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Certain Geometric Properties of an Integral Operator Involving Bessel Functions

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Abstract. In this article, we introduce a new integral operator involving normalized Bessel functions of the first kind and we obtain a set of sufficient conditions for univalence. Our results contain some interesting corollaries as special cases. Further, as particular cases, we improve some of the univalence conditions proved in [2].

1. Introduction and Preliminaries

Let $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ be an open unit disk and let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in $\mathcal{U}$ and satisfying the normalization condition $f(0) = f'(0)-1 = 0$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions f in $\mathcal{U}$. Further, by $\mathcal{P}$ we denote the class of analytic functions p in $\mathcal{U}$ with $p(0) = 1$ and $Re\, p(z) > 0$ for all $z \in \mathcal{U}$.

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The Bessel function of the first kind of order κ ($\kappa \in \mathbb{R}$) is given by

$$w(z) = J_\kappa(z) = \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(\kappa+n+1)}\left(\frac{z}{2}\right)^{2n+\kappa}, \quad z \in \mathbb{C}. \tag{1.2}$$

We consider the normalized Bessel function of the first kind $\varphi_\kappa : \mathcal{U} \longrightarrow \mathbb{C}$ defined by

$$\varphi_\kappa(z) = 2^\kappa \Gamma(\kappa+1) z^{1-\kappa/2} J_\kappa(z^{1/2}) = z + \sum_{n=1}^{\infty} \frac{(-1)^n\, z^{n+1}}{4^n\, n!\,(\kappa+1)\cdots(\kappa+n)}, \tag{1.3}$$

where $\kappa \neq -1, -2, \ldots$. It is easy to verify that $\varphi_{1/2}(z) = \sqrt{z}\, sin\sqrt{z}$ and $\varphi_{-1/2}(z) = z\, cos\sqrt{z}$. Recently, R. Szász and P. A. Kupán [16] and Á. Baricz [1] have studied the univalence of the normalized Bessel function of the first kind $\varphi_\kappa(z)$.

Very recently several authors studied the problem of integral operators which preserve the class $\mathcal{S}$. For example, H. M. Srivastava et al. [10] and D. Breaz et al. [4] extended univalence conditions for a family of integral operators. L. F. Stanciu et al. [14] obtained some sufficient conditions for certain families of integral operators. H. M. Srivastava et al. [12] gave a set of sufficient conditions for the univalence, starlikeness and convexity of a certain newly-defined general family of integral operators in the open unit disk. E. Deniz et al. [5] gave sufficient conditions for certain families of integral operators, which are defined by means of the normalized Bessel functions, to be univalent in the open unit disk. For recent investigations on normalized Bessel functions of first kind and on the univalence of integral operators we refer to [3, 6, 11, 13, 17].

We now introduce an integral operator $F_{\kappa_1,\ldots,\kappa_n,\alpha_1,\ldots,\alpha_n,\beta,p_1,\ldots,p_m} : \mathcal{U} \longrightarrow \mathbb{C}$, involving normalized Bessel functions of the first kind, defined by

$$F(z) = F_{\kappa_1,\ldots,\kappa_n,\alpha_1,\ldots,\alpha_n,\beta,p_1,\ldots,p_m}(z) = \left\{\beta\int_0^z t^{\beta-1}\prod_{i=1}^{n}\left(\frac{\varphi_{\kappa_i}(t)}{t}\right)^{1/\alpha_i}\prod_{j=1}^{m} p_j(t)\,dt\right\}^{1/\beta}, \tag{1.4}$$

where α_i, β are nonzero complex numbers, $\kappa_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$, $p_j(z) \in \mathcal{P}$ for all $j = 1, 2, \ldots, m$ and n, m are positive integers. We remark here that if $p_1(z) = p_2(z) = \cdots = p_m(z) = 1$, then we obtain the integral operator $F_{\kappa_1,\ldots,\kappa_n,\alpha_1,\ldots,\alpha_n,\beta}(z)$ defined in [2].

As remarked, the integral operator $F(z)$ in (1.4) generalizes the integral operator defined in [2]. The objective of defining this new integral operator is to find out whether the univalence is preserved by the operator if the integrand is multiplied by a finite product of analytic functions $p(z) \in \mathcal{P}$. As a consequence, we obtain some sufficient conditions for the integral operator $F(z)$, defined by the equation (1.4), to be in the class $\mathcal{S}$. In this article, our aim is to improve the results of Á. Baricz and B.A. Frasin from [2] by giving sufficient conditions for the parameters of the integral operator $F(z)$ to be univalent in the open unit disk. To prove our results we need the following lemmas.

Lemma 1.1.([8]) *Let υ be a complex number such that $Re\,\upsilon > 0$ and let $h \in \mathcal{A}$. If*

$$\frac{1-|z|^{2\,Re\,\upsilon}}{Re\,\upsilon}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1$$

for all $z \in \mathcal{U}$, then for any complex number ω with $Re\,\omega \geq Re\,\upsilon$ the function

$$F_\omega(z) = \left\{\omega \int_0^z t^{\omega-1} h'(t) dt\right\}^{1/\omega}$$

is in the class $\mathcal{S}$.

Lemma 1.2.([9]) *Let ω be a complex number with $Re\omega > 0$ and c be a complex number such that $|c| \leq 1, \quad c \neq -1$. If $h \in \mathcal{A}$ satisfies the inequality*

$$\left|c|z|^{2\omega} + \left(1-|z|^{2\omega}\right)\frac{zh''(z)}{\omega h'(z)}\right| \leq 1$$

for all $z \in \mathcal{U}$, then the function

$$F_\omega(z) = \left\{\omega \int_0^z t^{\omega-1} h'(z)\, dt\right\}^{1/\omega}$$

is in the class $\mathcal{S}$.

Lemma 1.3.(Miller-Mocanu, [7]) *Let $w(z) = 1 + a_n z^n + \ldots$ be analytic in U with $w(z) \not\equiv 1$. If $\operatorname{Re} w(z) \not> 0,\ z \in U$, then there is a point $z_0 \in U$, and there are two real numbers $x, y \in \mathbb{R}$ such that*

(i) $w(z_0) = ix$,

(ii) $z_0 w'(z_0) = y \leq -\frac{(x^2+1)}{2}$,

(iii) $\operatorname{Re} z_0^2 w''(z_0) + z_0 w'(z_0) \leq 0$.

The following results are crucial facts in the proofs of our main results.

Lemma 1.4.([15]) *Let $\kappa \geq \kappa^* \simeq -0.7745\ldots$, and let $\varphi_\kappa(z)$ be the normalized Bessel function of the first kind defined by (1.3). If $z \in \mathcal{U}$, then we have*

$$\left|\frac{z\,\varphi_\kappa'(z)}{\varphi_\kappa(z)} - 1\right| \leq 1 - \frac{\varphi_\kappa'(1)}{\varphi_\kappa(1)}$$

and the result is sharp.

Lemma 1.5.([15]) *The mapping $u : (-1, \infty) \longrightarrow \mathbb{R}$ defined by $u(\kappa) = 1 - \varphi_\kappa'(1)/\varphi_\kappa(1)$ is strictly decreasing.*

Remark 1.6. In [2] Á. Baricz and B.A. Frasin proved the following result:
Let $\kappa \geq (-5+\sqrt{5})/4$ and consider the normalized Bessel function of the first kind $\varphi_\kappa : \mathcal{U} \to \mathbb{C}$ defined by (1.3). If $z \in \mathcal{U}$ then we have

$$\left| \frac{z\,\varphi_\kappa'(z)}{\varphi_\kappa(z)} - 1 \right| \leq \frac{\kappa+2}{4\kappa^2+10\kappa+5}, \qquad z \in \mathcal{U}.$$

This result and Lemma 1.4 imply that

$$1 - \frac{\varphi_\kappa'(1)}{\varphi_\kappa(1)} \leq \frac{\kappa+2}{4\kappa^2+10\kappa+5}, \quad \text{provided} \quad \kappa > \kappa^* \simeq -0.7745\cdots.$$

2. Main Results

Theorem 2.1. *Let $\delta \in \mathbb{R}$ with $\delta > 0$, let $\kappa_1, \ldots, \kappa_n > \kappa^* \simeq -0.7745\ldots$ and $\kappa = min\{\kappa_1, \ldots, \kappa_n\}$. Also, let $\alpha_i \in \mathbb{C} \setminus \{0\}$ for $i = 1, \ldots, n$ and let $M_j \geq 0$ and $p_j \in \mathcal{P}$ for $j = 1, \ldots, m$. If*

$$\left| \frac{z\,p_j'(z)}{p_j(z)} \right| \leq M_j, \qquad \forall\, j = 1, \ldots, m \quad (z \in \mathcal{U}) \tag{2.1}$$

$$\text{and} \qquad \left(1 - \frac{\varphi_\kappa'(1)}{\varphi_\kappa(1)}\right) \sum_{i=1}^{n} \frac{1}{|\alpha_i|} + \sum_{j=1}^{m} M_j \leq \delta, \tag{2.2}$$

then for every complex number β with $Re\,\beta \geq \delta > 0$, the function $F(z)$ defined by (1.4) is univalent in $\mathcal{U}$.

Proof. Let us consider the function

$$f(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{\varphi_{\kappa_i}(t)}{t}\right)^{1/\alpha_i} \prod_{j=1}^{m} p_j(t)\,dt.$$

Clearly $f(z) \in \mathcal{A}$, since $\varphi_{\kappa_i} \in \mathcal{A}$ for all $i = 1, \ldots, n$ and $h_j \in \mathcal{P}$ for all $j = 1, \ldots, m$. Then we have

$$f'(z) = \prod_{i=1}^{n} \left(\frac{\varphi_{\kappa_i}(z)}{z}\right)^{1/\alpha_i} \prod_{j=1}^{m} p_j(z) \quad \text{and}$$

$$\frac{z\,f''(z)}{f'(z)} = \sum_{i=1}^{n} \frac{1}{\alpha_i} \left(\frac{z\,\varphi_{\kappa_i}'(z)}{\varphi_{\kappa_i}(z)} - 1\right) + \sum_{j=1}^{m} \frac{z\,p_j'(z)}{p_j(z)}. \tag{2.3}$$

Using (2.1) and (2.3) along with Lemma 1.4 and Lemma 1.5 we obtain

$$\begin{aligned}\frac{1-|z|^{2\delta}}{\delta}\left|\frac{zf''(z)}{f'(z)}\right| &\le \frac{1-|z|^{2\delta}}{\delta}\left[\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left|\frac{z\,\varphi'_{\kappa_i}(z)}{\varphi_{\kappa_i}(z)}-1\right|+\sum_{j=1}^{m}\left|\frac{z\,p'_j(z)}{p_j(z)}\right|\right]\\ &\le \frac{1-|z|^{2\delta}}{\delta}\left[\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left(1-\frac{\varphi'_{\kappa_i}(1)}{\varphi_{\kappa_i}(1)}\right)+\sum_{j=1}^{m}M_j\right]\\ &\le \frac{1-|z|^{2\delta}}{\delta}\left[\left(1-\frac{\varphi'_{\kappa}(1)}{\varphi_{\kappa}(1)}\right)\sum_{i=1}^{n}\frac{1}{|\alpha_i|}+\sum_{j=1}^{m}M_j\right]\end{aligned}$$

and by (2.2) we have

$$\frac{1-|z|^{2\delta}}{\delta}\left|\frac{zf''(z)}{f'(z)}\right|\le 1, \qquad \forall\, z\in\mathcal{U}.$$

Consequently, in view of Lemma 1.1, we obtain that $F(z)$ is in the class $\mathcal{S}$. □

By taking $m=1$ in Theorem 2.1 we obtain the following result.

Corollary 2.2. *Let $\delta\in\mathbb{R}$ with $\delta>0$, $M\ge 0$, $p\in\mathcal{P}$, $\kappa_1,\dots,\kappa_n>\kappa^*$ and $\kappa=min\{\kappa_1,\dots,\kappa_n\}$. Also, let $\alpha_i\in\mathbb{C}\setminus\{0\}$ for $i=1,\dots,n$. If*

$$\left|\frac{z\,p'(z)}{p(z)}\right|\le M, \qquad (z\in\mathcal{U}),$$

$$\text{and}\qquad \left(1-\frac{\varphi'_{\kappa}(1)}{\varphi_{\kappa}(1)}\right)\sum_{i=1}^{n}\frac{1}{|\alpha_i|}+M\le\delta,$$

then for every complex number β with $Re\,\beta\ge\delta$, the integral operator

(2.4)

$$F_{\kappa_1,\dots,\kappa_n,\alpha_1,\dots,\alpha_n,\beta,p}(z)=\left\{\beta\int_0^z t^{\beta-1}\prod_{i=1}^{n}\left(\frac{\varphi_{\kappa_i}(t)}{t}\right)^{1/\alpha_i}p(t)\,dt\right\}^{1/\beta}, \quad (z\in\mathcal{U})$$

is in the class $\mathcal{S}$.

By choosing $n=1$ in Theorem 2.1 we have the following.

Corollary 2.3. *Let $\delta\in\mathbb{R}$ with $\delta>0$, $\kappa>\kappa^*$ and let α be a nonzero complex number. Also, let $M_j\ge 0$ and $p_j\in\mathcal{P}$ for $j=1,\dots,m$. If*

$$\left|\frac{z\,p'_j(z)}{p_j(z)}\right|\le M_j, \qquad \forall\, j=1,\dots,m \quad (z\in\mathcal{U})$$

$$\text{and}\qquad \left(1-\frac{\varphi'_{\kappa}(1)}{\varphi_{\kappa}(1)}\right)\frac{1}{|\alpha|}+\sum_{j=1}^{m}M_j\le\delta,$$

then for every complex number β with $Re\,\beta \geq \delta$, the integral operator

$$F_{\kappa,\,\alpha,\,\beta,\,p_1,\dots,\,p_m}(z) = \left\{\beta \int_0^z t^{\beta-1}\left(\frac{\varphi_\kappa(t)}{t}\right)^{1/\alpha} \prod_{j=1}^{m} p_j(t)\,dt\right\}^{1/\beta}, \quad (z \in \mathcal{U}) \tag{2.5}$$

is in the class $\mathcal{S}$.

Putting $p_j(z) = 1$, $M_j = 0$ for $j = 1, \dots, m$ and $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \frac{1}{\alpha}$ in Theorem 2.1 we have

Corollary 2.4. *Let $\delta \in \mathbb{R}$ with $\delta > 0$, let $\kappa_1, \dots, \kappa_n > \kappa^*$ and $\kappa = min\{\kappa_1, \dots, \kappa_n\}$. If $\alpha \in \mathbb{C}$ with $Re\,\alpha > 0$ and*

$$\left(1 - \frac{\varphi_\kappa'(1)}{\varphi_\kappa(1)}\right)|\alpha| \leq \frac{\delta}{n},$$

then for every complex number β with $Re\,\beta \geq \delta$, the function

$$F_{\kappa_1,\dots,\,\kappa_n,\,\alpha,\,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n}\left(\frac{\varphi_{\kappa_i}(t)}{t}\right)^{\alpha} dt\right\}^{1/\beta},$$

is univalent in $\mathcal{U}$.

The following corollary gives a better result than Theorem 2 of [2]. Replacing β by $1 + n\alpha$ in Corollary 2.4 we have

Corollary 2.5. *Let $n \in \mathbb{N}$, $\kappa_1, \dots, \kappa_n > \kappa^*$ and $\kappa = min\{\kappa_1, \dots, \kappa_n\}$. If $\alpha \in \mathbb{C}$ with $Re\,\alpha > 0$ and*

$$\left(1 - \frac{\varphi_\kappa'(1)}{\varphi_\kappa(1)}\right)|\alpha| \leq \frac{1}{n} Re\,\alpha,$$

then the function

$$F_{\kappa_1,\dots,\,\kappa_n,\,\alpha,\,n}(z) = \left\{(n\alpha + 1) \int_0^z \prod_{i=1}^{n}\left(\varphi_{\kappa_i}(t)\right)^{\alpha} dt\right\}^{1/(n\alpha+1)},$$

is univalent in $\mathcal{U}$.

We now present some simple sufficient conditions for univalence of integral operators which involve sine and cosine functions. By choosing $n = 1$ in Corollary 2.5 we have the following particular cases.

Corollary 2.6. *Let $\alpha \neq 0$ be complex number and let $\kappa > \kappa^*$. If $0 < \left(1 - \frac{\varphi_\kappa'(1)}{\varphi_\kappa(1)}\right)|\alpha| \leq$ $\mathrm{Re}\,\alpha$, then the integral operator*

$$F_{\kappa,\,\alpha}(z) = \left\{(\alpha + 1) \int_0^z \left(\varphi_\kappa(t)\right)^{\alpha} dt\right\}^{1/(\alpha+1)}, \quad (z \in \mathcal{U}) \tag{2.6}$$

is in the class $\mathcal{S}$*. In particular, if* $\frac{\sin 1-\cos 1}{2\sin 1}|\alpha| \le \operatorname{Re}\alpha$*, then the integral operator*

$$F_{1/2,\,\alpha}(z) = \left\{(\alpha+1)\int_0^z \left(\sqrt{t}\sin\sqrt{t}\right)^{\alpha} dt\right\}^{1/(\alpha+1)}, \quad (z \in \mathcal{U})$$

is in the class $\mathcal{S}$*. Moreover, if* $\frac{\sin 1}{2\cos 1}|\alpha| \le \operatorname{Re}\alpha$*, then the integral operator*

$$F_{-1/2,\,\alpha}(z) = \left\{(\alpha+1)\int_0^z \left(t\cos\sqrt{t}\right)^{\alpha} dt\right\}^{1/(\alpha+1)}, \quad (z \in \mathcal{U})$$

is in the class $\mathcal{S}$*.*

By using Lemma 1.2 , we now present another set of sufficient conditions for the integral operator $F(z)$ to be in the class $\mathcal{S}$. This theorem improves the results of Theorem 1 given in [2].

Theorem 2.7. *Let* $\kappa_1,\ldots,\kappa_n > \kappa^* \simeq -0.7745\ldots$ *and* $\kappa = min\{\kappa_1,\ldots,\kappa_n\}$*. Let* $\alpha_i \in \mathbb{C}\setminus\{0\}$*, for* $i = 1,\ldots,n$*,* $\beta \in \mathbb{C}$ *with* $Re\,\beta > 0$*. If*

$$\left|\frac{z\,p_j'(z)}{p_j(z)}\right| \le M_j, \quad M_j \ge 0,\ p_j \in \mathcal{P},\ \forall\ j = 1,\ldots,m \quad (z \in \mathcal{U}) \tag{2.7}$$

and

$$\frac{1}{|\beta|}\left[\left(1-\frac{\varphi_\kappa'(1)}{\varphi_\kappa(1)}\right)\sum_{i=1}^{n}\frac{1}{|\alpha_i|} + \sum_{j=1}^{m} M_j\right] \le 1 \tag{2.8}$$

then the function $F(z)$ *defined by (1.4) is univalent in* $\mathcal{U}$*.*

Proof. Consider

$$f(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{\varphi_{\kappa_i}(t)}{t}\right)^{1/\alpha_i}\prod_{j=1}^{m} p_j(t)\,dt.$$

Then it follows from (2.3) that

$$\left|\frac{z\,f''(z)}{f'(z)}\right| \le \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left|\frac{z\,\varphi_{\kappa_i}'(z)}{\varphi_{\kappa_i}(z)}-1\right| + \sum_{j=1}^{m}\left|\frac{z\,p_j'(z)}{p_j(z)}\right|. \tag{2.9}$$

Now, using (2.7), (2.8) and (2.9) along with Lemma 1.4 and Lemma 1.5 , we obtain

$$\begin{aligned}
&\left| |z|^{2\beta} + \left(1-|z|^{2\beta}\right)\frac{zf''(z)}{\beta f'(z)} \right| \\
&\quad \le |z|^{2\beta} + (1-|z|^{2\beta})\frac{1}{|\beta|}\left(\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left|\frac{z\,\varphi'_{\kappa_i}(z)}{\varphi_{\kappa_i}(z)} - 1\right| + \sum_{j=1}^{m}\left|\frac{z\,p'_j(z)}{p_j(z)}\right|\right) \\
&\quad \le |z|^{2\beta} + (1-|z|^{2\beta})\frac{1}{|\beta|}\left[\sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left(1-\frac{\varphi'_{\kappa_i}(1)}{\varphi_{\kappa_i}(1)}\right) + \sum_{j=1}^{m}M_j\right] \\
&\quad \le |z|^{2\beta} + (1-|z|^{2\beta})\frac{1}{|\beta|}\left[\left(1-\frac{\varphi'_{\kappa}(1)}{\varphi_{\kappa}(1)}\right)\sum_{i=1}^{n}\frac{1}{|\alpha_i|} + \sum_{j=1}^{m}M_j\right] \\
&\quad \le |z|^{2\beta} + (1-|z|^{2\beta}) = 1.
\end{aligned}$$

Finally by applying Lemma 1.2 with $c = 1$, we conclude that $F(z) \in \mathcal{S}$. $\square$

By taking $m = 1$ in Theorem 2.7 we obtain the following result.

Corollary 2.8. *Let $\kappa_1, \dots, \kappa_n > \kappa^*$ and $\kappa = min\{\kappa_1, \dots, \kappa_n\}$. Let $\alpha_i \in \mathbb{C}\setminus\{0\}$, for $i = 1, \dots, n$, $\beta \in \mathbb{C}$ with $Re\,\beta > 0$. If*

$$\left|\frac{z\,p'(z)}{p(z)}\right| \le M, \quad M \ge 0,\ p \in \mathcal{P},\ z \in \mathcal{U}$$

$$\text{and} \qquad \frac{1}{|\beta|}\left[\left(1-\frac{\varphi'_{\kappa}(1)}{\varphi_{\kappa}(1)}\right)\sum_{i=1}^{n}\frac{1}{|\alpha_i|} + M\right] \le 1,$$

then the integral operator $F_{\kappa_1,\dots,\kappa_n,\alpha_1,\dots,\alpha_n,\beta,p}(z)$ defined by (2.4) is in the class $\mathcal{S}$.

Choosing $n = 1$ in Theorem 2.7 we have the following.

Corollary 2.9. *Let α be a nonzero complex number, $\beta \in \mathbb{C}$ with $Re\,\beta > 0$ and let $\kappa > \kappa^*$. If*

$$\left|\frac{z\,p'_j(z)}{p_j(z)}\right| \le M_j, \quad M_j \ge 0,\ p_j \in \mathcal{P},\ \forall\, j = 1, \dots, m \quad (z \in \mathcal{U})$$

$$\text{and} \qquad \frac{1}{|\beta|}\left[\frac{1}{|\alpha|}\left(1-\frac{\varphi'_{\kappa}(1)}{\varphi_{\kappa}(1)}\right) + \sum_{j=1}^{m}M_j\right] \le 1,$$

then the integral operator $F_{\kappa,\alpha,\beta,p_1,\dots,p_m}(z)$ defined by (2.5) is in the class $\mathcal{S}$.

Now by choosing $n = 1$ and $p_j(z) = 1$, $M_j = 0$ for $j = 1, \dots, m$ in Theorem 2.7 we have the following results. It is interesting to note that these results improve the conditions for univalence of the integral operators $F_{\kappa,\alpha,\beta}(z)$, $F_{1/2,\alpha,\beta}(z)$ and $F_{-1/2,\alpha,\beta}(z)$ given in Corollary 2 of [2].

Corollary 2.10. *Let α be a nonzero complex number, $\beta \in \mathbb{C}$ with $Re\,\beta > 0$ and let $\kappa > \kappa^*$. If*

$$\left(1 - \frac{\varphi_\kappa'(1)}{\varphi_\kappa(1)}\right) \leq |\alpha\beta|,$$

then the integral operator $F_{\kappa,\,\alpha,\,\beta}(z)$ defined by

$$F_{\kappa,\,\alpha,\,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1}\left(\frac{\varphi_\kappa(t)}{t}\right)^{1/\alpha} dt\right\}^{1/\beta}, \quad (z \in \mathcal{U})$$

is in the class $\mathcal{S}$.
In particular, if $\frac{\sin 1 - \cos 1}{2\sin 1} \leq |\alpha\beta|$, then the integral operator

$$F_{1/2,\,\alpha,\,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1}\left(\frac{sin\sqrt{t}}{\sqrt{t}}\right)^{1/\alpha} dt\right\}^{1/\beta}, \quad (z \in \mathcal{U})$$

is in the class $\mathcal{S}$.
Moreover, if $\frac{\sin 1}{2\cos 1} \leq |\alpha\beta|$, then the integral operator

$$F_{-1/2,\,\alpha,\,\beta}(z) = \left\{\beta \int_0^z t^{\beta-1}\left(cos\sqrt{t}\right)^{1/\alpha} dt\right\}^{1/\beta}, \quad (z \in \mathcal{U})$$

is in the class $\mathcal{S}$.

The following theorem shows that if the condition "$\beta \in \mathbb{C}$ with $Re\,\beta > 0$" is replaced by a stronger one "$\beta \in (0,\infty)$" in Theorem 2.7, then we get a much better result.

Theorem 2.11. *Let $\kappa_1, \ldots, \kappa_n > \kappa^* \simeq -0.7745\ldots$ and $\kappa = min\{\kappa_1, \ldots, \kappa_n\}$. Let $\alpha_i \in \mathbb{C} \setminus \{0\}$, for $i = 1, \ldots, n$, and $\beta \in (0,\infty)$. If*

$$\left|\frac{z\,p_j'(z)}{p_j(z)}\right| \leq M_j, \quad M_j \geq 0,\ p_j \in \mathcal{P},\ \forall\ j = 1, \ldots, m, \quad (z \in \mathcal{U}),$$

and

$$\frac{1}{|\beta|}\left[\left(1 - \frac{\varphi_\kappa'(1)}{\varphi_\kappa(1)}\right)\sum_{i=1}^n \frac{1}{|\alpha_i|} + \sum_{j=1}^m M_j\right] \leq 1 \tag{2.10}$$

then the function $F(z)$ defined by (1.4) is starlike-univalent in $\mathcal{U}$. Further, if $\beta = 1$, then the function $F(z)$ is convex-univalent in $\mathcal{U}$.

Proof. Let ϕ be the function defined by $\phi(z) = \frac{zF'(z)}{F(z)}$. Then from (1.4) we get

$$\phi(z)\beta + \frac{z\phi'(z)}{\phi(z)} = \beta + \sum_{i=1}^n \frac{1}{\alpha_i}\left(\frac{z\,\varphi_{\kappa_i}'(z)}{\varphi_{\kappa_i}(z)} - 1\right) + \sum_{j=1}^m \frac{z\,p_j'(z)}{p_j(z)}. \tag{2.11}$$

It is easily seen that (2.10) implies

$$\beta+\sum_{i=1}^{n}\frac{1}{\alpha_i}\left(\frac{z\,\varphi'_{\kappa_i}(z)}{\varphi_{\kappa_i}(z)}-1\right)+\sum_{j=1}^{m}\frac{z\,p'_j(z)}{p_j(z)}\geq 0,\ \ z\in\mathcal{U}. \tag{2.12}$$

Thus (2.11), (2.12) and the minimum principle for harmonic functions imply that

$$\phi(z)\beta+\frac{z\phi'(z)}{\phi(z)}>0,\quad z\in\mathcal{U}. \tag{2.13}$$

If the inequality $\operatorname{Re}\phi(z)>0,\ \ z\in\mathcal{U}$, does not hold, then according to the Miller-Mocanu lemma there is a point $z_0\in U$ and there are two real numbers $x,y\in\mathbb{R}$ such that

(i) $\phi(z_0)=ix$

(ii) $z_0\phi'(z_0)=y\leq-\frac{(x^2+1)}{2}$.

Thus we get $\operatorname{Re}\left(\phi(z_0)\beta+\frac{z_0\phi'(z_0)}{\phi(z_0)}\right)=\operatorname{Re}\left(ix+\frac{y}{ix}\right)=0$ and this inequality contradicts (2.13) and consequently F is starlike-univalent in $\mathcal{U}$.

If $\beta=1$, then we get

$$1+\frac{zF''(z)}{F'(z)}=1+\sum_{i=1}^{n}\frac{1}{\alpha_i}\left(\frac{z\,\varphi'_{\kappa_i}(z)}{\varphi_{\kappa_i}(z)}-1\right)+\sum_{j=1}^{m}\frac{z\,p'_j(z)}{p_j(z)}.$$

This equality along with (2.12) implies that $1+\frac{zF''(z)}{F'(z)}>0$, $(z\in\mathcal{U})$ and consequently F is convex when $\beta=1$. □

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References

[1] Á. Baricz, *Geometric properties of generalized Bessel functions*, Publ. Math. Debrecen, **73(1-2)**(2008), 155–178.

[2] Á. Baricz and B. A. Frasin, *Univalence of integral operators involving Bessel functions*, Appl. Math. Lett., **23(4)**(2010), 371–376.

[3] Á. Baricz and S. Ponnusamy, *Starlikeness and convexity of generalized Bessel functions*, Integral Transforms Spec. Func., **21(9)**(2010), 641–653.

[4] D. Breaz, N. Breaz and H. M. Srivastava, *An extension of the univalent condition for a family of integral operators*, Appl. Math. Lett., **22**(2009), no. 1, 41–44.

[5] E. Deniz, H. Orhan and H. M. Srivastava, *Some sufficient conditions for univalence of certain families of integral operators involving generalized Bessel functions*, Taiwanese J. Math., **15(2)**(2011), 883–917.

[6] B. A. Frasin, *Sufficient conditions for integral operator defined by Bessel functions*, J. Math. Inequal., **4(2)**(2010), 301–306.

[7] S. S. Miller and P. T. Mocanu, *Differential subordinations*, Monographs and Textbooks in Pure and Applied Mathematics **225**, Marcel Dekker, Inc., New York, 2000.

[8] N. N. Pascu, *An improvement of Becker's univalence criterion*, Proceedings of the Commemorative Session: Simion Stoïlow (Braşov, 1987), 43–48, Univ. Braşov, Braşov, 1987.

[9] V. Pescar, *A new generalization of Ahlfors's and Becker's criterion of univalence*, Bull. Malaysian Math. Soc. (2), **19(2)**(1996), 53–54.

[10] H. M. Srivastava, E. Deniz and H. Orhan, *Some general univalence criteria for a family of integral operators*, Appl. Math. Comput., **215(10)**(2010), 3696–3701.

[11] H. M. Srivastava, B. A. Frasin and V. Pescar, *Univalence of integral operators involving Mittag-Leffler functions*, Appl. Math. Inf. Sci., **11(3)**(2017), 635–641.

[12] H. M. Srivastava, J. K. Prajapat, G. I. Oros and R. Sendrutiu, *Geometric properties of a certain general family of integral operators*, Filomat, **28(4)**(2014), 745–754.

[13] H. M. Srivastava, K. A. Selvakumaran and S. D. Purohit, *Inclusion properties for certain subclasses of analytic functions defined by using the generalized Bessel functions*, Malaya J. Mat., **3(3)**(2015), 360–367.

[14] L. F. Stanciu, D. Breaz and H. M. Srivastava, *Some criteria for univalence of a certain integral operator*, Novi Sad J. Math., **43(2)**(2013), 51–57.

[15] R. Szász, *About the starlikeness of Bessel functions*, Integral Transforms Spec. Funct., **25(9)**(2014), 750–755.

[16] R. Szász and P. A. Kupán, *About the univalence of the Bessel functions*, Stud. Univ. Babeş-Bolyai Math., **54(1)** (2009), 127–132.

[17] N. Ularu, Two integral operators defined with Bessel functions on the class $N(\beta)$, J. Basic Apll. Sci., **9**(2013), 57–59.