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Uniqueness of Entire Functions Sharing Polynomials with Their Derivatives

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Abstract. In this paper, we investigate the uniqueness problem of entire functions sharing two polynomials with their k-th derivatives. We look into the conjecture given by Lü, Li and Yang [Bull. Korean Math. Soc., **51**(2014), 1281–1289] for the case $F = f^n P(f)$, where f is a transcendental entire function and $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ldots + a_1 z + a_0 (\not\equiv 0)$, m is a nonnegative integer, $a_m, a_{m-1}, \ldots, a_1, a_0$ are complex constants and obtain a result which improves and generalizes many previous results. We also provide some examples to show that the conditions taken in our result are best possible.

1. Introduction, Definitions and Results

In this paper, by meromorphic (entire) function we shall always mean meromorphic (entire) function in the complex plane. We assume that the reader is familiar with the standard notations of Nevanlinna's theory of meromorphic functions as explained in [7, 9, 17]. For a nonconstant meromorphic function f, we denote by $T(r, f)$ the Nevanlinna Characteristic function of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ for all r outside a possible exceptional set of finite logarithmic measure. The meromorphic function a is called a small function of f, if $T(r, a) = S(r, f)$, where $r \to \infty$ outside a possible exceptional set of finite measure.

Let k be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N_{k)}(r, a; f)$ the counting function of all those a-points of f whose multiplicities are not greater than k and by $N_{(k+1}(r, a; f)$ the counting function of all those a-points of f whose

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multiplicities are greater than k.

Let f and g be two nonconstant meromorphic functions and Q_1, Q_2 be two polynomials or complex numbers. If $f-Q_1$ and $g-Q_2$ have the same zeros with the same multiplicities, then we say that $f-Q_1$ and $g-Q_2$ share the value 0 CM. Especially, if $Q_1=Q_2=a$, where $a\in\mathbb{C}\cup\{\infty\}$, then we say that f and g share the value a CM, when $f-a$ and $g-a$ have the same zeros with the same multiplicities. The uniqueness problem of entire and meromorphic functions sharing values, small functions, polynomials with their derivatives is an interesting topic of value distribution theory. Many mathematicians (see [5, 8, 16, 19, 20, 21]) worked on this topic and they gave many conjectures and results. In 1976, Rubel and Yang [14] first proved a result which is as follows.

Theorem A. *If a nonconstant entire function f and its derivative f' share two distinct finite values CM, then $f=f'$.*

Theorem A suggests the following question.

Question 1. *What can be said if a nonconstant entire function f shares one finite value CM with its derivative f'?*

In 1996, Brück [2] presented the following conjecture relating to Question 1.

Conjecture 1. *Let f be a nonconstant entire function. Suppose that $\rho_1(f)$, the first iterated order of f, is not a positive integer or infinite where $\rho_1(f)=\limsup_{r\to\infty}\dfrac{\log\log T(r,f)}{\log r}$ and if f and f' share one finite value a CM, then $\frac{f'-a}{f-a}=c$, for some nonzero constant c.*

In 1996, Brück [2] proved that the conjecture is true if $a=0$ or $N(r,0;f')=S(r,f)$. In 1998, Gundersen and Yang [6] proved that the conjecture is true if f is of finite order and fails, in general, for meromorphic functions. In 2004, Chen and Shon [3] proved that the conjecture is true for entire function of order $\rho_1(f)<\frac{1}{2}$. In 2005, Al-Khaladi [1] proved that the conjecture is true for meromorphic function f when $N(r,0;f')=S(r,f)$.

Now it is natural to ask the following question.

Question 2. *Whether Brück Conjecture holds if the function f is replaced by its n-th power f^n?*

In 2008, Yang and Zhang [18] answered the above question by proving the following result.

Theorem B. *Let f be a nonconstant entire function, $n\geq 7$ be an integer and let $F=f^n$. If F and F' share 1 CM, then $F=F'$ and f assumes the form $f(z)=ce^{\frac{1}{n}z}$, where c is a nonzero constant.*

In 2010, Zhang and Yang [22] further improved Theorem B by considering k-th derivative of f^n as follows.

Theorem C. *Let f be a nonconstant entire function and n, k be two positive integers such that $n \geq k+1$. If f^n and $(f^n)^{(k)}$ share the value 1 CM, then $f^n = (f^n)^{(k)}$ and f assumes the form $f(z) = ce^{\frac{\lambda}{n}z}$, where c, λ are nonzero constants and $\lambda^k = 1$.*

In 2011, Lü and Yi [11] considered polynomial sharing instead of value sharing and proved the following result.

Theorem D. *Let f be a transcendental entire function, n, k be two positive integers and $Q \not\equiv 0$ be a polynomial. If $f^n - Q$ and $(f^n)^{(k)} - Q$ share the value 0 CM and $n \geq k+1$, then $f^n = (f^n)^{(k)}$ and f assumes the form $f(z) = ce^{\frac{\lambda}{n}z}$, where c, λ are nonzero constants and $\lambda^k = 1$.*

Regarding Theorem D one may ask the following question.

Question 3. *What can be said if $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share the value 0 CM, where Q_1, Q_2 are polynomials with $Q_1Q_2 \not\equiv 0$?*

In 2014, Lü, Li and Yang [10] answered the above question for $k = 1$ and obtained the following result.

Theorem E. *Let f be a transcendental entire function and $n \geq 2$ be an integer. If $f^n - Q_1$ and $(f^n)' - Q_2$ share the value 0 CM, then $\frac{Q_2}{Q_1}$ is a polynomial and $f' = \frac{Q_2}{nQ_1}f$. Furthermore, if $Q_1 = Q_2$, then $f(z) = ce^{\frac{1}{n}z}$, where Q_1, Q_2 are polynomials with $Q_1Q_2 \not\equiv 0$, and c is a nonzero constant.*

In the same paper the authors posed the following conjecture.

Conjecture 2. *Let f be a transcendental entire function and n, k be two positive integers such that $n \geq k+1$. If $f^n - Q_1$ and $(f^n)^{(k)} - Q_2$ share the value 0 CM, then $(f^n)^{(k)} = \frac{Q_2}{Q_1}f^n$. Furthermore, if $Q_1 = Q_2$, then $f(z) = ce^{\frac{\lambda}{n}z}$, where Q_1, Q_2 are polynomials with $Q_1Q_2 \not\equiv 0$, and c, λ are nonzero constants such that $\lambda^k = 1$.*

Recently Majumder [12] showed that the above conjecture is true for any positive integer k. The following two examples given in [12] respectively shows that the condition $n \geq k+1$ and f is transcendental in Conjecture 2 are essential.

Example 1. Let $f(z) = e^{2z} + z$. Then $f - Q_1$ and $f' - Q_2$ share 0 CM, but $f' \not\equiv \frac{Q_2}{Q_1}f$, where $Q_1(z) = z+1$ and $Q_2(z) = 3$.

Example 2. Let $f(z) = z$. Then $f^2 - Q_1$ and $(f^2)' - Q_2$ share 0 CM, but $(f^2)' \not\equiv \frac{Q_2}{Q_1}f^2$, where $Q_1(z) = 2z^2 + z$ and $Q_2(z) = 2z^2 + 4z$.

In [10] the authors posed the following two questions.

Question 4. *What can be said if in Conjecture 2 the condition "f^n" be replaced by "$P(f)$" where $P(z) = \sum_{i=0}^{n} a_i z^i$?*

Question 5. *What can be said if in Conjecture 2 the condition "f^n" be replaced by "$f(z+c_1)f(z+c_2)\ldots f(z+c_n)$" where c_j $(j = 1, 2, \ldots, n)$ are constants?*

Our aim to write this paper is to investigate the Conjecture due to Lü, Li and Yang by considering the function $F = f^n P(f)$ where f is a transcendental entire function and $P(z) = \sum_{i=0}^{m} a_i z^i$, $a_0, a_1, \ldots, a_m (\not\equiv 0)$ are complex constants. Though we are able to find out an affirmative solution of Question 4 as far as we know Question 5 remains open. The following is the main result of the paper.

Theorem 1. *Let f be transcendental entire function and n, m, k be positive integers such that $n \geq m+k+1$. If $f^n P(f) - Q_1$ and $(f^n P(f))^{(k)} - Q_2$ share 0 CM, then $P(z)$ reduces to a nonzero monomial, namely $P(z) = a_i z^i$ for some $i \in \{0, 1, \ldots, m\}$ and $(f^{n+i})^{(k)} = \frac{Q_2}{Q_1} f^{n+i}$. Furthermore, if $Q_1 = Q_2$, then f assumes the form $f(z) = ce^{\frac{\lambda}{n+i} z}$, where Q_1, Q_2 are polynomials with $Q_1 Q_2 \not\equiv 0$ and c, λ are nonzero constants such that $\lambda^k = 1$.*

The condition $n \geq m+k+1$ in Theorem 1 is essential as shown by the following example.

Example 3. Let $f(z) = e^z - 1$ and $P(f) = f^2 + 3f + 3$. Then $fP(f) - Q_1$ and $(fP(f))' - Q_2$ share 0 CM, but $(fP(f))' \not\equiv \frac{Q_2}{Q_1} fP(f)$, where $Q_1(z) = z + 1$ and $Q_2(z) = 3z + 6$.

The following example shows that the hypothesis of transcendental of f in Theorem 1 is necessary.

Example 4. Let $f(z) = z - 1$ and $P(f) = f + 1$. Then $f^3 P(f) - Q_1$ and $(f^3 P(f))' - Q_2$ share 0 CM, but $(f^3 P(f))' \not\equiv \frac{Q_2}{Q_1} f^3 P(f)$, where $Q_1(z) = z^4 + 3z^2$ and $Q_2(z) = -14z^3 - 9z^2 - 1$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1.([15]) *Let f be a nonconstant meromorphic function and $a_n(z)(\not\equiv 0)$, $a_{n-1}(z), \ldots, a_1(z), a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.([4]) *Suppose that f is a transcendental meromorphic function and that*

$$f^n P_*(f) = Q_*(f),$$

where $P_*(f)$ and $Q_*(f)$ are differential polynomials in f with functions of small proximity related to f as the coefficients and the degree of $Q_*(f)$ is at most n. Then

$$m(r, \infty; P_*(f)) = S(r, f).$$

Lemma 3.([7]) *Let f be a nonconstant meromorphic function and let $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r, a_i) = S(r, f)$, $i = 1, 2$. Then*

$$T(r, f) \leq \overline{N}(r, \infty; f) + \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + S(r, f).$$

Lemma 4.([13]) *Let f be a nonconstant meromorphic function and n, k, m be positive integers such that $n \geq k+1$. If $f^n P(f) = \{f^n P(f)\}^{(k)}$, then $P(z)$ reduces to a nonzero monomial, namely $P(z) = a_i z^i$ for some $i \in \{0, 1, \ldots, m\}$; and $f^{n+i} \equiv (f^{n+i})^{(k)}$, where f assumes the form $f(z) = ce^{\frac{\lambda}{n+i}z}$, where c is a nonzero constant and $\lambda^k = 1$.*

3. Proof of the Theorem

Proof of the Theorem 1. Let $F_* = \frac{F}{Q_1}$, $G_* = \frac{G}{Q_2}$, where $F = f^n P(f)$ and $G = (f^n P(f))^{(k)}$. Clearly F_* and G_* share 1 CM except for the zeros of $Q_i(z)$, where $i = 1, 2$ and so $\overline{N}(r, 1; F_*) = \overline{N}(r, 1; G_*) + S(r, f)$. Let

$$W = \frac{F'_*(F_* - G_*)}{F_*(F_* - 1)}. \tag{3.1}$$

We now consider the following two cases.

Case 1. Let $W \not\equiv 0$. It is obvious that $m(r, \infty; W) = S(r, f)$.

Let z_0 be zero of f with multiplicity $p_0(\geq 1)$ which is a zero of $P(f)$ with multiplicity $q_0(\geq 1)$ such that $Q_i(z_0) \neq 0$, where $i = 1, 2$. Then from (3.1), we obtain

$$W(z) = O\left((z - z_0)^{np_0 + q_0 - k - 1}\right). \tag{3.2}$$

Since $n \geq m+k+1$ and f is transcendental entire, we see that $N(r, \infty; W) = S(r, f)$. Consequently $T(r, W) = S(r, f)$.

Now from (3.1) we see that

$$\frac{1}{F_*} = \frac{1}{W} \frac{F'_*}{F_*(F_* - 1)} \left(1 - \frac{G_*}{F_*}\right).$$

Therefore, it follows from above that $m(r, 0; F_*) = S(r, f)$, and hence $m(r, 0; f) = S(r, f)$. Also

$$N(r, 0; f) \leq N(r, 0; W) \leq T(r, \frac{1}{W}) \leq T(r, W) + O(1) = S(r, f).$$

Hence $T(r, f) = S(r, f)$, a contradiction.

Let z_1 be zero of f with multiplicity $p_1(\geq 1)$ which is not a zero of $P(f)$ and $Q_i(z_1) \neq 0$ for $i = 1, 2$. Then as before we obtain

$$W(z) = O\left((z - z_1)^{np_1 - k - 1}\right), \tag{3.3}$$

$$T(r,W)=S(r,f)\ \ and\ \ m(r,0;f)=S(r,f). \tag{3.4}$$

We now discuss the following two subcases.
Subcase 1. Let $n>m+k+1$. Then from (3.3), we obtain

$$N(r,0;f)\leq N(r,0;W)\leq T(r,\frac{1}{W})\leq T(r,W)+O(1)=S(r,f). \tag{3.5}$$

Hence from (3.4) and (3.5), we get $T(r,f)=S(r,f)$, a contradiction.
Subcase 2. Let $n=m+k+1$. Then from (3.3) we see that

$$N_{(2}(r,0;f)\leq N(r,0;W)\leq T(r,\frac{1}{W})\leq T(r,W)+O(1)=S(r,f).$$

Then (3.4) gives

$$T(r,f)=N_{1)}(r,0;f)+S(r,f). \tag{3.6}$$

Let

$$\begin{aligned} F &= f^nP(f)\\ &= a_mf^{n+m}+a_{m-1}f^{n+m-1}+\ \ldots\ +a_1f^{n+1}+a_0f^n\\ &= F_m+F_{m-1}+\ \ldots\ +F_1+F_0,\quad say. \end{aligned} \tag{3.7}$$

Since $F-Q_1$ and $F^{(k)}-Q_2$ share 0 CM, there exists an entire function α, such that

$$F^{(k)}-Q_2=e^{\alpha}(F-Q_1). \tag{3.8}$$

First we assume that e^{α} is not constant. Differentiating (3.8), we get

$$F^{(k+1)}-Q_2'=\alpha'e^{\alpha}(F-Q_1)+e^{\alpha}(F'-Q_1'). \tag{3.9}$$

From (3.8) and (3.9) we obtain

$$\begin{aligned} & F^{(k+1)}F-\alpha'F^{(k)}F-F^{(k)}F'\\ = & Q_1F^{(k+1)}-(\alpha'Q_1+Q_1')F^{(k)}-Q_2F'+(Q_2'-\alpha'Q_2)F\\ & +\alpha'Q_1Q_2+Q_2Q_1'-Q_1Q_2'. \end{aligned} \tag{3.10}$$

From (3.8) we see that

$$T(r,e^{\alpha})\leq(k+2)T(r,F)+O(\log r)+S(r,F)=(n+m)(k+2)T(r,f)+S(r,f).$$

Since $T(r,\alpha')=S(r,e^{\alpha})$, it follows that $T(r,\alpha')=S(r,f)$. Now from (3.7), we deduce for $i\in\{0,\ 1,\ \ldots,\ m\}$ that

$$\begin{aligned} F_i' &= a_i(n+i)f^{n+i-1}f'\\ &= f^m\{a_i(n+i)f^{n+i-m-1}f'\} \end{aligned}$$

$$\begin{aligned} F_i'' &= a_i(n+i)(n+i-1)f^{n+i-2}(f')^2 + a_i(n+i)f^{n+i-1}f'' \\ &= f^m\{a_i(n+i)(n+i-1)f^{n+i-m-2}(f')^2 + a_i(n+i)f^{n+i-m-1}f''\} \end{aligned}$$

$$\begin{aligned} F_i''' &= a_i(n+i)(n+i-1)(n+i-2)f^{n+i-3}(f')^3 \\ &\quad +3a_i(n+i)(n+i-1)f^{n+i-2}f'f'' + a_i(n+i)f^{n+i-1}f''' \\ &= f^m\{a_i(n+i)(n+i-1)(n+i-2)f^{n+i-m-3}(f')^3 \\ &\quad +3a_i(n+i)(n+i-1)f^{n+i-m-2}f'f'' + a_i(n+i)f^{n+i-m-1}f'''\} \end{aligned}$$

and so on.

Thus in general we have

$$F_i^{(k)} = f^m \sum_{\lambda^i} a_{\lambda^i} f^{l_0^{\lambda^i}} (f')^{l_1^{\lambda^i}} \ldots (f^{(k)})^{l_k^{\lambda^i}},$$

where $l_0^{\lambda^i},\ l_1^{\lambda^i}, \ldots,\ l_k^{\lambda^i}$ are nonnegative integers satisfying $\sum_{j=0}^{k} l_j^{\lambda^i} = n+i-m$, $n+i-m-k \le l_0^{\lambda^i} \le n+i-m-1$ and a_{λ^i} are constants for $i \in \{0,\ 1,\ \ldots,\ m\}$. Also we have

$$F_i^{(k+1)} = f^m \sum_{\lambda^i} b_{\lambda^i} f^{p_0^{\lambda^i}} (f')^{p_1^{\lambda^i}} \ldots (f^{(k+1)})^{p_{k+1}^{\lambda^i}},$$

where $p_0^{\lambda^i},\ p_1^{\lambda^i}, \ldots,\ p_{k+1}^{\lambda^i}$ are nonnegative integers satisfying $\sum_{j=0}^{k+1} p_j^{\lambda^i} = n+i-m$, $n+i-m-k-1 \le p_0^{\lambda^i} \le n+i-m-1$ and b_{λ^i} are constants for $i \in \{0,\ 1,\ \ldots,\ m\}$.

Thus we have from (3.7)

$$\begin{aligned} F^{(k)} &= f^m\Big\{\sum_{\lambda^m} a_{\lambda^m} f^{l_0^{\lambda^m}} (f')^{l_1^{\lambda^m}} \ldots (f^{(k)})^{l_k^{\lambda^m}} \\ &\quad + \sum_{\lambda^{m-1}} a_{\lambda^{m-1}} f^{l_0^{\lambda^{m-1}}} (f')^{l_1^{\lambda^{m-1}}} \ldots (f^{(k)})^{l_k^{\lambda^{m-1}}} + \ldots \\ &\quad + \sum_{\lambda^1} a_{\lambda^1} f^{l_0^{\lambda^1}} (f')^{l_1^{\lambda^1}} \ldots (f^{(k)})^{l_k^{\lambda^1}} + \sum_{\lambda^0} a_{\lambda^0} f^{l_0^{\lambda^0}} (f')^{l_1^{\lambda^0}} \ldots (f^{(k)})^{l_k^{\lambda^0}}\Big\} \end{aligned} \tag{3.11}$$

and

$$
\begin{aligned}
F^{(k+1)} \quad = \quad & f^{m}\Big\{\sum_{\lambda^{m}} b_{\lambda^{m}} f^{p_0^{\lambda^m}} (f')^{p_1^{\lambda^m}} \ldots (f^{(k+1)})^{p_{k+1}^{\lambda^m}} \\
& + \sum_{\lambda^{m-1}} b_{\lambda^{m-1}} f^{p_0^{\lambda^{m-1}}} (f')^{p_1^{\lambda^{m-1}}} \ldots (f^{(k+1)})^{p_{k+1}^{\lambda^{m-1}}} \\
& + \ldots + \sum_{\lambda^{1}} b_{\lambda^{1}} f^{p_0^{\lambda^1}} (f')^{p_1^{\lambda^1}} \ldots (f^{(k+1)})^{p_{k+1}^{\lambda^1}} \\
& + \sum_{\lambda^{0}} b_{\lambda^{0}} f^{p_0^{\lambda^0}} (f')^{p_1^{\lambda^0}} \ldots (f^{(k+1)})^{p_{k+1}^{\lambda^0}} \Big\}. \qquad (3.12)
\end{aligned}
$$

Using (3.7), (3.11) and (3.12) in (3.10), we obtain

$$
f^{n+m} P_*(f) = Q_*(f), \qquad (3.13)
$$

where $Q_*(f)$ is a differential polynomial in f of degree $n+m$ and

$$
\begin{aligned}
P_*(f) \quad = \quad & \Big\{\sum_{\lambda^{m}} b_{\lambda^{m}} f^{p_0^{\lambda^m}} (f')^{p_1^{\lambda^m}} \ldots (f^{(k+1)})^{p_{k+1}^{\lambda^m}} \\
& + \sum_{\lambda^{m-1}} b_{\lambda^{m-1}} f^{p_0^{\lambda^{m-1}}} (f')^{p_1^{\lambda^{m-1}}} \ldots (f^{(k+1)})^{p_{k+1}^{\lambda^{m-1}}} \\
& + \ldots + \sum_{\lambda^{1}} b_{\lambda^{1}} f^{p_0^{\lambda^1}} (f')^{p_1^{\lambda^1}} \ldots (f^{(k+1)})^{p_{k+1}^{\lambda^1}} \\
& + \sum_{\lambda^{0}} b_{\lambda^{0}} f^{p_0^{\lambda^0}} (f')^{p_1^{\lambda^0}} \ldots (f^{(k+1)})^{p_{k+1}^{\lambda^0}} \Big\} P(f) \\
& - \alpha' \Big\{\sum_{\lambda^{m}} a_{\lambda^{m}} f^{l_0^{\lambda^m}} (f')^{l_1^{\lambda^m}} \ldots (f^{(k)})^{l_k^{\lambda^m}} \\
& + \sum_{\lambda^{m-1}} a_{\lambda^{m-1}} f^{l_0^{\lambda^{m-1}}} (f')^{l_1^{\lambda^{m-1}}} \ldots (f^{(k)})^{l_k^{\lambda^{m-1}}} \\
& + \ldots + \sum_{\lambda^{1}} a_{\lambda^{1}} f^{l_0^{\lambda^1}} (f')^{l_1^{\lambda^1}} \ldots (f^{(k)})^{l_k^{\lambda^1}} \\
& + \sum_{\lambda^{0}} a_{\lambda^{0}} f^{l_0^{\lambda^0}} (f')^{l_1^{\lambda^0}} \ldots (f^{(k)})^{l_k^{\lambda^0}} \Big\} P(f) \\
& - f' \Big\{\sum_{\lambda^{m}} a_{\lambda^{m}} f^{l_0^{\lambda^m}-1} (f')^{l_1^{\lambda^m}} \ldots (f^{(k)})^{l_k^{\lambda^m}} \\
& + \sum_{\lambda^{m-1}} a_{\lambda^{m-1}} f^{l_0^{\lambda^{m-1}}-1} (f')^{l_1^{\lambda^{m-1}}} \ldots (f^{(k)})^{l_k^{\lambda^{m-1}}} \\
& + \ldots + \sum_{\lambda^{1}} a_{\lambda^{1}} f^{l_0^{\lambda^1}-1} (f')^{l_1^{\lambda^1}} \ldots (f^{(k)})^{l_k^{\lambda^1}}
\end{aligned}
$$

$$+\sum_{\lambda^0} a_{\lambda^0} f^{l_0^{\lambda^0}-1}(f')^{l_1^{\lambda^0}} \dots (f^{(k)})^{l_k^{\lambda^0}} \Big\} P_1(f)$$

$$= A(f')^{k+1} + R_*(f) \tag{3.14}$$

is a differential polynomial in f of degree $n+m$, where A is a suitable constant, $P_1(f) = a_m(n+m)f^m + a_{m-1}(n+m-1)f^{m-1} + \dots + a_1(n+1)f + a_0 n$ and $R_*(f)$ is a differential polynomial in f. Actually every monomial of $R_*(f)$ has the form

$$R_i(\alpha') f^{q_0^{\lambda^i}} (f')^{q_1^{\lambda^i}} \dots (f^{(k+1)})^{q_{k+1}^{\lambda^i}},$$

where $q_0^{\lambda^i}, q_1^{\lambda^i}, \dots, q_{k+1}^{\lambda^i}$ are nonnegative integers satisfying $\sum_{j=0}^{k+1} q_j^{\lambda^i} = n+2i-m$, $n+2i-m-k \le q_0^{\lambda^i} \le n+2i-m-1$ and $R_i(\alpha')$ are polynomials in α' with constant coefficients for $i \in \{0, 1, \dots, m\}$.

First we suppose that $P_*(f) \not\equiv 0$. Then by Lemma 2, we get $m(r,\infty;P_*) = S(r,f)$ and so $T(r,P_*) = S(r,f)$. Consequently, $T(r,P_*') = S(r,f)$.

Note that from (3.14)

$$P_*'(f) = A(k+1)(f')^k f'' + B\alpha'(f')^{k+1} + S_*(f) \tag{3.15}$$

is a differential polynomial in f, where B is a suitable constant and $S_*(f)$ is a differential polynomial in f. Actually every monomial of $S_*(f)$ has the form

$$S_i(\alpha') f^{r_0^{\lambda^i}} (f')^{r_1^{\lambda^i}} \dots (f^{(k+1)})^{r_{k+1}^{\lambda^i}},$$

where $r_0^{\lambda^i}, r_1^{\lambda^i}, \dots, r_{k+1}^{\lambda^i}$ are nonnegative integers satisfying $\sum_{j=0}^{k+1} r_j^{\lambda^i} = n+2i-m$, $n+2i-m-k \le r_0^{\lambda^i} \le n+2i-m-1$ and $S_i(\alpha')$ are polynomials in α' with constant coefficients for $i \in \{0, 1, \dots, m\}$.

Let z_2 be simple zero of f. Then from (3.14) and (3.15), we obtain

$$P_*(f(z_2)) = A(f'(z_2))^{k+1},$$

and

$$P_*'(f(z_2)) = A(k+1)(f'(z_2))^k f''(z_2) + B\alpha'(f'(z_2))^{k+1}.$$

This shows that z_2 is a zero of $P_* f'' - (c_1 P_*' - c_2\alpha' P_*)f'$, where c_1 and c_2 are suitable constants. Let

$$\Phi = \frac{P_* f'' - (c_1 P_*' - c_2\alpha' P_*)f'}{f}. \tag{3.16}$$

Clearly $\Phi \not\equiv 0$ and $T(r,\Phi) = S(r,f)$.

From (3.16) we have

$$f'' = \alpha_1 f + \beta_1 f', \tag{3.17}$$

where

$$\alpha_1 = \frac{\Phi}{P_*}, \quad \beta_1 = c_1 \frac{P_*'}{P_*} - c_2\alpha' \tag{3.18}$$

and

$$T(r,\alpha_1) = S(r,f), \quad T(r,\beta_1) = S(r,f).$$

(3.14) and (3.18) together gives

$$\begin{aligned} P_*' &= \left(\frac{\beta_1}{c_1} + \frac{c_2}{c_1}\alpha'\right) P_* \\ &= A\left(\frac{\beta_1}{c_1} + \frac{c_2}{c_1}\alpha'\right)(f')^{k+1} + \left(\frac{\beta_1}{c_1} + \frac{c_2}{c_1}\alpha'\right) R_*(f). \end{aligned} \tag{3.19}$$

Using (3.15) and (3.17), we get

$$P_*' = A(k+1)\alpha_1 f (f')^k + \{A(k+1)\beta_1 + B\alpha'\}(f')^{k+1} + S_*(f). \tag{3.20}$$

By (3.19) and (3.20), we have

$$\begin{aligned} &\left(\frac{A}{c_1}\beta_1 - A(k+1)\beta_1 + A\frac{c_2}{c_1}\alpha' - B\alpha'\right)(f')^{k+1} - A(k+1)\alpha_1 f (f')^k \\ &+ \left(\frac{\beta_1}{c_1} + \frac{c_2}{c_1}\alpha'\right) R_*(f) - S_*(f) \equiv 0. \end{aligned} \tag{3.21}$$

Since $\alpha_1 \not\equiv 0$, from (3.21) we get

$$N_{1)}(r,0;f) = S(r,f). \tag{3.22}$$

Therefore from (3.6) and (3.22) we have

$$T(r,f) = S(r,f),$$

a contradiction.

Next we suppose that $P_*(f) \equiv 0$. Then $Q_*(f) \equiv 0$ by (3.13), where

$$\begin{aligned} Q_*(f) &= Q_1 F^{(k+1)} - (\alpha' Q_1 + Q_1')F^{(k)} - Q_2 F' + (Q_2' - \alpha' Q_2)F \\ &\quad + \alpha' Q_1 Q_2 + Q_2 Q_1' - Q_1 Q_2'. \end{aligned}$$

So from (3.10) it follows that

$$F^{(k+1)}F - \alpha' F^{(k)} F - F^{(k)} F' \equiv 0,$$

i.e.,

$$\frac{F^{(k+1)}}{F^{(k)}} \equiv \alpha' + \frac{F'}{F}. \tag{3.23}$$

Integrating we obtain $F^{(k)} = c_3 F e^{\alpha}$, where c_3 is a nonzero constant. Substituting the values of F and $F^{(k)}$ into (3.8) we obtain

$$(c_3 - 1) f^n P(f) = \frac{Q_2 - Q_1 e^{\alpha}}{e^{\alpha}}.$$

Clearly $c_3 \neq 1$ and all zeros of $Q_2 - Q_1 e^{\alpha}$ have the multiplicities at least n. Since $n = m + k + 1$, by Lemma 3 we get

$$\begin{aligned} T(r, e^{\alpha}) &\leq \overline{N}(r, 0; e^{\alpha}) + \overline{N}(r, \infty; e^{\alpha}) + \overline{N}\left(r, \frac{Q_2}{Q_1}; e^{\alpha}\right) + S(r, e^{\alpha}) \\ &\leq \frac{1}{n} N\left(r, \frac{Q_2}{Q_1}; e^{\alpha}\right) + S(r, e^{\alpha}) \\ &\leq \frac{1}{n} T(r, e^{\alpha}) + S(r, e^{\alpha}), \end{aligned}$$

which contradicts to the assumption that e^{α} is a nonconstant entire function.

Next we assume that e^{α} is a constant, say c_4. Then from (3.8), we have

$$F^{(k)} - c_4 F \equiv Q_2 - c_4 Q_1. \tag{3.24}$$

Since $n = m + k + 1$, it follows that

$$N(r, 0; f) = S(r, f)$$

and hence by (3.4) we have

$$T(r, f) = S(r, f),$$

a contradiction.

Case 2. Let $W \equiv 0$. Then from (3.1) we get $F_* = G_*$,

$$\textit{i.e.,} \quad (f^n P(f))^{(k)} = \frac{Q_2}{Q_1} f^n P(f). \tag{3.25}$$

If Q_1 and Q_2 are same polynomials then using Lemma 4 we can get the conclusion of the theorem. Next we assume that Q_1 and Q_2 are distinct. We show that $P(z)$ reduces to a nonzero monomial of the form $P(z) = a_i z^i$ for some $i \in \{0,\ 1,\ \ldots,\ m\}$. If not, we may assume that $P(z) = a_m z^m + a_{m-1} z^{m-1} + \ \ldots\ + a_1 z + a_0$, where at least two of $a_m,\ a_{m-1},\ \ldots, a_1,\ a_0$, namely $a_p,\ a_q,\ p \neq q$ are nonzero. As f is entire and $n \geq m + k + 1$, from (3.25) we see that 0 is a Picard exceptional value of f. So we have $f(z) = e^{\alpha}$, where α is a nonconstant entire function. It is easy to see that for $i \in \{0,\ 1,\ \ldots,\ m\}$,

$$\begin{aligned} a_i\{(f^{n+i})^{(k)} - \frac{Q_2}{Q_1} f^{n+i}\} &= \left[t_i(\alpha',\ \alpha'',\ \ldots,\ \alpha^{(k)}) - a_i \frac{Q_2}{Q_1}\right] e^{(n+i)\alpha} \\ &= s_i(\alpha',\ \alpha'',\ \ldots,\ \alpha^{(k)}) e^{(n+i)\alpha}, \end{aligned} \tag{3.26}$$

where $s_i(\alpha',\ \alpha'',\ \ldots,\ \alpha^{(k)})$ are differential polynomials in $\alpha',\ \alpha'',\ \ldots,\ \alpha^{(k)}$ with rational coefficients. Using (3.25) and (3.26), we obtain

$$\begin{aligned} & s_m(\alpha',\ \alpha'',\ \ldots,\ \alpha^{(k)}) e^{m\alpha} + s_{m-1}(\alpha',\ \alpha'',\ \ldots,\ \alpha^{(k)}) e^{(m-1)\alpha} + \ \ldots \\ & + s_1(\alpha',\ \alpha'',\ \ldots,\ \alpha^{(k)}) e^{\alpha} + s_0(\alpha',\ \alpha'',\ \ldots,\ \alpha^{(k)}) \equiv 0. \end{aligned} \tag{3.27}$$

Since $T(r, s_i) = S(r, f)$ $(i = 0,\ 1,\ \ldots,\ m)$, by Borel theorem on the combination of entire functions (see Theorem 1.52 of [17]), (3.27) gives $s_i = 0$ for $i \in \{0,\ 1,\ \ldots,\ m\}$. As $a_p,\ a_q \neq 0$, from (3.26), we have

$$(f^{n+p})^{(k)} = \frac{Q_2}{Q_1} f^{n+p} \ and \ (f^{n+q})^{(k)} = \frac{Q_2}{Q_1} f^{n+q}.$$

Thus we get two different forms of f simultaneously, a contradiction. Hence $P(z) = a_i z^i$ for some $i \in \{0,\ 1,\ \ldots,m\}$. Therefore, $(f^{n+i})^{(k)} = \frac{Q_2}{Q_1} f^{n+i}$ for some $i \in \{0,\ 1,\ \ldots,\ m\}$. This completes the proof of Theorem 1. □

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References

[1] A. Al-Khaladi, *On meromorphic functions that share one value with their derivatives*, Analysis, **25**(2005), 131–140.

[2] R. Brück, *On entire functions which share one value CM with their first derivative*, Results Math., **30**(1996), 21–24.

[3] Z. X. Chen and K. H. Shon, *On conjecture of R. Brück concerning entire function sharing one value CM with its derivative*, Taiwanese J. Math., **8**(2004), 235–244.

[4] J. Clunie, *On integral and meromorphic functions*, J. London Math. Soc., **37**(1962), 17–27.

[5] G. G. Gundersen, *Meromorphic functions that share finite values with their derivatives*, J. Math. Anal. Appl., **75**(1980), 441–446.

[6] G. G. Gundersen and L. Z. Yang, *Entire functions that share one value with one or two of their derivatives*, J. Math. Anal. Appl., **223**(1998), 88–95.

[7] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.

[8] G. Jank, E. Mues and L. Volkmann, *Meromorphe Funktionen, die mit ihrer ersten und zweiten Ableitung einen endchen Wert teilen*, Complex Variables Theory Appl., **6**(1986), 51–71.

[9] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin/New York, 1993.

[10] W. Lü, Q. Li and C. C. Yang, *On the transcendental entire solutions of a class of differential equations*, Bull. Korean Math. Soc., **51**(2014), 1281–1289.

[11] F. Lü and H.-X. Yi, *The Brück conjecture and entire functions sharing polynomials with their k-th derivatives*, J. Korean Math. Soc., **48**(2011), 499–512.

[12] S. Majumder, *A result on a conjecture of W. Lü, Q. Li and C. C. Yang*, Bull. Korean Math. Soc., **53**(2016), 411–421.

[13] S. Majumder, *Values shared by meromorphic functions and their derivatives*, Arab J. Math. Sci., **22**(2016), 265–274.

[14] L. A. Rubel and C. C. Yang, *Values shared by an entire function and its derivative*, Complex Analysis (Proc. Conf., Univ. Kentucky, Lexington, Ky.) (1976), 101–103.

[15] C. C. Yang, *On deficiencies of differential polynomials II*, Math. Z., **125**(1972), 107–112.

[16] L. Z. Yang, *Entire functions that share finite values with their derivatives*, Bull. Aust. Math. Soc., **41**(1990), 337–342.

[17] C.-C. Yang and H.-X. Yi, *Uniqueness theory of meromorphic functions*, Science Press, Beijing, 1995.

[18] L.-Z. Yang and J.-L. Zhang, *Non-existence of meromorphic solutions of Fermat Type functional equation*, Aequationes Math., **76**(2008), 140–150.

[19] H. X. Yi, *On characteristic function of a meromorphic function and its derivative*, Indian J. Math., **33**(1991), 119–133.

[20] Q. C. Zhang, *Meromorphic function that shares one small function with its derivative*, JIPAM. J. Inequal. Pure. Appl. Math., **6**(2005), Article 116, 13 pp.

[21] J.-L. Zhang and L.-Z. Yang, *Some results related to a conjecture of R. Brück*, JIPAM. Inequal. Pure Appl. Math., **8**(2007), Article 18, 11 pp.

[22] J.-L. Zhang and L.-Z. Yang, *A Power of an entire function sharing one value with its derivative*, Comput. Math. Appl., **60**(2010), 2153–2160.