KYUNGPOOK Math. J. 58(2018), 559-571
https://doi.org/10.5666/KMJ.2018.58.3.559
pISSN 1225-6951 eISSN 0454-8124
© Kyungpook Mathematical Journal

Stability and Constant Boundary-Value Problems of f-Harmonic Maps with Potential

Bouazza Kacimi and Ahmed Mohammed Cherif*
Department of Mathematics, University Mustapha Stambouli, Mascara, 29000, Algeria
e-mail : kacimibouazza@yahoo.fr *and* med_cherif_ahmed@yahoo.fr

Abstract. In this paper, we give some results on the stability of f-harmonic maps with potential from or into spheres and any Riemannian manifold. We study the constant boundary-value problems of such maps defined on a specific Cartan-Hadamard manifolds, and obtain a Liouville-type theorem. It can also be applied to the static Landau-Lifshitz equations. We also prove a Liouville theorem for f-harmonic maps with finite f-energy or slowly divergent f-energy.

1. Preliminaries and Notations

We give some definitions.
(1) Let (M, g) be a Riemannian manifold. The divergence of $(0, p)$-tensor α on M is defined by

$$(\operatorname{div}^M \alpha)(X_1, ..., X_{p-1}) = (\nabla^M_{e_i} \alpha)(e_i, X_1, ..., X_{p-1}), \tag{1.1}$$

where ∇^M is the Levi-Civita connection with respect to g, $X_1, ..., X_{p-1} \in \Gamma(TM)$, and $\{e_i\}$ is an orthonormal frame. Given a smooth function λ on M, the gradient of λ is defined by

$$g(\operatorname{grad}^M \lambda, X) = X(\lambda), \tag{1.2}$$

the Hessian of λ is defined by

$$(\operatorname{Hess}^M \lambda)(X, Y) = g(\nabla^M_X \operatorname{grad} \lambda, Y), \tag{1.3}$$

where $X, Y \in \Gamma(TM)$, the Laplacian of λ is defined by

$$\Delta^M(\lambda) = \operatorname{trace}_g \operatorname{Hess}^M \lambda, \tag{1.4}$$

* Corresponding Author.
Received October 3, 2017; accepted July 18, 2018.
2010 Mathematics Subject Classification: 58E20, 53C21.
Key words and phrases: f-harmonic maps with potential, stability, boundary-value.
This work was supported by National Agency Scientific Research of Algeria.

(see [11]).
(2) Let $\varphi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds, $\tau(\varphi)$ the tension field of φ (see [1, 2, 6]), f a smooth positive function on M, and let H be a smooth function on N, the (f, H)-tension field of φ is given by

$$\tau_{f,H}(\varphi) = f\tau(\varphi) + d\varphi(\operatorname{grad}^M f) + (\operatorname{grad}^N H) \circ \varphi, \tag{1.5}$$

where grad^M (resp. grad^N) denotes the gradient operator with respect to g (resp. h). Then φ is called f-harmonic with potential H if the (f, H)-tension field vanishes, i.e. $\tau_{f,H}(\varphi) = 0$ (for more details on the concept of f-harmonic maps with potential H see [7]). The notion of f-harmonic with potential H is a generalization of harmonic maps with potential H if $f \equiv 1$, f-harmonic maps if $H = 0$ and the usual harmonic maps if $f \equiv 1$ and $H = 0$. We define the index form for f-harmonic maps with potential H by

$$I^{\varphi}_{f,H}(v, w) = \int_M h(J^{\varphi}_{f,H}(v), w)v^M, \tag{1.6}$$

for all $v, w \in \Gamma(\varphi^{-1}TN)$, where

$$\begin{aligned} J^{\varphi}_{f,H}(v) &= -f \operatorname{trace}_g R^N(v, d\varphi)d\varphi - \operatorname{trace}_g \nabla^{\varphi} f \nabla^{\varphi} v \\ &\quad -(\nabla^N_v \operatorname{grad}^N H) \circ \varphi, \end{aligned} \tag{1.7}$$

R^N is the curvature tensor of (N, h), ∇^N is the Levi-Civita connection of (N, h), ∇^{φ} denote the pull-back connection on $\varphi^{-1}TN$, and v^M is the volume form of (M, g) (see [1, 11]). If φ be a f-harmonic map with potential H and for any vector field v along φ, the index form satisfies $I^{\varphi}_{f,H}(v, v) \geq 0$, then φ is called a stable f-harmonic map with potential H. Note that, the definition of stable f-harmonic maps with potential H is a generalization of stable harmonic maps if $f = 1$ on M and $H = 0$ on N (see [4, 16]).

For the smooth map $\varphi : (M, g) \to (N, h)$, S. Ouakkas et al. introduced in [12] the f-stress energy tensor S_f of φ associated to the f-energy functional

$$E_f(\varphi) = \int_M e_f(\varphi)v^g, \tag{1.8}$$

is given by

$$S_f(\varphi) = e_f(\varphi)g - f\varphi^* h,$$

where $e_f(\varphi) = \frac{1}{2}f|d\varphi|^2$ is the f-energy density of φ. For any vector field X on M (see [12]), we have

$$\operatorname{div}^M S_f(\varphi)(X) = -h(\tau_f(\varphi), d\varphi(X)) + \frac{1}{2}X(f)|d\varphi|^2, \tag{1.9}$$

where $\tau_f(\varphi) = f\tau(\varphi) + d\varphi(\operatorname{grad}^M f)$. If φ is a f-harmonic map with potential H, it follows that

$$\operatorname{div}^M S_f(\varphi)(X) = h((\operatorname{grad}^N H) \circ \varphi, d\varphi(X)) + \frac{1}{2}X(f)|d\varphi|^2. \tag{1.10}$$

2. Stable f-harmonic Maps with Potential on Sphere

Theorem 2.1. *Let φ be a stable f-harmonic map with potential H from sphere $(\mathbb{S}^n, g)$ $(n > 2)$ to Riemannian manifold (N, h), where f is a smooth positive function on $\mathbb{S}^n$ satisfying* $\mathrm{trace}_g\, h((\nabla d\varphi)(\cdot, \mathrm{grad}^{\mathbb{S}^n} f), d\varphi(\cdot)) \geq 0$, *and H is a smooth function on N. Then, φ is constant.*

Proof. Choose a normal orthonormal frame $\{e_i\}$ at point x_0 in $\mathbb{S}^n$. Set

$$\lambda(x) = <\alpha, x>_{\mathbb{R}^{n+1}},$$

for all $x \in \mathbb{S}^n$, where $\alpha \in \mathbb{R}^{n+1}$ and let $v = \mathrm{grad}^{\mathbb{S}^n} \lambda$. Note that

$$v = <\alpha, e_i>_{\mathbb{R}^{n+1}} e_i,\ \nabla^{\mathbb{S}^n}_X v = -\lambda X, \text{ for all } X \in \Gamma(T\mathbb{S}^n),$$

$$\mathrm{trace}_g(\nabla^{\mathbb{S}^n})^2 v = \nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_{e_i} v - \nabla^{\mathbb{S}^n}_{\nabla^{\mathbb{S}^n}_{e_i} e_i} v = -v,$$

where $\nabla^{\mathbb{S}^n}$ is the Levi-Civita connection on $\mathbb{S}^n$ with respect to the standard metric g of the sphere (see [16]). At point x_0, we have

$$\nabla^{\varphi}_{e_i} f \nabla^{\varphi}_{e_i} d\varphi(v) = \nabla^{\varphi}_{\mathrm{grad}^{\mathbb{S}^n} f} d\varphi(v) + f \nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} d\varphi(v), \tag{2.1}$$

the first term of (2.1) is given by

$$\begin{aligned} \nabla^{\varphi}_{\mathrm{grad}^{\mathbb{S}^n} f} d\varphi(v) &= \nabla^{\varphi}_v d\varphi(\mathrm{grad}^{\mathbb{S}^n} f) + d\varphi([\mathrm{grad}^{\mathbb{S}^n} f, v]) \\ &= \nabla^{\varphi}_v d\varphi(\mathrm{grad}^{\mathbb{S}^n} f) + d\varphi(\nabla^{\mathbb{S}^n}_{\mathrm{grad}^{\mathbb{S}^n} f} v) \\ &\quad - d\varphi(\nabla^{\mathbb{S}^n}_v \mathrm{grad}^{\mathbb{S}^n} f), \end{aligned} \tag{2.2}$$

the seconde term of (2.1) is given by

$$\begin{aligned} f \nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} d\varphi(v) &= f \nabla^{\varphi}_{e_i} \nabla^{\varphi}_v d\varphi(e_i) + f \nabla^{\varphi}_{e_i} d\varphi([e_i, v]) \\ &= f R^N(d\varphi(e_i), d\varphi(v)) d\varphi(e_i) + f \nabla^{\varphi}_v \nabla^{\varphi}_{e_i} d\varphi(e_i) \\ &\quad + f d\varphi([e_i, [e_i, v]]) + 2 f \nabla^{\varphi}_{[e_i, v]} d\varphi(e_i), \end{aligned} \tag{2.3}$$

from the definition of tension field, we get

$$\begin{aligned} f \nabla^{\varphi}_{e_i} \nabla^{\varphi}_{e_i} d\varphi(v) &= -f R^N(d\varphi(v), d\varphi(e_i)) d\varphi(e_i) + f \nabla^{\varphi}_v \tau(\varphi) \\ &\quad + f \nabla^{\varphi}_v d\varphi(\nabla^{\mathbb{S}^n}_{e_i} e_i) + f d\varphi(\nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_{e_i} v) \\ &\quad - f d\varphi(\nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_v e_i) + 2 f \nabla^{\varphi}_{[e_i, v]} d\varphi(e_i) \\ &= -f R^N(d\varphi(v), d\varphi(e_i)) d\varphi(e_i) + \nabla^{\varphi}_v f \tau(\varphi) - v(f) \tau(\varphi) \\ &\quad + f \nabla^{\varphi}_v d\varphi(\nabla^{\mathbb{S}^n}_{e_i} e_i) + f d\varphi(\nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_{e_i} v) \\ &\quad - f d\varphi(\nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_v e_i) + 2 f \nabla^{\varphi}_{[e_i, v]} d\varphi(e_i), \end{aligned} \tag{2.4}$$

by equations (2.1), (2.2), (2.4), and the f-harmonicity with potential H condition of φ, we have

$$\begin{aligned} \nabla^{\varphi}_{e_i} f \nabla^{\varphi}_{e_i} d\varphi(v) &= d\varphi(\nabla^{\mathbb{S}^n}_{\operatorname{grad}^{\mathbb{S}^n} f} v) - d\varphi(\nabla^{\mathbb{S}^n}_v \operatorname{grad}^{\mathbb{S}^n} f) \\ &\quad - f R^N(d\varphi(v), d\varphi(e_i)) d\varphi(e_i) \\ &\quad - \nabla^{\varphi}_v (\operatorname{grad}^N H) \circ \varphi - v(f)\tau(\varphi) \\ &\quad + f d\varphi(\nabla^{\mathbb{S}^n}_v \nabla^{\mathbb{S}^n}_{e_i} e_i) + f d\varphi(\nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_{e_i} v) \\ &\quad - f d\varphi(\nabla^{\mathbb{S}^n}_{e_i} \nabla^{\mathbb{S}^n}_v e_i) + 2f \nabla^{\varphi}_{\nabla^{\mathbb{S}^n}_{e_i} v} d\varphi(e_i), \end{aligned} \tag{2.5}$$

by the definition of Ricci tensor, we get

$$\begin{aligned} \nabla^{\varphi}_{e_i} f \nabla^{\varphi}_{e_i} d\varphi(v) &= d\varphi(\nabla^{\mathbb{S}^n}_{\operatorname{grad}^{\mathbb{S}^n} f} v) - d\varphi(\nabla^{\mathbb{S}^n}_v \operatorname{grad}^{\mathbb{S}^n} f) \\ &\quad - f R^N(d\varphi(v), d\varphi(e_i)) d\varphi(e_i) - \nabla^{\varphi}_v (\operatorname{grad}^N H) \circ \varphi \\ &\quad - v(f)\tau(\varphi) + f d\varphi(\operatorname{Ricci}^{\mathbb{S}^n} v) + f d\varphi(\operatorname{trace}(\nabla^{\mathbb{S}^n})^2 v) \\ &\quad + 2f \nabla^{\varphi}_{\nabla^{\mathbb{S}^n}_{e_i} v} d\varphi(e_i), \end{aligned} \tag{2.6}$$

from the property $\nabla^{\mathbb{S}^n}_X v = -\lambda X$, we obtain

$$\begin{aligned} \nabla^{\varphi}_{e_i} f \nabla^{\varphi}_{e_i} d\varphi(v) &= -\lambda d\varphi(\operatorname{grad}^{\mathbb{S}^n} f) - d\varphi(\nabla^{\mathbb{S}^n}_v \operatorname{grad}^{\mathbb{S}^n} f) \\ &\quad - f R^N(d\varphi(v), d\varphi(e_i)) d\varphi(e_i) \\ &\quad - \nabla^{\varphi}_v (\operatorname{grad}^N H) \circ \varphi \\ &\quad - v(f)\tau(\varphi) + f d\varphi(\operatorname{Ricci}^{\mathbb{S}^n} v) \\ &\quad + f d\varphi(\operatorname{trace}(\nabla^{\mathbb{S}^n})^2 v) - 2\lambda f \tau(\varphi). \end{aligned} \tag{2.7}$$

From the definition of Jacobi operator (1.7) and equation (2.7) we have

$$\begin{aligned} J^f_{\varphi}(d\varphi(v)) &= \lambda d\varphi(\operatorname{grad}^{\mathbb{S}^n} f) + d\varphi(\nabla^{\mathbb{S}^n}_v \operatorname{grad}^{\mathbb{S}^n} f) + v(f)\tau(\varphi) \\ &\quad - f d\varphi(\operatorname{Ricci}^{\mathbb{S}^n} v) - f d\varphi(\operatorname{trace}(\nabla^{\mathbb{S}^n})^2 v) + 2\lambda f \tau(\varphi), \end{aligned} \tag{2.8}$$

since $\operatorname{trace}_g(\nabla^{\mathbb{S}^n})^2 v = -v$ and $\operatorname{Ricci}^{\mathbb{S}^n} v = (n-1)v$ (see [1, 16]), we conclude

$$\begin{aligned} h(J^f_{\varphi}(d\varphi(v)), d\varphi(v)) &= \lambda h(d\varphi(\operatorname{grad}^{\mathbb{S}^n} f), d\varphi(v)) \\ &\quad + h(d\varphi(\nabla^{\mathbb{S}^n}_v \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(v)) \\ &\quad + v(f) h(\tau(\varphi), d\varphi(v)) \\ &\quad - (n-2) f h(d\varphi(v), d\varphi(v)) \\ &\quad + 2\lambda f h(\tau(\varphi), d\varphi(v)), \end{aligned} \tag{2.9}$$

by (2.9) and the f-harmonicity with potential H condition of φ, it follows that

$$\begin{aligned} \operatorname{trace}_\alpha h(J^f_{\varphi}(d\varphi(v)), d\varphi(v)) &= h(d\varphi(\nabla^{\mathbb{S}^n}_{e_j} \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) \\ &\quad + h(\tau(\varphi), d\varphi(\operatorname{grad}^{\mathbb{S}^n} f)) - (n-2) f |d\varphi|^2, \end{aligned} \tag{2.10}$$

note that

$$\begin{aligned} h(\tau(\varphi), d\varphi(\operatorname{grad}^{\mathbb{S}^n} f)) &= h(\nabla^{\varphi}_{e_i} d\varphi(e_i), d\varphi(\operatorname{grad}^{\mathbb{S}^n} f)) \\ &= \operatorname{div}^{\mathbb{S}^n} \eta - h(d\varphi(e_i), \nabla^{\varphi}_{e_i} d\varphi(\operatorname{grad}^{\mathbb{S}^n} f)), \end{aligned}$$

with $\eta(X) = h(d\varphi(X), d\varphi(\operatorname{grad}^{\mathbb{S}^n} f))$, $\forall X \in \Gamma(T\mathbb{S}^n)$. We obtain

$$\begin{aligned} \operatorname{trace}_\alpha h(J^f_\varphi(d\varphi(v)), d\varphi(v)) &= -h((\nabla d\varphi)(e_j, \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) \\ &\quad + \operatorname{div}^{\mathbb{S}^n} \eta - (n-2) f |d\varphi|^2, \end{aligned} \tag{2.11}$$

since $h((\nabla d\varphi)(e_j, \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) \geq 0$, from the stable f-harmonic with potential H condition, and equation (2.11), we get

$$\begin{aligned} 0 \leq \operatorname{trace}_\alpha I^\varphi_f(d\varphi(v), d\varphi(v)) &+ \int_{\mathbb{S}^n} h((\nabla d\varphi)(e_j, \operatorname{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) v^{\mathbb{S}^n} \\ &= -(n-2) \int_{\mathbb{S}^n} f |d\varphi|^2 v^{\mathbb{S}^n} \leq 0. \end{aligned}$$

Consequently, $|d\varphi| = 0$, that is φ is constant, because $n > 2$. □

If $f = 1$ on $\mathbb{S}^n$, we get the following result:

Corollary 2.2.([14]) *Any stable harmonic map φ from sphere $\mathbb{S}^n$ $(n > 2)$ to Riemannian manifold (N, h) is constant.*

Corollary 2.3.([3]) *Any stable harmonic map with potential from sphere $\mathbb{S}^n$ $(n > 2)$ to Riemannian manifold (N, h) is constant.*

Using the similar technique we have:

Theorem 2.4. *Let (M, g) be a compact Riemannian manifold, and $\varphi : M \to \mathbb{S}^n$ a stable f-harmonic map with potential H, where f is a smooth positive function on M, and H is a smooth function on $\mathbb{S}^n$ satisfying $(\Delta^{\mathbb{S}^n} H) \circ \varphi \geq 0$. Then, φ is constant.*

Proof. Choose a normal orthonormal frame $\{e_i\}$ at point x_0 in M. When the same data of previous proof, we have

$$\nabla^\varphi_{e_i} f \nabla^\varphi_{e_i}(v \circ \varphi) = \nabla^\varphi_{\operatorname{grad}^M f}(v \circ \varphi) + f \nabla^\varphi_{e_i} \nabla^\varphi_{e_i}(v \circ \varphi), \tag{2.12}$$

the first term of (2.12) is given by

$$\nabla^\varphi_{\operatorname{grad}^M f}(v \circ \varphi) = -(\lambda \circ \varphi) d\varphi(\operatorname{grad}^M f), \tag{2.13}$$

the seconde term of (2.12) is given by

$$\begin{aligned} f \nabla^\varphi_{e_i} \nabla^\varphi_{e_i}(v \circ \varphi) &= -f \nabla^\varphi_{e_i}(\lambda \circ \varphi) d\varphi(e_i) \\ &= -f d\varphi(\operatorname{grad}^M(\lambda \circ \varphi)) - (\lambda \circ \varphi) f \tau(\varphi), \end{aligned} \tag{2.14}$$

by the definition of gradient operator, we get

$$-fd\varphi(\mathrm{grad}^M(\lambda\circ\varphi)) = -f < d\varphi(e_i), v\circ\varphi > d\varphi(e_i), \tag{2.15}$$

substituting the formulas (2.13), (2.14), (2.15) into (2.12) gives

$$\begin{aligned}\nabla^\varphi_{e_i} f\nabla^\varphi_{e_i}(v\circ\varphi) &= -(\lambda\circ\varphi)d\varphi(\mathrm{grad}^M f) - f < d\varphi(e_i), v\circ\varphi > d\varphi(e_i) \\ &\quad -(\lambda\circ\varphi)f\tau(\varphi),\end{aligned} \tag{2.16}$$

from the f-harmonicity with potential H condition of φ, and equation (2.16), we have

$$\begin{aligned}< \nabla^\varphi_{e_i} f\nabla^\varphi_{e_i}(v\circ\varphi), v\circ\varphi > &= -f < d\varphi(e_i), v\circ\varphi >< d\varphi(e_i), v\circ\varphi > \\ &\quad +(\lambda\circ\varphi) < (\mathrm{grad}^{\mathbb{S}^n} H)\circ\varphi, v\circ\varphi >,\end{aligned} \tag{2.17}$$

since the sphere $\mathbb{S}^n$ has constant curvature, we obtain

$$\begin{aligned}< fR^{\mathbb{S}^n}(v\circ\varphi, d\varphi(e_i))d\varphi(e_i), v\circ\varphi > &= f|d\varphi|^2 < v\circ\varphi, v\circ\varphi > \\ &\quad -f < d\varphi(e_i), v\circ\varphi >< d\varphi(e_i), v\circ\varphi >,\end{aligned} \tag{2.18}$$

by the definition of Jacobi operator and equations (2.17), (2.18), we get

$$\begin{aligned}< J^f_\varphi(v\circ\varphi), v\circ\varphi > &= 2f < d\varphi(e_i), v\circ\varphi >< d\varphi(e_i), v\circ\varphi > \\ &\quad -f|d\varphi|^2 < v\circ\varphi, v\circ\varphi > \\ &\quad -(\lambda\circ\varphi) < (\mathrm{grad}^{\mathbb{S}^n} H)\circ\varphi, v\circ\varphi > \\ &\quad - < (\nabla^{\mathbb{S}^n}_{v\circ\varphi}\mathrm{grad}^{\mathbb{S}^n} H)\circ\varphi, v\circ\varphi >,\end{aligned}$$

so that

$$\mathrm{trace}_\alpha < J^\varphi_{f,H}(v\circ\varphi), v\circ\varphi > = (2-n)f|d\varphi|^2 - (\Delta^{\mathbb{S}^n} H)\circ\varphi, \tag{2.19}$$

and then

$$\mathrm{trace}_\alpha I^\varphi_{f,H}(v\circ\varphi, v\circ\varphi) = (2-n)\int_M f|d\varphi|^2 v^M - \int_M [(\Delta^{\mathbb{S}^n} H)\circ\varphi]v^M \tag{2.20}$$

Hence Theorem 2.4 follows from (2.20) and the stable f-harmonicity with potential H condition of φ with $n > 2$ and $(\Delta^{\mathbb{S}^n} H)\circ\varphi \geq 0$. □

From Theorem 2.4, we deduce:

Corollary 2.5.([14]) *Let (M, g) be a compact Riemannian manifold. When $n > 2$, any stable harmonic map $\varphi : M \to \mathbb{S}^n$ must be constant.*

Corollary 2.6.([5]) *Let (M, g) be a compact Riemannian manifold. When $n > 2$, any stable f-harmonic map $\varphi : M \to \mathbb{S}^n$ must be constant, where f is a smooth* *positive function on M.*

3. Liouville Theorems

Let $\varphi : (M, g) \to (N, h)$ be a smooth map. For any fixed $x_0 \in M$, by $r(x)$ we denote the distance function from x_0 to x, and by $B_R(x_0)$ the geodesic ball with radius R and center x_0. We say that the f-energy of φ is divergent slowly if there exists a positive function $\psi(t)$ with $\int_{R_0}^{\infty} \frac{dt}{t\psi(t)} = \infty$ $(R_0 > 0)$, such that

$$\lim_{R\to\infty} \int_{B_R(x_0)} \frac{e_f(\varphi)(x)}{\psi(r(x))} < \infty,$$

(see [7]). The next lemma is very useful in the sequel.

Lemma 3.1. *Let $\varphi : (M^m, g) \to (N^n, h)$ be a smooth map, $D \subset M$ a compact domain such that ∂D is a smooth hypersurface in M. Let n denotes the unit normal vector of ∂D. Let X be any vector field in M with compact support. Then*

(3.1)
$$\int_{\partial D} e_f(\varphi)g(X, n) = \int_{\partial D} fh(d\varphi(X), d\varphi(n)) + \int_D \operatorname{div}^M S_f(\varphi)(X) + \int_D \langle S_f(\varphi), \nabla X\rangle.$$

Here $\langle , \rangle$ denote the inner product on $T^*M \otimes T^*M$.

Proof. Choosing a local orthonormal frame field $\{e_i\}$ on M, and define $\nabla X(e_i, e_j) = g(\nabla^M_X e_i, e_j)$, then

$$\begin{aligned}
\operatorname{div}^M(e_f(\varphi)X) &= g(\nabla^M_{e_i}(e_f(\varphi)X), e_i) \\
&= g(\nabla^M_{e_i}(e_f(\varphi))X, e_i) + e_f(\varphi)g(\nabla^M_{e_i}X, e_i) \\
&= \nabla^M_X e_f(\varphi) + e_f(\varphi)g(\nabla^M_{e_i}X, e_i),
\end{aligned}$$

and

$$\begin{aligned}
\nabla^M_X e_f(\varphi) &= \frac{1}{2}\nabla^M_X(fh(d\varphi(e_i), d\varphi(e_i))) \\
&= \frac{1}{2}X(f)|d\varphi|^2 + fh((\nabla_X d\varphi)e_i, d\varphi(e_i)) \\
&= \frac{1}{2}X(f)|d\varphi|^2 + fh((\nabla_{e_i} d\varphi)X, d\varphi(e_i)) \\
&= \frac{1}{2}X(f)|d\varphi|^2 + h(\nabla^{\varphi}_{e_i} d\varphi(X), fd\varphi(e_i)) - fh(d\varphi(\nabla^M_{e_i}X), d\varphi(e_i)) \\
&= \frac{1}{2}X(f)|d\varphi|^2 + \nabla^M_{e_i} h(d\varphi(X), fd\varphi(e_i)) - h(d\varphi(X), (\nabla_{e_i}(fd\varphi))e_i) \\
&\quad -fh(d\varphi(\nabla^M_{e_i}X), d\varphi(e_i)) \\
&= \frac{1}{2}X(f)|d\varphi|^2 + \operatorname{div}^M(fh(d\varphi(X), d\varphi(e_i))e_i) - h(d\varphi(X), \tau_f(\varphi)) \\
&\quad -f\langle \nabla X, \varphi^*h\rangle.
\end{aligned}$$

Hence we obtain

$$\begin{aligned} \operatorname{div}^M(e_f(\varphi)X) &= \frac{1}{2}X(f)|d\varphi|^2 + \operatorname{div}^M(fh(d\varphi(X), d\varphi(e_i))e_i) - h(d\varphi(X), \tau_f(\varphi)\rangle \\ &\quad -f\langle \nabla X, \varphi^* h\rangle + e_f(\varphi) g(\nabla^M_{e_i} X, e_i) \\ &= \frac{1}{2}X(f)|d\varphi|^2 + \operatorname{div}^M(fh(d\varphi(X), d\varphi(e_i))e_i) - h(d\varphi(X), \tau_f(\varphi)) \\ &\quad +\langle S_f(\varphi), \nabla X\rangle. \end{aligned} \tag{3.2}$$

Now, for compact domain D in M with its smooth hypersurface ∂D, taking local orthonormal frame field $\{e_i\}$ on M along ∂D, such that $\{e_1, ..., e_{m-1}\} \in \Gamma(T\partial D)$, and $e_m = n$ be the unit normal vector of ∂D. Since $SuppX$ is compact, integrating the formula (3.2) on D, by means of Green's theorem and using (1.9), we have the desired formula. □

Theorem 3.2. *Let M be an $m-$dimensional complete, simply connected Riemannian manifold with non-positive sectional curvature K^M, $m > 2$. Assuming that K^M satisfies*

(1) $-a^2 < \mathrm{K}^M < -b^2$, *where* $a > 0$, $b > 0$ *and* $\frac{(m-1)b}{2} \geq a$*; or*

(2) $\frac{-A}{1+r^2} \leq \mathrm{K}^M \leq 0$, *where* $0 < A < \frac{m(m-2)}{4}$,

assume that φ is a f-harmonic map with potential H from $B_R(x_0)$ to any Riemannian manifold N with $\varphi\,|_{\partial B_R(x_0)} = P$, where $P \in N$ satisfies $H(P) = \max_{y\in N} H(y)$, and $X(f) \geq 0$ such that $X = r\frac{\partial}{\partial r}$. Then φ must be constant in $B_R(x_0)$.

Proof. First of all, from the definition of $S_f(\varphi)$, we obtain

$$\begin{aligned} \langle S_f(\varphi), \nabla X\rangle &= (e_f(\varphi)g(e_\alpha, e_\beta) - fh(d\varphi(e_\alpha), d\varphi(e_\beta)))g(\nabla^M_{e_\alpha} X, e_\beta) \\ &= e_f(\varphi)g(\nabla^M_{e_\alpha} X, e_\alpha) - fh(d\varphi(e_\alpha), d\varphi(e_\beta))g(\nabla^M_{e_\alpha} X, e_\beta). \end{aligned} \tag{3.3}$$

Let $e_\alpha = \{e_s, \frac{\partial}{\partial r}\}$ be the orthonormal frame field of $B_R(x_0)$ and $X = r\frac{\partial}{\partial r}$, then

$$\nabla^M_{\frac{\partial}{\partial r}} X = \frac{\partial}{\partial r}, \tag{3.4}$$

$$\nabla^M_{e_s} X = r\nabla^M_{e_s}\frac{\partial}{\partial r} = r\,\mathrm{Hess}^M(r)(e_s, e_t)e_t, \tag{3.5}$$

$$\operatorname{div}^M X = g(\nabla^M_{e_\alpha} X, e_\alpha) = 1 + r\,\mathrm{Hess}^M(r)(e_s, e_s). \tag{3.6}$$

Substituting (3.4), (3.5) and (3.6) into (3.3), we get

$$\begin{aligned}\langle S_f(\varphi), \nabla X\rangle &= e_f(\varphi)(1 + r\,\mathrm{Hess}^M(r)(e_s, e_s)) - fh(d\varphi(e_s), d\varphi(e_t))g(\nabla^M_{e_s} X, e_t) \\ &\quad - fh(d\varphi(\frac{\partial}{\partial r}), d\varphi(\frac{\partial}{\partial r}))g(\nabla^M_{\frac{\partial}{\partial r}} X, \frac{\partial}{\partial r}) - fh(d\varphi(\frac{\partial}{\partial r}), d\varphi(e_t)) \\ &\quad g(\nabla^M_{\frac{\partial}{\partial r}} X, e_t) - fh(d\varphi(e_s), d\varphi(\frac{\partial}{\partial r}))g(\nabla^M_{e_s} X, \frac{\partial}{\partial r}) \\ &= e_f(\varphi)(1 + r\,\mathrm{Hess}^M(r)(e_s, e_s)) - fh(d\varphi(e_s), d\varphi(e_t)) \\ &\quad r\,\mathrm{Hess}^M(r)(e_s, e_t) - fh(d\varphi(\frac{\partial}{\partial r}), d\varphi(\frac{\partial}{\partial r})). \end{aligned} \tag{3.7}$$

Under the assumption (1) in Theorem 3.2, from Hessian comparison theorem (see [8]) we have

$$b\coth(br)(g - dr \otimes dr) \le \mathrm{Hess}^M(r) \le a\coth(ar)(g - dr \otimes dr). \tag{3.8}$$

Therefore, (3.7) becomes

$$\begin{aligned}\langle S_f(\varphi), \nabla X\rangle &\ge e_f(\varphi)(1 + (m-1)(br)\coth(br)) - f(ar)\coth(ar)h(d\varphi(e_s), d\varphi(e_s)) \\ &\quad - fh(d\varphi(\frac{\partial}{\partial r}), d\varphi(\frac{\partial}{\partial r})) \\ &= f(\frac{m-1}{2}(br)\coth(br) - \frac{1}{2})h(d\varphi(\frac{\partial}{\partial r}), d\varphi(\frac{\partial}{\partial r})) \\ &\quad + f(\frac{1}{2} + \frac{m-1}{2}(br)\coth(br) - (ar)\coth(ar))h(d\varphi(e_s), d\varphi(e_s)) \\ &\ge \frac{m-2}{2} fh(d\varphi(\frac{\partial}{\partial r}), d\varphi(\frac{\partial}{\partial r})) + f(\frac{1}{2} + r\coth(br)(\frac{m-1}{2}b - a)) \\ &\quad h(d\varphi(e_s), d\varphi(e_s)). \end{aligned} \tag{3.9}$$

Hence, when $\frac{(m-1)b}{2} \ge a$, it follows from (3.9)

$$\langle S_f(\varphi), \nabla X\rangle \ge \delta e_f(\varphi),$$

where $\delta > 0$.

Under the assumption (2), also by Hessian comparison theorem (see [8]) we have

$$\frac{1}{r}(g - dr \otimes dr) \le \mathrm{Hess}^M(r) \le \frac{\beta}{r}(g - dr \otimes dr),$$

where $\beta = \frac{1}{2} + \frac{1}{2}(1 + 4A)^{\frac{1}{2}}$. By (3.7), it follows that

$$\begin{aligned}\langle S_f(\varphi), \nabla X\rangle &\ge me_f(\varphi) - f\beta h(d\varphi(e_s), d\varphi(e_s)) - fh(d\varphi(\frac{\partial}{\partial r}), d\varphi(\frac{\partial}{\partial r})) \\ &= \frac{m-2}{2} fh(d\varphi(\frac{\partial}{\partial r}), d\varphi(\frac{\partial}{\partial r})) + \frac{m-2\beta}{2} fh(d\varphi(e_s), d\varphi(e_s)) \\ &\ge \delta e_f(u). \end{aligned}$$

Then, under the two assumptions of Theorem 3.2 we obtain

$$\langle S_f(\varphi), \nabla X\rangle \geq \delta e_f(\varphi), \tag{3.10}$$

where $\delta > 0$. Now choosing the geodesic polar coordinates (θ, r) in $B_R(x_0)$ and a local orthonormal frame field $\{e_1, ..., e_{m-1}, \frac{\partial}{\partial r}\}$ on M. After applying $D = B_R(x_0)$, $X = r\frac{\partial}{\partial r}$ and $n = \frac{\partial}{\partial r}$ to (3.1), we get

$$\begin{aligned} R\int_{\partial B_R(x_0)} e_f(\varphi) &= R\int_{\partial B_R(x_0)} f|d\varphi(\frac{\partial}{\partial r})|^2 + \int_{B_R(x_0)} \operatorname{div}^M S_f(\varphi)(r\frac{\partial}{\partial r}) \\ &\quad + \int_{B_R(x_0)} \langle S_f(\varphi), \nabla X\rangle. \end{aligned} \tag{3.11}$$

Noting that φ is f-harmonic map with potential H, and using (1.10), we have

$$\operatorname{div}^M S_f(\varphi)(r\frac{\partial}{\partial r}) = \int_{B_R(x_0)} r\frac{\partial(H\circ\varphi)}{\partial r} + \frac{1}{2}\int_{B_R(x_0)} r\frac{\partial f}{\partial r}|d\varphi|^2,$$

so, (3.11) becomes

$$\begin{aligned} R\int_{\partial B_R(x_0)} e_f(\varphi) &= R\int_{\partial B_R(x_0)} f|d\varphi(\frac{\partial}{\partial r})|^2 + \int_{B_R(x_0)} r\frac{\partial(H\circ\varphi)}{\partial r} + \\ &\quad \frac{1}{2}\int_{B_R(x_0)} r\frac{\partial f}{\partial r}|d\varphi|^2 + \int_{B_R(x_0)} \langle S_f(\varphi), \nabla X\rangle. \end{aligned} \tag{3.12}$$

Since φ is constant at $\partial B_R(x_0)$, by (3.10) and (3.12), we have

$$\int_{B_R(x_0)} r\frac{\partial(H\circ\varphi)}{\partial r} + \frac{1}{2}\int_{B_R(x_0)} r\frac{\partial f}{\partial r}|d\varphi|^2 + \delta\int_{B_R(x_0)} e_f(\varphi) \leq 0. \tag{3.13}$$

Denote $J(\theta, r)d\theta dr$ the volume element of $B_R(x_0)$ in polar coordinates around x_0. Since $\frac{\partial}{\partial r}(rJ(\theta, r)) > 0$ (see [3]), we obtain

$$\begin{aligned} \int_0^R r\frac{\partial(H\circ\varphi)}{\partial r}J(\theta, r)dr &= RJ(\theta, R)H(P) - \int_0^R H\circ\varphi(\theta, r)\frac{\partial(rJ(\theta, r))}{\partial r}dr \\ &\geq RJ(\theta, R)H(P) - H(P)\int_0^R \frac{\partial(rJ(\theta, r))}{\partial r}dr \\ &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{B_R(x_0)} r\frac{\partial(H\circ\varphi)}{\partial r} &= \int_{\partial B_R(x_0)} \Big(\int_0^R r\frac{\partial(H\circ\varphi)}{\partial r}J(\theta, r)dr\Big)d\theta \\ &\geq 0. \end{aligned} \tag{3.14}$$

By (3.13) and (3.14) and $X(f) \geq 0$, we immediately conclude that $e_f(\varphi) \equiv 0$ in $B_R(x_0)$, namely, φ is constant in $B_R(x_0)$, which completes the proof of Theorem 3.2. □

Remark 3.3. Consider the following static Landau-Lifshitz equation

$$\Delta\varphi + \varphi|d\varphi|^2 - < H_0, \varphi >_{\mathbb{R}^3} \varphi + H_0 = 0, \tag{3.15}$$

where $|\varphi(x)|^2 = 1$, $x \in \Omega \subset \mathbb{R}^m$, $H_0 \neq 0$ is a constant vector in $\mathbb{R}^3$. Then the solution φ of (3.15) can be seen as a harmonic map with potential: $\Omega \to \mathbb{S}^2$ with the potential $H(y) =< H_0, y >_{\mathbb{R}^3}$, $y \in \mathbb{S}^2$ (see [3]). Moreover, Hong [9] asserted that the static Landau-Lifshitz equation (3.15) with constant boundary-value problem $\varphi\,|_{\partial\Omega}= \frac{H_0}{|H_0|}$, has only constant solution, if $\Omega = B^3$, where B^3 denote the unit ball in $\mathbb{R}^3$. On the other hand, if we choose $M = \mathbb{R}^m (m > 2)$, $N = \mathbb{S}^2$, $H(y) =< H_0, y >_{\mathbb{R}^3}$, $y \in \mathbb{S}^2$, then Theorem 3.2 for $f \equiv 1$ leads to a conclusion for the static Landau-Lifshitz equation, in particular, when $m = 3$, it is just the result of Hong. Theorem 3.2 also generalizes the result of [10] for the usual harmonic maps and Theorem 3 in [3] for the harmonic maps with potential.

For f-harmonic maps, we have

Theorem 3.4. *Let M be as in Theorem 3.2. If φ is a f-harmonic map from M whose f-energy is finite or divergent slowly. Then φ must be a constant map when $X(f) \geq 0$.*

Proof. By setting $D = B_R(x_0)$, $X = r\frac{\partial}{\partial r}$ and $n = \frac{\partial}{\partial r}$ in (3.1), we obtain

$$\begin{aligned}\int_{B_R(x_0)} (\operatorname{div}^M S_f(\varphi))(X) \quad + \quad & \int_{B_R(x_0)} \langle S_f(\varphi), \nabla X\rangle = \\ & R\int_{\partial B_R(x_0)} e_f(\varphi) - R\int_{\partial B_R(x_0)} f|d\varphi(\frac{\partial}{\partial r})|^2 \\ \leq \quad & R\int_{\partial B_R(x_0)} e_f(\varphi).\end{aligned} \tag{3.16}$$

According to (1.9), (3.10) and (3.16), for a f-harmonic map φ, we get

$$\begin{aligned} R\int_{\partial B_R(x_0)} e_f(\varphi) \quad & \geq \quad \delta\int_{B_R(x_0)} e_f(\varphi) + \frac{1}{2}\int_{B_R(x_0)} X(f)|d\varphi|^2 \\ & \geq \quad \delta\int_{B_R(x_0)} e_f(\varphi).\end{aligned} \tag{3.17}$$

Now suppose that φ is a nonconstant map, i.e. the f-energy density $e_f(\varphi)$ does not vanish everywhere, so there exists $R_0 > 0$ such that for $R > R_0$,

$$\int_{B_R(x_0)} e_f(\varphi) \geq C_0, \tag{3.18}$$

where C_0 be a positive constant. So when $R > R_0$, we have from (3.17) and (3.18)

$$\int_{\partial B_R(x_0)} e_f(\varphi) \geq \frac{\delta C_0}{R}, \tag{3.19}$$

therefore, (3.19) will imply

$$\begin{aligned} E_f(\varphi) > \int_{B_R(x_0)} e_f(\varphi) &= \int_0^R \Big(\int_{\partial B_R(x_0)} e_f(\varphi) \Big) dr \geq \int_{R_0}^R \Big(\int_{\partial B_R(x_0)} e_f(\varphi) \Big) dr \\ &\geq \int_{R_0}^R \frac{\delta C_0}{r} dr = \delta C_0 \ln \frac{R}{R_0}. \end{aligned}$$

Let $R \longrightarrow \infty$, this contradicts the assumption of the finite f-energy, then φ is constant. If the f-energy of φ divergent slowly, therefore (3.19) leads to

$$\begin{aligned} \lim_{R\to\infty} \int_{B_R(x_0)} \frac{e_f(\varphi)}{\psi(r(x))} &= \int_0^\infty \frac{dr}{\psi(r)} \int_{\partial B_R(x_0)} e_f(\varphi) \\ &\geq \delta C_0 \int_0^\infty \frac{dr}{r\psi(r)} \\ &\geq \delta C_0 \int_{R_0}^\infty \frac{dr}{r\psi(r)} = \infty. \end{aligned}$$

Which is in contradiction with f-energy of φ being slowly divergent. So φ must be a constant map. □

Remark 3.5. When $f \equiv 1$, it is clear that Theorem 3.4 recovers the results due to Sealey [13] and Xin [16] as special cases. If the manifold M in the Theorems 3.2 and 3.4 satisfies $-a^2 < \mathrm{K}^M < 0$ and $\mathrm{Ric}^M < -b^2 < 0$ with $b > 2a$, then this two theorems remain true. Note that this kind of manifolds includes the bounded symmetric domains and complex hyperbolic spaces see ([15]).

Acknowledgements. The authors would like to thank the reviewers for their useful remarks and suggestions. The authors are supported by National Agency Scientific Research of Algeria and Laboratory of Geometry, Analysis, Controle and Applications, Algeria.

References

[1] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannain manifolds*, Clarendon Press, Oxford University press, Oxford. 2003.

[2] R. Caddeo and S. Montaldo, C. Oniciuc, *Biharmonic submanifolds of* $\mathbb{S}^3$, Internat. J. Math., **12**(2001), 867–876.

[3] Q. Chen, *Stability and constant boundary-value problems of harmonic maps with potential*, J. Austral. Math. Soc. Ser. A, **68**(2000), 145–154.

[4] A. M. Cherif and M. Djaa, *Geometry of energy and bienergy variations between Riemannian Manifolds*, Kyungpook Math. J., **55**(2015), 715–730.

[5] A. M. Cherif, M. Djaa and K. Zegga, *Stable f-harmonic maps on sphere*, Commun. Korean Math. Soc., **30(4)**(2015), 471–479.

[6] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., **86**(1964), 109–160.

[7] S. Feng and Y. Han, *Liouville type theorems of f-harmonic maps with potential*, Results Math., **66**(2014), 43–64.

[8] R. E. Greene and H. Wu, *Function theory on manifolds which possess a pole*, Lecture Notes in Mathematics 699, Springer-Verlag, Berlin, 1979.

[9] M. C. Hong and L. Lemaire, *Multiple solutions of the static Landau-Lifshitz equation from B^2 into S^2*, Math. Z., **220**(1995), 295–306.

[10] J. Karcher and J. C. Wood, *Nonexistence results and growth properties for harmonic maps and forms*, J. Reine Angew. Math., **353**(1984), 165–180.

[11] B. O'Neil, *Semi-Riemannian Geometry*, Academic Press, New York. 1983.

[12] S. Ouakkas, R. Nasri and M. Djaa, *On the f-harmonic and f-biharmonic maps*, JP J. Geom. Topol., **10(1)**(2010), 11–27.

[13] H. C. J. Sealey, *Some conditions ensuring the vanishing of harmonic differential forms with applications to harmonic maps and Yang–Mills theory*, Math. Proc. Cambridge Philos. Soc., **91**(1982), 441–452.

[14] Y. Xin, *Some results on stable harmonic maps*, Duke Math. J., **47**(1980), 609-613.

[15] Y. Xin, *Harmonic maps of bounded symmetric domains*, Math. Ann., **303**(1995), 417-433.

[16] Y. Xin, *Geometry of harmonic maps*, Birkhäuser Boston, Boston, MA, 1996.