem (1.4)-(1.5) by using a variety of fixed point theorems.

3.1. Existence and uniqueness results

Our first existence and uniqueness result is based on Banach's fixed point theorem.

Theorem 3.2. *Assume that:*

(H_1) *there exists a constant* $L > 0$ *such that* $|f(t,x) - f(t,y)| \leq L|x - y|$ *for each* $t \in [a,b]$ *and* $x, y \in \mathbb{R}$.

If

$$L\Omega < 1, \tag{3.11}$$

where Ω *is defined by* (3.10), *then the boundary value problem* (1.4)-(1.5) *has a unique solution on* $[a,b]$.

Proof. We transform the boundary value problem (1.4)-(1.5) into a fixed point problem, $x = \mathcal{A}x$, where the operator $\mathcal{A}$ is defined as in (3.9). Observe that the fixed points of the operator $\mathcal{A}$ are solutions of problem (1.4)-(1.5). Applying the Banach contraction mapping principle, we shall show that $\mathcal{A}$ has a unique fixed point.

To construct a neighborhood of radius r, we let $\sup_{t\in[a,b]} |f(t,0)| = M < \infty$, and choose

$$r \geq \frac{M\Omega}{1 - L\Omega}. \tag{3.12}$$

Now, we show that $\mathcal{A}B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. For any $x \in B_r$, we have

$$\begin{aligned}
&|(\mathcal{A}x)(t)| \\
&\leq \sup_{t\in[a,b]} \left\{ \frac{(t-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha}|f(s,x(s))|(b) \right. \\
&\quad \left. + \frac{(t-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |\delta_i| I^{\alpha+\varphi_i}|f(s,x(s))|(\xi_i) + I^{\alpha}|f(s,x(s))|(t) \right\} \\
&\leq \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha} \left(|f(s,x(s)) - f(s,0)| + |f(s,0)|\right)(b) \\
&\quad + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |\delta_i| I^{\alpha+\varphi_i}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)(\xi_i) \\
&\quad + I^{\alpha}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)(b) \\
&\leq (L\|x\| + M) \left\{ \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha}(1)(b) + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |\delta_i| I^{\alpha+\varphi_i}(1)(\xi_i) + I^{\alpha}(1)(b) \right\} \\
&\leq (Lr + M) \left(\frac{(b-a)^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} \frac{|\delta_i|(\xi_i - a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \right) \\
&= (Lr + M)\Omega \leq r,
\end{aligned}$$

which implies that $\mathcal{A}B_r \subset B_r$.

Next, we let $x, y \in \mathcal{C}$. Then for $t \in [a,b]$, we have

$$\begin{aligned}
&|(\mathcal{A}x)(t) - (\mathcal{A}y)(t)| \\
\le\ & \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha}|f(s,x(s)) - f(s,y(s))|(b) \\
&+ \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |\delta_i| I^{\alpha+\varphi_i}|f(s,x(s)) - f(s,y(s))|(\xi_i) \\
&+ I^{\alpha}|f(s,x(s)) - f(s,y(s))|(b) \\
\le\ & L\left(\frac{(b-a)^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} \frac{|\delta_i|(\xi_i-a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \right) \|x-y\| \\
=\ & L\Omega\|x-y\|,
\end{aligned}$$

which implies that $\|\mathcal{A}x - \mathcal{A}y\| \le L\Omega\|x-y\|$. As $L\Omega < 1$, $\mathcal{A}$ is a contraction. Therefore, we deduce by the Banach's contraction mapping principle, that $\mathcal{A}$ has a fixed point which is the unique solution of the boundary value problem (1.4)-(1.5). The proof is completed. □

Remark 3.3. We would like point out that the condition $L\Omega < 1$ can be deleted if we use the well-known Bielecki's renorming method.

Now, we give some special cases of the above theorem by setting constants Ω_0 and Ω_1 with $\beta = 0$ and $\beta = 1$, respectively, as

$$\Omega_0 := \frac{(b-a)^{2\alpha-1}}{|\Lambda_0|\Gamma(\alpha)\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha-1}}{|\Lambda_0|\Gamma(\alpha)} \sum_{i=1}^{m} \frac{|\delta_i|(\xi_i-a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}, \tag{3.13}$$

$$\Omega_1 := \frac{(b-a)^{\alpha+1}}{|\Lambda_1|\Gamma(\alpha+1)} + \frac{(b-a)}{|\Lambda_1|} \sum_{i=1}^{m} \frac{|\delta_i|(\xi_i-a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}, \tag{3.14}$$

where

$$\Lambda_0 = \sum_{i=1}^{m} \frac{\delta_i(\xi_i-a)^{\alpha+\varphi_i-1}}{\Gamma(\alpha+\varphi_i)} - \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}, \tag{3.15}$$

$$\Lambda_1 = \sum_{i=1}^{m} \frac{\delta_i(\xi_i-a)^{\varphi_i+1}}{\Gamma(\varphi_i+2)} - (b-a). \tag{3.16}$$

Thus, we have the following corollaries.

Corollary 3.4. *Suppose that the condition* (H_1) *holds. If* $L\Omega_0 < 1$, *where* Ω_0 *is defined by* (3.13), *then the problem* (2.1)-(1.5) *has a unique solution on* $[a,b]$.

Corollary 3.5. *Assume that the condition* (H_1) *is satisfied. If* $L\Omega_1 < 1$, *where* Ω_1 *is defined by* (3.14), *then the problem* (2.2)-(1.5) *has a unique solution on* $[a,b]$.

Our second existence and uniqueness result is proved by using Banach's fixed point theorem together with Hölder inequality.

Theorem 3.6. *Suppose that $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following assumption*:

(H_2) $|f(t,x) - f(t,y)| \le \theta(t)|x-y|$ *for* $t \in [a,b]$, $x, y \in \mathbb{R}$ *and* $\theta \in L^{1/\sigma}([a,b], \mathbb{R}^+)$, $\sigma \in (0,1)$.

Denote $\|\theta\| = \left(\int_a^b |\theta(s)|^{1/\sigma} ds\right)^\sigma$ *and*

$$\omega = \frac{(b-a)^{\gamma+\alpha-\sigma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha)}\left(\frac{1-\sigma}{\alpha-\sigma}\right)^{1-\sigma} + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^m \frac{|\delta_i|(\xi_i-a)^{\alpha+\varphi_i-\sigma}}{\Gamma(\alpha+\varphi)}\left(\frac{1-\sigma}{\alpha+\varphi_i-\sigma}\right)^{1-\sigma} + \frac{(b-a)^{\alpha-\sigma}}{\Gamma(\alpha)}\left(\frac{1-\sigma}{\alpha-\sigma}\right)^{1-\sigma}.$$

If

$$\|\theta\|\omega < 1, \tag{3.17}$$

then the boundary value problem (1.4)-(1.5) *has a unique solution on* $[a,b]$.

Proof. For $x, y \in \mathcal{C}$ and for each $t \in [a,b]$, by Hölder's inequality, we have

$$\begin{aligned}
&|(\mathcal{A}x)(t) - (\mathcal{A}y)(t)| \\
\le\ & \frac{(b-s)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^\alpha |f(s,x(s)) - f(s,y(s))|(b) \\
&+ \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^m |\delta_i| I^{\alpha+\varphi_i}|f(s,x(s)) - f(s,y(s))|(\xi_i) \\
&+ I^\alpha |f(s,x(s)) - f(s,y(s))|(b) \\
=\ & \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\theta(s)|x(s)-y(s)|ds \\
&+ \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^m \frac{|\delta_i|}{\Gamma(\alpha+\varphi_i)}\int_a^{\xi_i}(\xi_i-s)^{\alpha+\varphi_i-1}\theta(s)|x(s)-y(s)|ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\theta(s)|x(s)-y(s)|ds \\
\le\ & \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha)}\left(\int_a^b \left((b-s)^{\alpha-1}\right)^{\frac{1}{1-\sigma}} ds\right)^{1-\sigma}\left(\int_a^b (\theta(s))^{\frac{1}{\sigma}}\right)^\sigma \|x-y\| \\
&+ \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^m \frac{|\delta_i|}{\Gamma(\alpha+\varphi_i)}\left(\int_a^{\xi_i}\left((\xi_i-s)^{\alpha+\varphi_i-1}\right)^{\frac{1}{1-\sigma}} ds\right)^{1-\sigma}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_a^{\xi_i} \left(\theta(s)^{\frac{1}{\sigma}} ds\right)^{\sigma}\right) \|x-y\| + \frac{1}{\Gamma(\alpha)} \left(\int_a^b \left((b-s)^{\alpha-1}\right)^{\frac{1}{1-\sigma}} ds\right)^{1-\sigma} \\
&\times \left(\int_a^b (\theta(s))^{\frac{1}{\sigma}}\right)^{\sigma} \|x-y\| \\
\leq\ & \|\theta\| \Bigg[\frac{(b-a)^{\gamma+\alpha-\sigma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha)} \left(\frac{1-\sigma}{\alpha-\sigma}\right)^{1-\sigma} \\
&+ \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} \frac{|\delta_i|\,(\xi_i-a)^{\alpha+\varphi_i-\sigma}}{\Gamma(\alpha+\varphi_i)} \left(\frac{1-\sigma}{\alpha+\varphi_i-\sigma}\right)^{1-\sigma} \\
&+ \frac{(b-a)^{\alpha-\sigma}}{\Gamma(\alpha)} \left(\frac{1-\sigma}{\alpha-\sigma}\right)^{1-\sigma}\Bigg] \|x-y\| = \|\theta\|\omega\|x-y\|.
\end{aligned}
$$

It follows, by (3.17), that $\mathcal{A}$ is a contraction mapping. Hence Banach's fixed point theorem implies that $\mathcal{A}$ has a unique fixed point, which is the unique solution of the boundary value problem (1.4)-(1.5) on $[a, b]$. This completes the proof. □

The next two special cases are established by setting constants as

$$
\begin{aligned}
\omega_0 :=\ & \frac{(b-a)^{2\alpha-\sigma-1}}{|\Lambda_0|\Gamma^2(\alpha)} \left(\frac{1-\sigma}{\alpha-\sigma}\right)^{1-\sigma} + \frac{(b-a)^{\alpha-\sigma}}{\Gamma(\alpha)} \left(\frac{1-\sigma}{\alpha-\sigma}\right)^{1-\sigma} \\
&+ \frac{(b-a)^{\alpha-1}}{|\Lambda_0|\Gamma(\alpha)} \sum_{i=1}^{m} \frac{|\delta_i|(\xi_i-a)^{\alpha+\varphi_i-\sigma}}{\Gamma(\alpha+\varphi)} \left(\frac{1-\sigma}{\alpha+\varphi_i-\sigma}\right)^{1-\sigma}, \qquad (3.18)
\end{aligned}
$$

$$
\begin{aligned}
\omega_1 :=\ & \frac{(b-a)^{\alpha-\sigma+1}}{|\Lambda_1|\Gamma(\alpha)} \left(\frac{1-\sigma}{\alpha-\sigma}\right)^{1-\sigma} + \frac{(b-a)^{\alpha-\sigma}}{\Gamma(\alpha)} \left(\frac{1-\sigma}{\alpha-\sigma}\right)^{1-\sigma} \\
&+ \frac{(b-a)}{|\Lambda_1|} \sum_{i=1}^{m} \frac{|\delta_i|(\xi_i-a)^{\alpha+\varphi_i-\sigma}}{\Gamma(\alpha+\varphi)} \left(\frac{1-\sigma}{\alpha+\varphi_i-\sigma}\right)^{1-\sigma}. \qquad (3.19)
\end{aligned}
$$

Corollary 3.7. *Suppose that the condition (H_2) holds. If $\|\theta\|\omega_0 < 1$, where ω_0 is define by* (3.18)*, then the problem* (2.1)-(1.5) *has a unique solution on* $[a, b]$.

Corollary 3.8. *Assume that the condition (H_2) is satisfied. If $\|\theta\|\omega_1 < 1$, where ω_1 is define by* (3.19)*, then the problem* (2.2)-(1.5) *has a unique solution on* $[a, b]$.

Now we give our third existence and uniqueness result via nonlinear contractions. Some preliminary facts are necessary.

Definition 3.9. Let E be a Banach space and let $\mathcal{A} : E \to E$ be a mapping. $\mathcal{A}$ is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\epsilon) < \epsilon$ for all $\epsilon > 0$ with the property:

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \Psi(\|x-y\|), \qquad \forall x, y \in E.$$

Lemma 3.10 (Boyd and Wong [1])**.** *Let E be a Banach space and let $\mathcal{A}: E \to E$ be a nonlinear contraction. Then $\mathcal{A}$ has a unique fixed point in E.*

Theorem 3.11. *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption:*

(H_3) $|f(t,x) - f(t,y)| \leq h(t)\dfrac{|x-y|}{H^* + |x-y|}$ *for $t \in [a,b], x, y \in \mathbb{R}$, where $h : [a,b] \to \mathbb{R}^+$ is a continuous function and the positive constant H^* is defined by*

$$H^* := \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^\alpha h(b) + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |\delta_i| I^{\alpha+\varphi_i} h(\xi_i) + I^\alpha h(b).$$

Then the boundary value problem (1.4)-(1.5) has a unique solution on $[a,b]$.

Proof. We define the operator $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ as in (3.9) and the continuous nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\Psi(\epsilon) = \frac{H^*\epsilon}{H^* + \epsilon}, \qquad \forall \epsilon \geq 0.$$

Note that the function Ψ satisfies $\Psi(0) = 0$ and $\Psi(\epsilon) < \epsilon$ for all $\epsilon > 0$.

For any $x, y \in \mathcal{C}$ and for each $t \in [a,b]$, we have

$$\begin{aligned}
|(\mathcal{A}x)(t) - (\mathcal{A}y)(t)| &\leq \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^\alpha |f(s,x(s)) - f(s,y(s))|(b) \\
&\quad + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |\delta_i| I^{\alpha+\varphi_i} |f(s,x(s)) - f(s,y(s))|(\xi_i) \\
&\quad + I^\alpha |f(s,x(s)) - f(s,y(s))|(t) \\
&\leq \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^\alpha \left(h(s)\frac{|x-y|}{H^* + |x-y|} \right)(b) \\
&\quad + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |\delta_i| I^{\alpha+\varphi_i} \left(h(s)\frac{|x-y|}{H^* + |x-y|} \right)(\xi_i) \\
&\quad + I^\alpha \left(h(s)\frac{|x-y|}{H^* + |x-y|} \right)(b) \\
&\leq \frac{\Psi(\|x-y\|)}{H^*} \Bigg(\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^\alpha h(b) \\
&\quad + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |\delta_i| I^{\alpha+\varphi_i} h(\xi_i) + I^\alpha h(b) \Bigg) \\
&= \Psi(\|x-y\|).
\end{aligned}$$

This implies that $\|\mathcal{A}x - \mathcal{A}y\| \leq \Psi(\|x-y\|)$. Therefore $\mathcal{A}$ is a nonlinear contraction. Hence, by Lemma 3.10 the operator $\mathcal{A}$ has a unique fixed point which is the unique solution of the boundary value problem (1.4)-(1.5). This completes the proof. □

In the special cases we set constants as

$$H_0^* := \frac{(b-a)^{\alpha-1}}{|\Lambda_0|\Gamma(\alpha)} I^{\alpha}h(b) + \frac{(b-a)^{\alpha-1}}{|\Lambda_0|\Gamma(\alpha)} \sum_{i=1}^{m} |\delta_i| I^{\alpha+\varphi_i}h(\xi_i) + I^{\alpha}h(b), \tag{3.20}$$

$$H_1^* := \frac{(b-a)}{|\Lambda_1|} I^{\alpha}h(b) + \frac{(b-a)}{|\Lambda_1|} \sum_{i=1}^{m} |\delta_i| I^{\alpha+\varphi_i}h(\xi_i) + I^{\alpha}h(b). \tag{3.21}$$

Corollary 3.12. *Suppose that the condition (H_3) holds with H^* replaced by H_0^*. Then the problem (2.1)-(1.5) has a unique solution on $[a,b]$.*

Corollary 3.13. *Assume that the condition (H_3) is satisfied with H^* replaced by H_1^*. Then the problem (2.2)-(1.5) has a unique solution on $[a,b]$.*

Example 3.14. Consider the nonlocal boundary value problem with Hilfer fractional differential equation

$$\begin{cases} {}^{H}D^{\frac{3}{2},\frac{1}{3}}x(t) = \dfrac{1}{2(3+2t)^2}\left(\dfrac{x^2(t)+2|x(t)|}{1+|x(t)|}\right) + \dfrac{3}{2}, \quad t \in [1/2, 5], \\ x\left(\dfrac{1}{2}\right) = 0, \ x(5) = I^{\frac{3}{4}}x(2) + \dfrac{3}{5} I^{\frac{5}{3}} x\left(\dfrac{7}{2}\right) + \dfrac{7}{3} I^{\frac{4}{5}} x\left(\dfrac{9}{5}\right). \end{cases} \tag{3.22}$$

Here $\alpha = 3/2$, $\beta = 1/3$, $\gamma = 5/3$, $a = 1/2$, $b = 5$, $\delta_1 = 1$, $\delta_2 = 3/5$, $\delta_3 = 7/3$, $\varphi_1 = 3/4$, $\varphi_2 = 5/3$, $\varphi_3 = 4/5$, $\xi_1 = 2$, $\xi_2 = 7/2$ and $\xi_3 = 9/5$. Since $|f(t,x) - f(t,y)| \leq (1/16)|x-y|$, then (H_1) is satisfied with $L = 1/16$. By direct computation, we have $\Omega = 15.20692499$.

Thus $L\Omega = 0.95043281 < 1$. Hence, by Theorem 3.2, the boundary value problem (3.22) has a unique solution on $[1/2, 5]$.

Example 3.15. Consider the nonlocal boundary value problem with Hilfer fractional differential equation

$$\begin{cases} {}^{H}D^{\frac{4}{3},\frac{5}{6}}x(t) = \left(\sqrt{t^3+1}\right)\left(\dfrac{|x(t)|}{105+|x(t)|}\right) + \dfrac{1}{2}, \quad t \in [1, 15/2], \\ x(1) = 0, \ x\left(\dfrac{15}{2}\right) = 3I^{\frac{1}{2}}x\left(\dfrac{3}{2}\right) + I^{\frac{2}{3}}x\left(\dfrac{5}{2}\right) + \dfrac{1}{2}I^{\frac{3}{4}}x\left(\dfrac{7}{2}\right) + 5I^{\frac{4}{5}}x\left(\dfrac{9}{2}\right). \end{cases} \tag{3.23}$$

Here $\alpha = 4/3$, $\beta = 5/6$, $\gamma = 17/9$, $a = 1$, $b = 15/2$, $\delta_1 = 3$, $\delta_2 = 1$, $\delta_3 = 1/2$, $\delta_4 = 5$, $\varphi_1 = 1/2$, $\varphi_2 = 2/3$, $\varphi_3 = 3/4$, $\varphi_4 = 4/5$, $\xi_1 = 3/2$, $\xi_2 = 5/2$, $\xi_3 = 7/2$ and $\xi_4 = 9/2$. We choose $h(t) = \sqrt{t^3+1}$ and find that

$$H^* = \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha}h(b) + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} |\delta_i| I^{\alpha+\varphi_i}h(\xi_i) + I^{\alpha}h(b) = 103.5822234.$$

Clearly,

$$|f(t,x)-f(t,y)| = \left(\sqrt{t^3+1}\right)\left(\frac{105(|x|-|y|)}{105^2+105|x|+105|y|+|x||y|}\right),$$
$$\leq \frac{|x-y|}{103.5822234+|x-y|}.$$

Hence, by Theorem 3.11, the boundary value problem (3.23) has a unique solution on $[1, 15/2]$.

3.2. Existence results

In this subsection we present some existence results. The first existence result is based on the well-known Krasnoselskii's fixed point theorem.

Lemma 3.16 (Krasnoselskii's fixed point theorem)**.** *Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that* (a) *$Ax + By \in M$ whenever $x, y \in M$*; (b) *A is compact and continuous*; (c) *B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.17. *Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (H_1). In addition we assume that*:

$$(H_4) \qquad |f(t,x)| \leq \varphi(t), \quad \forall (t,x) \in [a,b]\times\mathbb{R}, \ \text{and } \varphi \in C([a,b],\mathbb{R}^+).$$

Then the boundary value problem (1.4)-(1.5) *has at least one solution on $[a, b]$ provided*

$$L\mu < 1, \tag{3.24}$$

where

$$\mu = \frac{(b-a)^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)}. \tag{3.25}$$

Proof. Setting $\sup_{t\in[a,b]} \varphi(t) = \|\varphi\|$ and choosing

$$\rho \geq \|\varphi\|\Omega, \tag{3.26}$$

(where Ω is defined by (3.10)), we consider $B_\rho = \{x \in \mathcal{C} : \|x\| \leq \rho\}$. We define the operators $\mathcal{A}_1$, $\mathcal{A}_2$ on B_ρ by

$$\mathcal{A}_1 x(t) = I^\alpha f(s, x(s))(t), \quad t \in [a, b],$$

and

$$\mathcal{A}_2 x(t) = \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)} I^\alpha f(s,x(s))(b) - \frac{(t-a)^{\gamma-1}}{\Lambda\Gamma(\gamma)}\sum_{i=1}^{m}\delta_i I^{\alpha+\varphi_i} f(s,x(s))(\xi_i), \quad t \in [a,b].$$

For any $x, y \in B_\rho$, we have

$$\begin{aligned}
&|(\mathcal{A}_1 x)(t) + (\mathcal{A}_2 y)(t)| \\
&\le \sup_{t\in[a,b]} \Bigg\{ I^\alpha |f(s,x(s))|(t) + \frac{(t-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^\alpha |f(s,y(s))|(b) \\
&\quad + \frac{(t-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^m |\delta_i| I^{\alpha+\varphi_i} |f(s,y(s))|(\xi_i) \Bigg\} \\
&\le \|\varphi\| \left(\frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^m \frac{|\delta_i|(\xi_i - a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)} \right) \\
&\le \|\varphi\|\Omega \le \rho.
\end{aligned}$$

This shows that $\mathcal{A}_1 x + \mathcal{A}_2 y \in B_\rho$. Therefore, the condition (a) in Lemma 3.16 is satisfied. It is easy to see, using (3.24), that $\mathcal{A}_2$ is a contraction mapping.

Continuity of f implies that the operator $\mathcal{A}_1$ is continuous. Also, $\mathcal{A}_1$ is uniformly bounded on B_ρ as

$$\|\mathcal{A}_1 x\| \le \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|\varphi\|.$$

Now we prove the compactness of the operator $\mathcal{A}_1$.

We define $\sup_{(t,x)\in[a,b]\times B_\rho} |f(t,x)| = \bar{f} < \infty$, and consequently we have

$$\begin{aligned}
|(\mathcal{A}_1 x)(t_2) - (\mathcal{A}_1 x)(t_1)| &= \frac{1}{\Gamma(\alpha)} \Bigg| \int_a^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] f(s,x(s)) ds \\
&\quad + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} f(s,x(s)) ds \Bigg| \\
&\le \frac{\bar{f}}{\Gamma(\alpha+1)} [2(t_2 - t_1)^\alpha + |(t_2 - a)^\alpha - (t_1 - a)^\alpha|],
\end{aligned}$$

which is independent of x and tend to zero as $t_2 - t_1 \to 0$. Thus, $\mathcal{A}_1$ is equicontinuous. So $\mathcal{A}_1$ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, $\mathcal{A}_1$ is compact on B_ρ. Thus all the assumptions of Lemma 3.16 are satisfied. So the conclusion of Lemma 3.16 implies that the boundary value problem (1.4)-(1.5) has at least one solution on $[a, b]$. □

Remark 3.18. In the above theorem we can interchange the roles of the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ to obtain a second result replacing (3.24) by the following condition:

$$L \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} < 1.$$

Next, we give two special cases by setting constants as

$$\mu_0 = \frac{(b-a)^{2\alpha-1}}{|\Lambda_0|\Gamma(\alpha)\Gamma(\alpha+1)} + \frac{(b-a)^{\alpha-1}}{|\Lambda_0|\Gamma(\alpha)} \sum_{i=1}^m \frac{|\delta_i|(\xi_i - a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)}, \tag{3.27}$$

$$\mu_1 = \frac{(b-a)^{\alpha+1}}{|\Lambda_1|\Gamma(\alpha+1)} + \frac{(b-a)}{|\Lambda_1|}\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)}. \tag{3.28}$$

Corollary 3.19. *Suppose that the condition* (H_4) *holds. Then the problem* (2.1)-(1.5) *has at least one solution on* $[a,b]$, *provided* $L\mu_0 < 1$.

Corollary 3.20. *Assume that the condition* (H_4) *is satisfied. Then the problem* (2.2)-(1.5) *has at least one solution on* $[a,b]$, *provided* $L\mu_1 < 1$.

The Leray-Schauder's Nonlinear Alternative is used for our next existence result.

Lemma 3.21 (Nonlinear alternative for single valued maps [4])**.** *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $\mathcal{A} : \bar{U} \to C$ is a continuous, compact (that is, $\mathcal{A}(\bar{U})$ is a relatively compact subset of C) map. Then either*

(i) *$\mathcal{A}$ has a fixed point in $\bar{U}$, or*
(ii) *there is a $x \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $x = \lambda\mathcal{A}(x)$.*

Theorem 3.22. *Assume that*:

(H_5) *there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([a,b], \mathbb{R}^+)$ such that*
$$|f(t,u)| \le p(t)\psi(\|x\|) \quad \text{for each} \quad (t,x) \in [a,b]\times\mathbb{R};$$
(H_6) *there exists a constant $M > 0$ such that*
$$\frac{M}{\psi(M)\|p\|\Omega} > 1,$$
where Ω is defined by (3.10).

Then the boundary value problem (1.4)-(1.5) *has at least one solution on* $[a,b]$.

Proof. Let the operator $\mathcal{A}$ be defined by (3.9). Firstly, we shall show that $\mathcal{A}$ maps bounded sets (balls) into bounded set in $\mathcal{C}$. For a number $r > 0$, let $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$ be a bounded ball in $\mathcal{C}$. Then for $t \in [a,b]$ we have

$$\begin{aligned}
&|(\mathcal{A}x)(t)| \\
\le& \sup_{t\in[a,b]}\left\{\frac{(t-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha}|f(s,x(s))|(b)\right. \\
&\left. + \frac{(t-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^{m}|\delta_i| I^{\alpha+\varphi_i}|f(s,x(s))|(\xi_i) + I^{\alpha}|f(s,x(s))|(t)\right\} \\
\le& \psi(\|x\|)\frac{(t-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha}p(s)(b) \\
&+ \psi(\|x\|)\frac{(t-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^{m}|\delta_i| I^{\alpha+\varphi_i}p(s)(\xi_i) + \psi(\|x\|)I^{\alpha}p(s)(b)
\end{aligned}$$

$$\le \psi(\|x\|)\|p\|\left(\frac{(b-a)^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)}+\frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)}+\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\right),$$

and consequently,

$$\|\mathcal{A}x\| \le \psi(r)\|p\|\Omega.$$

This means that $\mathcal{A}(B_r)$ is uniformly bounded. Next we will show that $\mathcal{A}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let $\tau_1, \tau_2 \in [a,b]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then we have

$$\begin{aligned}
&|(\mathcal{A}x)(\tau_2)-(\mathcal{A}x)(\tau_1)| \\
&\le \frac{(\tau_2-a)^{\gamma-1}-(\tau_1-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} I^{\alpha}|f(s,x(s))|(b) \\
&\quad + \frac{(\tau_2-a)^{\gamma-1}-(\tau_1-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\sum_{i=1}^{m}|\delta_i| I^{\alpha+\varphi_i}|f(s,x(s))|(\xi_i) \\
&\quad + \frac{1}{\Gamma(\alpha)}\bigg|\int_a^{\tau_1}[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}]f(s,x(s))ds \\
&\quad + \int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}f(s,x(s))ds\bigg| \\
&\le \frac{(\tau_2-a)^{\gamma-1}-(\tau_1-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\psi(r) I^{\alpha}p(s)(b) \\
&\quad + \frac{(\tau_2-a)^{\gamma-1}-(\tau_1-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\psi(r)\sum_{i=1}^{m}|\delta_i| I^{\alpha+\varphi_i}p(s)(\xi_i) \\
&\quad + \frac{\psi(r)}{\Gamma(\alpha)}\bigg|\int_a^{\tau_1}[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}]p(s)ds \\
&\quad + \int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}p(s)ds\bigg| \\
&\le \frac{(\tau_2-a)^{\gamma-1}-(\tau_1-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)}\|p\|\psi(r)\left\{\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}+\sum_{i=1}^{m}\frac{|\delta_i|(\xi_i-a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)}\right\} \\
&\quad + \frac{\|p\|\psi(r)}{\Gamma(\alpha+1)}[2(t_2-t_1)^{\alpha}+|(t_2-a)^{\alpha}-(t_1-a)^{\alpha}|].
\end{aligned}$$

As $\tau_2-\tau_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. It follows that the set $\mathcal{A}(B_r)$ is an equicontinuous set. Therefore, by the Arzelá-Ascoli theorem, the set $\mathcal{A}(B_r)$ is relatively compact which implies that the operator $\mathcal{A}: \mathcal{C} \to \mathcal{C}$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative (Lemma 3.21) once we have proved the boundedness of the set of all solutions to equations $x = \lambda\mathcal{A}x$ for $\lambda \in (0,1)$.

Let x be a solution of (1.4)-(1.5). Then, for $t \in [a,b]$, and following the similar computations as in the first step, we have

$$|x(t)| \leq \psi(\|x\|)\|p\|\Omega,$$

which leads to

$$\frac{\|x\|}{\psi(\|x\|)\|p\|\Omega} \leq 1.$$

In view of (H_6), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C([a,b],\mathbb{R}) : \|x\| < M\}.$$

We see that the operator $\mathcal{A} : \bar{U} \to \mathcal{C}$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \lambda \mathcal{A} x$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.21), we deduce that $\mathcal{A}$ has a fixed point $x \in \bar{U}$ which is a solution of the boundary value problem (1.4)-(1.5). This completes the proof. □

Corollary 3.23. *Suppose that the condition* (H_5) *and* (H_6) *holds. If*

$$\frac{M}{\psi(M)\|p\|\Omega_0} > 1,$$

where Ω_0 *is define by* (3.13)*, then the problem* (2.1)-(1.5) *has at least one solution on* $[a,b]$.

Corollary 3.24. *Assume that the condition* (H_5) *and* (H_6) *are satisfied. If*

$$\frac{M}{\psi(M)\|p\|\Omega_1} > 1,$$

where Ω_1 *is define by* (3.14)*, then the problem* (2.2)-(1.5) *has at least one solution on* $[a,b]$.

Corollary 3.25. *Suppose that the continuous function* f *satisfies* $|f(t,x)| \leq \kappa|x| + M, \kappa \geq 0$ *and* $M > 0$. *Then*:

(i) *If* $\kappa < \Omega^{-1}$, *the problem* (1.4)-(1.5) *has at least one solution on* $[a,b]$.
(ii) *If* $\kappa < \Omega_0^{-1}$, *the problem* (2.1)-(1.5) *has at least one solution on* $[a,b]$.
(iii) *If* $\kappa < \Omega_1^{-1}$, *the problem* (2.2)-(1.5) *has at least one solution on* $[a,b]$.

Example 3.26. Consider the nonlocal boundary value problem with Hilfer fractional differential equation

$$\begin{cases} {}^{H}D^{\frac{5}{4},\frac{3}{7}}x(t) = \dfrac{e^{-t}\sin x}{5+t^2} + \dfrac{7}{3}, \quad t \in [3/4, 8], \\ x\left(\dfrac{3}{4}\right) = 0,\ x(8) = \dfrac{7}{3} I^{\frac{4}{3}} x(1) + \dfrac{10}{3} I^{\frac{7}{3}} x\left(\dfrac{5}{4}\right) + \dfrac{11}{3} I^{\frac{8}{3}} x(5). \end{cases} \tag{3.29}$$

Here $\alpha = 5/4,\ \beta = 3/7,\ \gamma = 11/7,\ a = 3/4,\ b = 8,\ \delta_1 = 7/3,\ \delta_2 = 10/3,$ $\delta_3 = 11/3,\ \varphi_1 = 4/3,\ \varphi_2 = 7/3,\ \varphi_3 = 8/3,\ \xi_1 = 1,\ \xi_2 = 5/4$ and $\xi_3 = 5$. Since

$|f(t,x)-f(t,y)| \leq (1/5)|x-y|$, (H_1) is satisfied with $L=1/5$. By using the given data we find that

$$\mu = \frac{(b-a)^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} \frac{|\delta_i|(\xi_i-a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)} \approx 4.648284127.$$

Thus, $L\mu < 1$. Clearly,

$$|f(t,x)| = \left| \frac{e^{-t}\sin x}{5+t^2} + \frac{7}{3} \right| \leq \frac{1}{5+t^2} + \frac{7}{3}.$$

Hence, by Theorem 3.17, the boundary value problem (3.29) has at least one solution on $[3/4, 8]$.

Example 3.27. Consider the nonlocal boundary value problem with Hilfer fractional differential equation

(3.30)
$$\begin{cases} {}^{H}D^{\frac{6}{5},\frac{11}{12}}x(t) = \left(\dfrac{1}{5+3t+t^2}\right)\left(\dfrac{1}{3}\cdot\dfrac{|x|^7}{1+|x|^6}+4\right), \quad t \in [1/2, 11/2], \\ x\left(\dfrac{1}{2}\right) = 0,\ x\left(\dfrac{11}{2}\right) = \dfrac{3}{2} I^{\frac{2}{3}} x\left(\dfrac{3}{2}\right) + \dfrac{2}{3} I^{\frac{4}{3}} x\left(\dfrac{5}{2}\right) + \dfrac{4}{7} I^{\frac{7}{3}} x\left(\dfrac{7}{2}\right) + 6 I^{\frac{8}{3}} x\left(\dfrac{9}{2}\right). \end{cases}$$

Here $\alpha = 6/5$, $\beta = 11/12$, $\gamma = 29/15$, $a = 1/2$, $b = 11/2$, $\delta_1 = 3/2$, $\delta_2 = 2/3$, $\delta_3 = 4/7$, $\delta_4 = 6$, $\varphi_1 = 2/3$, $\varphi_2 = 4/3$, $\varphi_3 = 7/3$, $\varphi_4 = 8/3$, $\xi_1 = 3/2$, $\xi_2 = 5/2$, $\xi_3 = 7/2$ and $\xi_4 = 9/2$. It is easy to verify that

$$\begin{aligned} \Omega &= \frac{(b-a)^{\alpha+\gamma-1}}{|\Lambda|\Gamma(\gamma)\Gamma(\alpha+1)} + \frac{(b-a)^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \sum_{i=1}^{m} \frac{|\delta_i|(\xi_i-a)^{\alpha+\varphi_i}}{\Gamma(\alpha+\varphi_i+1)} + \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} \\ &\approx 11.52794978. \end{aligned}$$

Clearly,

$$|f(t,x)| \leq \frac{1}{5+3t+t^2}\left(\frac{1}{3}|x|+4\right).$$

Choosing $p(t) = 1/(5+3t+t^2)$ and $\psi(|x|) = (1/3)|x| + 4$, we can show that there exists a constant $M > 39.84256810$ such that

$$\frac{M}{\psi(M)\|p\|\Omega} > 1.$$

Hence, by Theorem 3.22, the boundary value problem (3.30) has at least one solution on $[1/2, 11/2]$.

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