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Equivalence of Cyclic p-squared Actions on Handlebodies

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Abstract. In this paper we consider all orientation-preserving $\mathbb{Z}_{p^2}$-actions on 3-dimensional handlebodies V_g of genus $g > 0$ for p an odd prime. To do so, we examine particular graphs of groups $(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v}))$ in canonical form for some 5-tuple $\mathrm{v} = (r, s, t, m, n)$ with $r + s + t + m > 0$. These graphs of groups correspond to the handlebody orbifolds $V(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v}))$ that are homeomorphic to the quotient spaces $V_g/\mathbb{Z}_{p^2}$ of genus less than or equal to g. This algebraic characterization is used to enumerate the total number of $\mathbb{Z}_{p^2}$-actions on such handlebodies, up to equivalence.

1. Introduction

A $\mathbf{G}$-action on a handlebody V_g, of genus $g > 0$, is a group monomorphism $\phi : \mathbf{G} \longrightarrow \mathrm{Homeo}^+(V_g)$, where $\mathrm{Homeo}^+(V_g)$ denotes the group of orientation-preserving homeomorphisms of V_g. Two actions ϕ_1 and ϕ_2 on V_g are said to be equivalent if and only if there exists an orientation-preserving homeomorphism h of V_g such that $\phi_2(x) = h \circ \phi_1(x) \circ h^{-1}$ for all $x \in \mathbf{G}$. A graph of groups $(\Gamma, \mathbf{G})$ is a connected graph Γ consisting of a collection of groups G_v and G_e indexed by the vertices and edges of Γ, respectively, together with a collection of edge-to-vertex monomorphisms $f_e : G_e \longrightarrow G_v$ associated from $\mathbf{G}$. As described in [3], we may construct a handlebody orbifold $V(\Gamma, \mathbf{G})$ using the graph of groups $(\Gamma, \mathbf{G})$, which satisfies a set of normalized conditions that allows for this construction, as a 'core'. Due to the work of Kalliongis and Miller found in [2], if V is the orbifold quotient of V_g under an orientation-preserving finite group action or V is homeomorphic to $V(\Gamma_0, \mathbf{G_0})$ where $(\Gamma_0, \mathbf{G_0})$ satisfies the normalized conditions, then V is homeomorphic to $V(\Gamma, \mathbf{G})$ for a graph of groups $(\Gamma, \mathbf{G})$ in canonical form. Note that $(\Gamma, \mathbf{G})$ is said to be in canonical form provided it has the form shown in Figure 1, where each Γ_k is a subgraph of Γ such that (1) no edges have trivial edge group and (2)

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if e is an edge in Γ_k then e is either a loop or f_e is not surjective. We also require the vertex group G_{v_0} be trivial and each edge not contained in one of the Γ_k's has trivial edge group. This form uniquely determines the homeomorphism type of its corresponding handlebody orbifold.

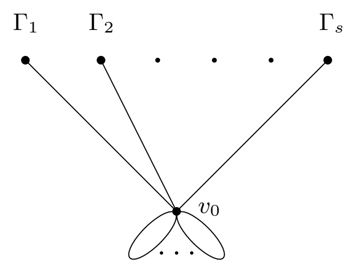

Figure 1: Canonical Form

Define Γ_1 to be the subgraph with one vertex v and one edge e that is a loop such that $G_v = G_e = \mathbb{Z}_{p^2}$ and define Γ_2 to be the subgraph with one vertex v such that $G_v = \mathbb{Z}_{p^2}$. Now repeat this process for Γ_3 and Γ_4 using the group $\mathbb{Z}_p$. Let $\mathrm{v} = (r, s, t, m, n)$ be an ordered 5-tuple of nonnegative integers such that the graph of groups $(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v}))$ in canonical form has r trivial loops, s copies of Γ_1, t copies of Γ_2, m copies of Γ_3, and n copies of Γ_4. (See [4] for the case $p = 2$ and $p^2 = 4$). Then $(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v}))$ satisfies the normalized conditions and determines a handlebody orbifold $V(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v}))$. The orbifold $V(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v}))$ is constructed in a similar manner as described in [2]. Note that the quotient of any $\mathbb{Z}_{p^2}$-action on V_g is an orbifold of this type, up to homeomorphism.

An explicit combinatorial enumeration of orientation-preserving $\mathbb{Z}_4$-actions on V_g, up to equivalence, is given in [4]. In this work we will be interested in examining the orientation-preserving geometric group actions on V_g for the group $\mathbb{Z}_{p^2}$ for p an odd prime. The results obtained here are potentially useful in studying finite group actions on compact 3-manifolds by way of their Heegaard decompositions into the sum of two handlebodies.

2. Combinatorial Argument

The orbifold fundamental group $\pi_1^{orb}(V(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v})))$ is an extension of the free group $\pi_1(V_g)$ by $\mathbf{G} = \mathbb{Z}_{p^2}$ so that the latter group is a normal subgroup of $\pi_1^{orb}(V(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v})))$ with quotient the free group. Using a version of van Kampen's theorem for orbifolds, we see that the fundamental group is the free product $G_1 * G_2 * G_3 * \cdots * G_{r+s+t+m+n}$, where G_i is isomorphic to either $\mathbb{Z}$, $\mathbb{Z}_{p^2} \times \mathbb{Z}$, $\mathbb{Z}_{p^2}$, $\mathbb{Z}_p \times \mathbb{Z}$, or $\mathbb{Z}_p$. We establish notation similar to [4] and denote the generators of the orbifold fundamental group by $\{a_i : 1 \leq i \leq r\} \cup \{b_j, c_j : 1 \leq j \leq s\} \cup \{d_k : 1 \leq k \leq t\} \cup \{e_l, f_l : 1 \leq l \leq m\} \cup \{g_q : 1 \leq q \leq n\}$ such that $b_j^{p^2} = d_k^{p^2} = 1$, $[b_j, c_j] = 1$,

$e_l^p = g_q^p = 1$, and $[e_l, f_l] = 1$.

Consider the set of pairs $((\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})), \lambda)$, where λ is a finite injective epimorphism from $\pi_1^{orb}(V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))$ onto $\mathbb{Z}_{p^2}$. We say λ is finite injective since the kernel of λ is a free group of rank g. We consider only finite injective epimorphisms such that ker(λ)=im(ν_*) for some orbifold covering $\nu : V \longrightarrow V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v}))$. Since V is a handlebody with torsion free fundamental group, V is homeomorphic to a handlebody V_g of genus $g = 1 - p^2\chi(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v}))$. Define an equivalence relation on this set of pairs by setting $((\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})), \lambda) \equiv ((\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})), \lambda')$ if and only if there exists an orbifold homeomorphism $h : V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})) \longrightarrow V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v}))$ such that $\lambda' = \lambda \circ h_*$. We define the set $\Delta(\mathbb{Z}_{p^2}, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))$ to be the set of equivalence classes $[((\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})), \lambda)]$ under this relation.

Denote the set set of equivalence classes $\mathscr{E}(\mathbb{Z}_{p^2}, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))$ to be the set $\{[\phi] \mid \phi : \mathbb{Z}_{p^2} \longrightarrow Homeo^+(V_g) \textit{ and } V_g/\phi \simeq V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v}))\}$. Note that given any $\mathbb{Z}_{p^2}$-action $\phi : \mathbb{Z}_{p^2} \longrightarrow Homeo^+(V_g)$, it must be the case that for some $V(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v}))$, $[\phi] \in \mathscr{E}(\mathbb{Z}_{p^2}, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))$. The following theorem has a similar proof technique as found in [2].

Theorem 2.1. *Let* $\mathbf{v} = (r, s, t, m, n)$. *The set* $\mathscr{E}(\mathbb{Z}_{p^2}, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))$ *is in one-to-one correspondence with the set* $\Delta(\mathbb{Z}_{p^2}, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))$ *for every* $g > 0$.

To prove the three main theorems in this paper (Theorems 3.4, 4.2, and 5.1), we count the number of elements in the delta set and use the one-to-one correspondence given in Theorem 2.1 to give the total count for the set $\mathscr{E}(\mathbb{Z}_{p^2}, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))$ as stated in Corrollary 6.1. To do so, recall that $\lambda(g_i) = y_i$ for $g_i \in \pi_1^{orb}(V(\Gamma(\mathrm{v}), \mathbf{G}(\mathbf{v})))$ and $y_i \in \mathbb{Z}_{p^2}$. In Lemmas 3.3 and 4.1 we will see that we can restrict our count to specific λ's that satisfy an ordering on a subset of the generators (or nongenerators) of $\mathbb{Z}_{p^2}$. We will use the following combinatorial argument along with Lemmas 3.3 and 4.1 to prove the three main theorems.

Let G be a finite group of n elements. Let $S(k) \subseteq G$ be a subset with k elements. Hence $S(k) = \{y_1, y_2, y_3, \ldots, y_k\}$ and $y_i \in G$. Define the set $O(j) \subseteq (S(k))^j$ to be the set of j-tuples ordered on the indices. That is, $O(j) = \{(y_{i_1}, y_{i_2}, y_{i_3}, \ldots, y_{i_j}) \mid i_1 \leq i_2 \leq i_3 \leq \cdots \leq i_j\}$. Now define the set $O(j, \ell)$ to be the set of j-tuples ordered on the indices such that the first index is the fixed quantity $k - (\ell - 1)$. That is, $O(j, \ell) = \{(y_{i_1}, y_{i_2}, y_{i_3}, \ldots, y_{i_j}) \mid i_1 \leq i_2 \leq i_3 \leq \cdots \leq i_j,\ i_1 = k - (\ell - 1)\}$.

Lemma 2.2. $O(j, \ell) \cap O(j', \ell) = \emptyset$ *if and only if* $j \neq j'$.

Now define the count of the set $O(j)$ to be the number $\mathscr{C}(j) = |O(j)|$ and the count of the set $O(j, \ell)$ to be the number $\mathscr{C}(j, \ell) = |O(j, \ell)|$. Then $\mathscr{C}(j) = \sum_{l=1}^{k} \mathscr{C}(j, l)$. Furthermore, the following lemma holds.

Lemma 2.3. $\mathscr{C}(j + 1, l) = \sum_{u=1}^{l} \mathscr{C}(j, u)$.

Theorem 2.4. $\mathscr{C}(j) = A_{k_j}$ *where* $A_{k_1} = k$ *if* $j = 1$, $A_{k_2} = \frac{k(k+1)}{2}$ *if* $j = 2$, *and* $A_{k_j} = \sum_{i=0}^{k-1} \left[\binom{j-3+i}{j-3} \sum_{u=1}^{k-i} u\right]$ *if* $j \geq 3$.

Proof. Let $j = 1$. Then $\mathscr{C}(1) = \sum_{l=1}^{k} \mathscr{C}(1,l) = \sum_{l=1}^{k} 1 = k$. Now let $j = 2$. Then $\mathscr{C}(2,l) = l$ so that $\mathscr{C}(2) = \sum_{l=1}^{k} \mathscr{C}(2,l) = \sum_{l=1}^{k} l = \frac{k(k+1)}{2}$. Finally let $j \geq 3$. Then $\mathscr{C}(j) = \sum_{i=0}^{k-1} \left[\binom{j-3+i}{j-3} \sum_{u=1}^{k-i}\right]$. To see this we will use induction on j. For the base case, let $j = 3$. Then

$$\begin{aligned}\mathscr{C}(3) &= \sum_{l=1}^{k} \mathscr{C}(3,l) = \sum_{l=1}^{k} \sum_{u=1}^{l} \mathscr{C}(2,u) = \sum_{l=1}^{k} \sum_{u=1}^{l} u = \sum_{u=1}^{1} u + \sum_{u=1}^{2} u + \cdots + \sum_{u=1}^{k} u \\ &= \sum_{u=1}^{k} u + \cdots + \sum_{u=1}^{2} u + \sum_{u=1}^{1} u = \sum_{i=0}^{k-1} \left[\binom{i}{0} \sum_{u=1}^{k-i} u\right].\end{aligned}$$

For the inductive step, assume $\mathscr{C}(j) = \sum_{i=0}^{k-1} \left[\binom{j-3+i}{j-3} \sum_{u=1}^{k-i} u\right]$. Then

$$\begin{aligned}\mathscr{C}(j+1) &= \sum_{l=1}^{k} \mathscr{C}(j+1,l) = \mathscr{C}(j+1,1) + \mathscr{C}(j+1,2) + \cdots + \mathscr{C}(j+1,k) \\ &= \sum_{q=1}^{1} \mathscr{C}(j,q) + \sum_{q=1}^{2} \mathscr{C}(j,q) + \cdots + \sum_{q=1}^{k} \mathscr{C}(j,q) = \sum_{i=0}^{0} \left[\binom{j-3+i}{j-3} \sum_{u=1}^{1-i} u\right] \\ &+ \sum_{i=0}^{1} \left[\binom{j-3+i}{j-3} \sum_{u=1}^{2-i} u\right] + \cdots + \sum_{i=0}^{k-1} \left[\binom{j-3+i}{j-3} \sum_{u=1}^{k-i} u\right] \\ &= \left[\binom{j-3}{j-3} + \binom{j-3+1}{j-3} + \binom{j-3+2}{j-3} + \cdots + \binom{j-3+(k-1)}{j-3}\right] \sum_{u=1}^{1} u \\ &+ \left[\binom{j-3}{j-3} + \binom{j-3+1}{j-3} + \cdots + \binom{j-3+(k-2)}{j-3}\right] \sum_{u=1}^{2} + \cdots \\ &= \left[\binom{j-2+(k-1)}{j-2}\right] \sum_{u=1}^{1} u. \qquad \square\end{aligned}$$

Note that we will define $A_{k_0} = 1$ for $j = 0$.

3. The 5-tuple v$= (r, s, t, m, n)$ with $s + t > 0$

We resort to the following lemma to help count the number of elements in the delta set. The proof is an adaptation from [2]. The element α in Lemma 3.1 is an element of an automorphism group of the (orbifold) fundamental group of the graph of groups that is associated with a group action on a handlebody whose fundamental group is free.

Lemma 3.1. *If* $\alpha \in Aut(\pi_1^{orb}(V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v}))))$, *then* $\alpha = h_*$ *for some orientation-*

preserving homeomorphism $h : V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})) \longrightarrow V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v}))$ *if and only if*

$$\begin{aligned}
\alpha(b_j) &= x_j b_{\sigma(j)}^{\varepsilon_j} x_j^{-1},\\
\alpha(c_j) &= x_j b_{\sigma(j)}^{v_j} c_{\sigma(j)}^{\varepsilon_j} x_j^{-1},\\
\alpha(d_k) &= y_k d_{\tau(k)}^{\delta_k} y_k^{-1},\\
\alpha(e_l) &= u_l e_{\gamma(l)}^{\varepsilon'_l} u_l^{-1},\\
\alpha(f_l) &= u_l e_{\gamma(l)}^{w_l} f_{\gamma(l)}^{\varepsilon'_l} u_l^{-1}, \text{and}\\
\alpha(g_q) &= z_q g_{\xi(q)}^{\delta'_q} z_q^{-1},
\end{aligned}$$

for some $x_j, y_k, u_l, z_q \in \pi_1^{orb}(V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))$*;* $\sigma \in \sum_s$*,* $\tau \in \sum_t$*,* $\gamma \in \sum_m$*,* $\xi \in \sum_n$*;* $\varepsilon_j, \delta_k, \varepsilon'_l, \delta'_q \in \{+1, -1\}$*; and* $0 \leq v_j < p^2$*,* $0 \leq w_l < p$*.*
Note that Σ_l *is the permutation group on* l *letters.*

Note that from [1], a generating set for the automorphisms of $\pi_1^{orb}(V(\Gamma(\mathrm{v}, \mathbf{G}(\mathbf{v}))))$ is the set of mappings $\{\rho_{ji}(x), \lambda_{ji}(x), \mu_{ji}(x), \omega_{ij}, \sigma_i, \phi_i\}$ whose definitions may be found in [1]. The first five maps are realizable. The realizable ϕ_i's are of the form found in Lemma 3.1 and will be used in the remaining arguments of this paper.

Lemma 3.2 *Let* $\mathbf{v} = (r, s, t, m, n)$ *for* $s + t > 0$ *and let* $\lambda : \pi_1^{orb}(V(\Gamma(v, \mathbf{G}(\mathbf{v})))) \longrightarrow \mathbb{Z}_{p^2}$ *be a finite injective epimorphism. There exists a finite injective epimorphism* $\lambda' : \pi_1^{orb}(V(\Gamma(v, \mathbf{G}(\mathbf{v})))) \longrightarrow \mathbb{Z}_{p^2}$ *equivalent to* λ *such that* $p \leq \lambda'(e_i) = u_i \leq (\frac{p-1}{2})p$ *with* $gcd(u_i, p^2) = p$ *and* $0 \leq \lambda'(f_i) \leq p - 1$*.*

Proof. Let $\lambda : \pi_1^{orb}(V(\Gamma(\mathrm{v}, \mathbf{G}(\mathbf{v})))) \longrightarrow \mathbb{Z}_{p^2}$ be a finite injective epimorphism such that $\lambda(e_i) = t_i p$ and $\lambda(f_i) = x_i$. For the first case assume that $1 \leq t_i \leq \frac{p-1}{2}$ and $x_i > p - 1$. Then we may write $x_i = s_i + w_i p$ where $0 \leq s_i \leq p - 1$ and $1 \leq w_i \leq p - 1$. Define the realizable automorphism $\prod \phi_i$, where each ϕ_i sends f_i to the element $e_i^{-w_i t_i^{-1}} f_i$ and fixes e_i. (Note that this is a Dehn twist). That is, $(\lambda \circ \phi_i)(f_i) = \lambda(e_i^{-w_i t_i^{-1}} f_i) = -w_i p + (s_i + w_i p) = s_i$ for all i. For the second case assume that $\frac{p-1}{2} < t_i \leq p - 1$ and $0 \leq x_i \leq p - 1$. In this case we may define a realizable automorphism $\prod \phi_i$, where each ϕ_i sends e_i to e_i^{-1} and sends f_i to $e_i^{t_i^{-1}} f_i^{-1}$. We can easily see that $(\lambda \circ \phi_i)(e_i) = -t_i p = (p - t_i)p$ and $1 \leq p - t_i \leq \frac{p-1}{2}$ for all i. Now $(\lambda \circ \phi_i)(f_i) = \lambda(e_i^{t_i^{-1}} f_i^{-1}) = p - x_i$ and $0 \leq p - x_i \leq p - 1$ holds for all i. Finally, assume that $\frac{p-1}{2} < t_i \leq p - 1$ and $x_i > p - 1$. In this case define a realizable automorphism $\prod \phi_i \circ \prod \phi'_i$, where each ϕ'_i sends e_i and f_i to their inverses and ϕ_i sends f_i to $e_i^{w_i t_i^{-1} + t_i^{-1}} f_i$ and fixes e_i. Again, it is easy to verify that $(\lambda \circ \phi_i)(e_i) = (p - t_i)p$ and $1 \leq p - t_i \leq \frac{p-1}{2}$ for all i. Now $(\lambda \circ \phi_i \circ \phi'_i)(f_i) = (\lambda \circ \phi_i)(f_i^{-1}) = \lambda((e_i^{w_i t_i^{-1} + t_i^{-1}} f_i)^{-1}) = -(s_i + w_i p) + w_i p + p = p - s_i$ and $0 \leq p - s_i \leq p - 1$ for all i. □

Lemma 3.3. *Let* $\mathbf{v} = (r, s, t, m, n)$ *for* $s+t > 0$ *and let* $\lambda : \pi_1^{orb}(V(\Gamma(v, \mathbf{G}(\mathbf{v})))) \longrightarrow \mathbb{Z}_{p^2}$ *be a finite injective epimorphism. There exists a finite injective epimorphism* $\lambda' : \pi_1^{orb}(V(\Gamma(v, \mathbf{G}(\mathbf{v})))) \longrightarrow \mathbb{Z}_{p^2}$ *equivalent to* λ *such that the following hold:*

(1) $\lambda'(a_1) = \cdots = \lambda'(a_r) = 0$.

(2) $1 \le \lambda'(b_1) = x_1 \le \lambda'(b_2) = x_2 \le \cdots \le \lambda'(b_s) = x_s \le \frac{p^2-1}{2}$ *and* $gcd(x_i, p^2) = 1$ *for* $1 \le i \le s$.

(3) $\lambda'(c_1) = \cdots = \lambda'(c_s) = 0$.

(4) $1 \le \lambda'(d_1) = y_1 \le \lambda'(d_2) = y_2 \le \cdots \le \lambda'(d_t) = y_t \le \frac{p^2-1}{2}$ *and* $gcd(y_j, p^2) = 1$ *for* $1 \le j \le t$.

(5) $p \le \lambda'(e_l) = u_l \le (\frac{p-1}{2})p$ *and* $gcd(u_l, p^2) = p$ *for all* $1 \le l \le m$.

(6) $0 \le \lambda'(f_l) \le p - 1$ *for all* $1 \le l \le m$.

(7) $p \le \lambda'(g_1) = z_1 \le \lambda'(g_2) = z_2 \le \cdots \le \lambda'(g_n) = z_n \le (\frac{p-1}{2})p$ *and* $gcd(z_k, p^2) = p$ *for* $1 \le k \le n$.

Proof. Let $\lambda : \pi_1^{orb}(V(\Gamma(\mathrm{v}, \mathbf{G}(\mathbf{v})))) \longrightarrow \mathbb{Z}_{p^2}$ be a finite injective epimorphism. Without loss of generality, assume that $s > 0$. Then there exists an element k_i such that $k_i\lambda(b_{r+1}) = \lambda(a_i)$ for each generator a_i. Note that if $s = 0$, choose d_{r+1}. Property (1) follows by composing λ with the realizable automorphism $\prod \lambda_{(r+1)i}((\lambda(b_{r+1}))^{k_i})$. (Note that this is a handle slide). That is, we have $(\lambda \circ \lambda_{(r+1)i}((\lambda(b_{r+1}))^{k_i}))(a_i) = \lambda((b_{r+1})^{-k_i}a_i) = -k_i\lambda(b_{r+1}) + \lambda(a_i) = -\lambda(a_i) + \lambda(a_i) = 0$ for all $1 \le i \le r$. Similarly, there exists an element ℓ_i such that $\ell_i\lambda(b_i) = \lambda(c_i)$ for all c_i since λ is finite injective. Property (3) follows by composing λ with the realizable automorphism $\prod \phi_i$, where each ϕ_i sends the generator c_i to the element $b_i^{-\ell_i}c_i$ and fixes b_i. (Note that this is a Dehn twist). That is, $(\lambda\circ\phi_i)(c_i) = \lambda(b_i^{-\ell_i}c_i) = -\ell_i\lambda(b_i)+\lambda(c_i) = -\lambda(c_i)+\lambda(c_i) = 0$ for all $1 \le i \le s$. Now cut the set $\mathbb{Z}_{p^2}-\{0\}$ in half to get the two sets $\{1, 2, \ldots, \frac{p^2-1}{2}\}$ and $\{\frac{p^2+1}{2}, \ldots, p^2-1\}$. Notice that each element in the first set is the inverse of an element in the second set. Property (2) follows by composing λ with the realizable automorphism $\prod \phi_i$, where ϕ_i sends the generator b_i to b_i^{-1} provided $\lambda(b_i) \in \{\frac{p^2+1}{2}, \ldots, p^2 - 1\}$. (Note that this is a spin). We may then compose with the realizable automorphism $\prod \omega_{ij}$, which interchanges handles if necessary. A similar argument shows Properties (4) and (7). Clearly Properties (5) and (6) follow from Lemma 3.2. □

Theorem 3.4. *Let* $\mathbf{v}= (r, s, t, m, n)$ *with* $s + t > 0$. *If* $\mathbb{Z}_{p^2}$ *acts on* V_g, *then* $g - 1 = p^2 (r + s + m - 1) + (p^2 - 1) t + (p^2 - p) n$. *The number of equivalence classes of* $\mathbb{Z}_{p^2}$*-actions on* V_g *such that* $V_g/\mathbb{Z}_{p^2} = V(\Gamma(\mathbf{v}), G(\mathbf{v}))$ *is the product* $A_{(\frac{p(p-1)}{2})_s} \cdot A_{(\frac{p(p-1)}{2})_t} \cdot A_{(\frac{p(p-1)}{2})_m} \cdot A_{(\frac{p-1}{2})_n}$.

Proof. Applying Theorem 2.1, and noting that the orbifold fundamental group is a free product, we see that the count of the delta set is the product of the count of distinct mappings λ satisfying the result of Lemma 3.3. Due to this, we only need to consider the generators b_i, d_i, e_i, f_i, and g_i. We will first count the number of distinct mappings restricted to the generator b_i. To do so, note that there are $\frac{p(p-1)}{2}$ generators of $\mathbb{Z}_{p^2}$ in the set $\{1, 2, \ldots, \frac{p^2-1}{2}\}$. Since $\lambda(b_i)$ are ordered, we may think of them as s-tuples in the set $O(\frac{p(p-1)}{2})$ from Section 2. Applying Theorem 2.4, we see that there are $A_{(\frac{p(p-1)}{2})_s}$ distinct λ's. A similar argument works for the generator d_i giving us $A_{(\frac{p(p-1)}{2})_s}$ distinct λ's. We will now count the number of distinct mappings restricted to the generator g_i. To do so, we may note that there are only $\frac{p-1}{2}$ multiples of p that satisfy Lemma 3.3. Again, since $\lambda(g_i)$ are ordered, we may think of them as n-tuples in the set $O(\frac{p-1}{2})$ from Section 2. Applying Theorem 2.4, we see that there are $A_{(\frac{p-1}{2})_n}$ distinct λ's. Now to count the number of distinct mappings for the generators e_i and f_i, we may think of them as the ordered pair (e_i, f_i). When $m = 1$, there are $\frac{p(p-1)}{2}$ distinct order pairs that satisfy Properties (5) and (6) from Lemma 3.3. We may relabel these ordered pairs by y_i and create an m-tuple in the set $O(\frac{p(p-1)}{2})$. Note that there exists a realizable action when doing this. Again, applying Theorem 2.4, we see that there are $A_{(\frac{p(p-1)}{2})_m}$ distinct λ's, proving the theorem. □

4. The 5-tuple v$= (r, 0, 0, m, n)$ with $r > 0$

We will now consider the 5-tuple v$= (r, 0, 0, m, n)$ with $s + t = 0$ and $r > 0$. For the following lemma, we will need to account for two cases: (1) there exists at least one generator f_i that is mapped to a generator of $\mathbb{Z}_{p^2}$ and (2) otherwise. Note that these maps are not equivalent. (This is a modification of Lemma 2.3 from [4]).

Lemma 4.1. *Let* $\mathbf{v} = (r, 0, 0, m, n)$ *for* $r > 0$ *and let* $\lambda : \pi_1^{orb}(V(\Gamma(v, \mathbf{G}(\mathbf{v})))) \longrightarrow \mathbb{Z}_{p^2}$ *be a finite injective epimorphism. There exists a finite injective epimorphism* $\lambda' : \pi_1^{orb}(V(\Gamma(v, \mathbf{G}(\mathbf{v})))) \longrightarrow \mathbb{Z}_{p^2}$ *equivalent to* λ *such that the following hold:*

(1) $\lambda'(a_1) = \cdots = \lambda'(a_r) = 0$.

(2) $p \leq \lambda'(e_l) = u_l \leq (\frac{p-1}{2})p$ *and* $gcd(u_l, p^2) = p$ *for all* $1 \leq l \leq m$.

(3) $1 \leq \lambda'(f_1) = y_1 \leq p - 1$.

(4) $0 \leq \lambda'(f_j) \leq p - 1$ *for all* $2 \leq j \leq m$.

(5) $p \leq \lambda'(g_1) = z_1 \leq \lambda'(g_2) = z_2 \leq \cdots \leq \lambda'(g_n) = z_n \leq (\frac{p-1}{2})p$ *and* $gcd(z_k, p^2) = p$ *for* $1 \leq k \leq n$.

OR

(1) $1 \leq \lambda'(a_1) = x_1 \leq \frac{p(p-1)}{2}$ *and* $gcd(x_1, p^2) = 1$.

(2) $\lambda'(a_2) = \cdots = \lambda'(a_r) = 0$.

(3) $p \leq \lambda'(e_l) = u_l \leq (\frac{p-1}{2})p$ *and* $gcd(u_l, p^2) = p$ *for all* $1 \leq l \leq m$.

(4) $\lambda'(f_1) = \cdots = \lambda'(f_m) = 0$.

(5) $p \leq \lambda'(g_1) = z_1 \leq \lambda'(g_2) = z_2 \leq \cdots \leq \lambda'(g_n) = z_n \leq (\frac{p-1}{2})p$ *and* $gcd(z_k, p^2) = p$ *for* $1 \leq k \leq n$.

Note that the first set of properties hold when there exists at least one generator f_i that is mapped to a generator of $\mathbb{Z}_{p^2}$. To show Properties (1)-(5) in both cases, we use similar techniques found in the proof of Lemma 3.3. That is, handle slides, spins, interchanging handles, and Dehn twists. From this we get the following theorem.

Theorem 4.2. *Let* $\mathbf{v}= (r, 0, 0, m, n)$ *with* $r > 0$. *If* $\mathbb{Z}_{p^2}$ *acts on* V_g, *then* $g - 1 = p^2 (r + m - 1) + \left(p^2 - p\right) n$. *The number of equivalence classes of* $\mathbb{Z}_{p^2}$*-actions on* V_g *such that* $V_g/\mathbb{Z}_{p^2} = V(\Gamma(\mathbf{v}), G(\mathbf{v}))$ *is the sum* $\frac{(p-1)^2}{2} \cdot A_{(\frac{p(p-1)}{2})_{m-1}} \cdot A_{(\frac{p-1}{2})_n} + \frac{p(p-1)}{2} \cdot A_{(\frac{p-1}{2})_m} \cdot A_{(\frac{p-1}{2})_n}$.

Proof. The first portion of the sum follows from the case there exists at least one generator f_i that is mapped to a generator of $\mathbb{Z}_{p^2}$. In this case we can again think of the ordered pair (e_i, f_i) as y_i exactly as we did in Theorem 3.4. We may note that there are $\frac{(p-1)^2}{2}$ possibilities for y_1. The count of the remaining $m - 1$ slots follows from Theorem 3.4. (Similarly for counting g_i). The second portion of the sum involves counting a_1. However, we can see that there are $\frac{p(p-1)}{2}$ possibilities. The remaining values follow from Theorem 3.4. □

5. The 5-tuple $\mathbf{v}= (0, 0, 0, m, n)$ with $m > 0$

Finally we will consider the 5-tuple v$= (0, 0, 0, m, n)$ with $r + s + t = 0$ and $m > 0$. Given a finite injective epimorphism λ, it is clear that we may obtain an equivalent λ' that satisfies Properties (2)-(5) of the first case of Lemma 4.1. The following theorem is a modification of Theorem 4.2.

Theorem 5.1. *Let* $\mathbf{v}= (0, 0, 0, m, n)$ *with* $m > 0$. *If* $\mathbb{Z}_{p^2}$ *acts on* V_g, *then* $g - 1 = p^2 (m - 1) + \left(p^2 - p\right) n$. *The number of equivalence classes of* $\mathbb{Z}_{p^2}$*-actions on* V_g *such that* $V_g/\mathbb{Z}_{p^2} = V(\Gamma(\mathbf{v}), G(\mathbf{v}))$ *is the product* $\frac{(p-1)^2}{2} \cdot A_{(\frac{p(p-1)}{2})_{m-1}} \cdot A_{(\frac{p-1}{2})_n}$.

6. The Number of Equivalence Classes of $\mathbb{Z}_{25}$-actions on V_{26}

Corollary 6.1 *Let* p *be a fixed odd prime and* g *a fixed natural number. Then the order of the set* $\mathscr{E}(\mathbb{Z}_{p^2}, V_g, V(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v})))$ *is the count for all 5-tuples* $\mathbf{v}= (r, s, t, m, n)$ *that satisfy the equation* $g = 1 - p^2\chi(\Gamma(\mathbf{v}), \mathbf{G}(\mathbf{v}))$.

To see this, fix $p = 5$ and $g = 26$. Now g must satisfy the genus equation $g = 1 - 25\chi(\Gamma, G)$. Therefore we see that $50 = 24t + 20n + 25(r + s + m)$. Solving this equation we see that $t = 0$, $n = 0$, and $r + s + m = 2$. This leads to the following ordered 5-tuples: (0,2,0,0,0), (2,0,0,0,0), (0,0,0,2,0), (1,1,0,0,0), (1,0,0,1,0), and (0,1,0,1,0). Using Theorems 3.4, 4.2, and 5.1, the counts for the following ordered 5-tuples are 55, 10, 55, 10, 18, and 100, respectively. Thus the total number of equivalence classes of $\mathbb{Z}_{25}$-actions on V_{26} is 55+10+55+10+18+100=248.

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