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Hyperinvariant Subspaces for Some 2×2 Operator Matrices

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Abstract. The first purpose of this note is to generalize two nice theorems of H. J. Kim concerning hyperinvariant subspaces for certain classes of operators on Hilbert space, proved by him by using the technique of "extremal vectors". Our generalization (Theorem 1.2) is obtained as a consequence of a new theorem of the present authors, and doesn't utilize the technique of extremal vectors. The second purpose is to use this theorem to obtain the existence of hyperinvariant subspaces for a class of 2×2 operator matrices (Theorem 3.2).

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, we write $\{T\}'$ for the commutant of T (i.e., for the algebra of all $S \in \mathcal{B}(\mathcal{H})$ such that $TS = ST$). A subspace $\mathcal{M} \subset \mathcal{H}$ is *invariant* for T in $\mathcal{B}(\mathcal{H})$ if $T\mathcal{M} \subset \mathcal{M}$, and a subspace $\mathcal{M}$ is *hyperinvariant* for T if it is an invariant subspace for all S in $\{T\}'$. The question whether every operator in $\mathcal{B}(\mathcal{H})$ has a nontrivial invariant subspace, which has been around since von Neumann studied it in the 1930's, is still an open problem.

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Moreover the question of whether every operator in $\mathcal{B}(\mathcal{H}) \setminus \mathbb{C}1_{\mathcal{H}}$ has a nontrivial hyperinvariant subspace is also open. The results in this note contribute to this circle of ideas.

In two recent papers [5] and [6], H. J. Kim, using the technique of "extremal vectors" introduced by Ansari and Enflo in [1] (for more information about this technique, see the book [3]), proved two nice theorems which we combine as

Theorem 1.1.(H. J. Kim) *Let $T \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ be given matricially as*

$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where A, B, and C are arbitrary operators in $\mathcal{B}(\mathcal{H})$ such that A is either a nonzero compact operator or a nonscalar normal operator. Then at least one of T and

$$\widetilde{T} = \begin{pmatrix} B & D \\ 0 & A \end{pmatrix},$$

where D is arbitrary operator in $\mathcal{B}(\mathcal{H})$, has a nontrivial hyperinvariant subspace (*notation: n.h.s.*).

The purpose of this note is first to give a short and simpler proof of a better theorem, and then to use this result to obtain the existence of n.h.s. for a class of 2×2 matrices with operator entries (Theorem 3.2). Our first result is the following.

Theorem 1.2. *Let A and B be operators in $\mathcal{B}(\mathcal{H})$ such that either A or B has a n.h.s. Then either*

i) *for every operator C in $\mathcal{B}(\mathcal{H})$, the operator $T_C \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ given matricially as*

$$T_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \tag{1.1}$$

has a n.h.s., or

ii) *for every operator D in $\mathcal{B}(\mathcal{H})$, the operator $\widetilde{T}_D$ given matricially as*

$$\widetilde{T}_D = \begin{pmatrix} B & D \\ 0 & A \end{pmatrix}, \tag{1.2}$$

has a n.h.s.

2. Preliminary Results

Before proving Theorem 1.2, we obtain two needed results. The first one is a new theorem of the present authors, and the second is an elementary proposition about transitive subalgebras of $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ (i.e., subalgebras $\mathcal{S}$ with the property that

$\mathcal{S}$ has no nontrivial invariant subspace). The first of these two results generalizes theorems from [4].

Theorem 2.1. *Suppose $T = T_C$ is as in* (1.1)*, where A, B, and C are arbitrary operators in $\mathcal{B}(\mathcal{H})$ and there exists X in $\mathcal{B}(\mathcal{H})$ satisfying $AX = XB$. If either*

a) *there exists a n.h.s. $\mathcal{M}$ for A such that $X\mathcal{H} \not\subset \mathcal{M}$, or*

b) *there exists a n.h.s. $\mathcal{N}$ for B such that* $\ker X \not\supset \mathcal{N}$,

then T has a n.h.s.

Proof. We prove b); the proof of a) is quite similar and left to the reader. Thus we are given a n.h.s. $\mathcal{N}$ for B and an operator X in $\mathcal{B}(\mathcal{H})$ such that $AX = XB$ and $\ker X \not\supset \mathcal{N}$. With T as in (1.1), denote its commutant by

$$\{T\}' = \left\{ \begin{pmatrix} L_\sigma & M_\sigma \\ N_\sigma & P_\sigma \end{pmatrix} : \sigma \in \Sigma \right\}. \tag{2.1}$$

Since every matrix in (2.1) commutes with T, upon doing the matrix multiplication, for each $\sigma \in \Sigma$ we obtain four equations, with the one corresponding to the $(2,1)$ entry of the product being

$$BN_\sigma = N_\sigma A, \qquad \sigma \in \Sigma. \tag{2.2}$$

Multiplication of (2.2) on the right by X gives

$$BN_\sigma X = N_\sigma AX = N_\sigma XB, \qquad \sigma \in \Sigma,$$

so for each $\sigma \in \Sigma$, $N_\sigma X$ commutes with B and therefore $N_\sigma X\mathcal{N} \subset \mathcal{N}$, $\sigma \in \Sigma$. By hypothesis there exists $y \in \mathcal{N}$ such that $Xy \neq 0$. Finally, let us suppose, to obtain a contradiction, that $\{T\}'$ is transitive. It then follows easily from Proposition 2.2 below (which is completely independent of this theorem) that $\{N_\sigma Xy : \sigma \in \Sigma\}^- = \mathcal{H}$, which contradicts the fact that $N_\sigma Xy \in \mathcal{N}$ for all $\sigma \in \Sigma$. Thus $\{T\}'$ is not transitive and the proof is complete. □

Proposition 2.2. *Suppose that*

$$\mathcal{S} = \left\{ \begin{pmatrix} L_\sigma & M_\sigma \\ N_\sigma & P_\sigma \end{pmatrix} : \sigma \in \Sigma \right\}$$

is a transitive subalgebra of $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$, and let x, $y \in \mathcal{H}$ with $x \neq 0$. Then for every $\varepsilon > 0$, there exists L_{σ_1} [respectively, M_{σ_2}, N_{σ_3}, P_{σ_4}] such that $\|L_{\sigma_1}x - y\| < \varepsilon$ [respectively, $\|M_{\sigma_2}x - y\| < \varepsilon$, $\|N_{\sigma_3}x - y\| < \varepsilon$, $\|P_{\sigma_4}x - y\| < \varepsilon$].

Proof. It is well-known (cf., e.g., [8]) that every transitive subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is 1-transitive, meaning that for every $x \neq 0$ and y in $\mathcal{H}$ and every $\varepsilon > 0$, there exists $A_\varepsilon \in \mathcal{A}$ such that $\|A_\varepsilon x - y\| < \varepsilon$. If we apply this fact to the transitive subalgebra $\mathcal{S} \subset \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ and the vectors $(x,0)^t$ and $(y,0)^t$, we obtain an element

$$\begin{pmatrix} L_\sigma & M_\sigma \\ N_\sigma & P_\sigma \end{pmatrix} \in \mathcal{S}$$

such that

$$\begin{aligned}\varepsilon &> \left\| \begin{pmatrix} L_\sigma & M_\sigma \\ N_\sigma & P_\sigma \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} - \begin{pmatrix} y \\ 0 \end{pmatrix} \right\| \\ &= \left(\|L_\sigma x - y\|^2 + \|N_\sigma x\|^2 \right)^{1/2} \geq \|L_\sigma x - y\| .\end{aligned}$$

By changing the positions of the vectors x and y in the direct sum $\mathcal{H} \oplus \mathcal{H}$, the other three desired inequalities follow similarly. □

Proof of Theorem 1.2. We shall prove i). The proof of ii) is essentially the same. To establish i) there are two cases - that in which A has a n.h.s. and that in which B has a n.h.s. Once again the proofs are virtually indistinguishable, so we shall content ourselves with proving i) under the condition that B has a n.h.s. $\mathcal{N}$. Thus, by virtue of Theorem 2.1 b), it is sufficient to exhibit an operator X in $\mathcal{B}(\mathcal{H})$ and a nonzero vector $v \in \mathcal{N}$ such that $AX = XB$ and $Xv \neq 0$. We may and do suppose that ii) is false, that is, there exists D_0 in $\mathcal{B}(\mathcal{H})$ such that $\widetilde{T}_{D_0}$ as in (1.2) has no n.h.s. We write the commutant of $\{\widetilde{T}_{D_0}\}$ as

$$\{\widetilde{T}_{D_0}\}' = \left\{ \begin{pmatrix} L'_\sigma & M'_\sigma \\ N'_\sigma & P'_\sigma \end{pmatrix} : \sigma \in \Sigma \right\}, \tag{2.3}$$

and we are given that the algebra $\{\widetilde{T}_{D_0}\}'$ is transitive. Since T_{D_0} commutes with every matrix in (2.3), we obtain, for each $\sigma \in \Sigma$, four equations, one of which is

$$AN'_\sigma = N'_\sigma B, \qquad \sigma \in \Sigma.$$

Let now v_0 be an arbitrary nonzero vector in $\mathcal{N}$. It is an easy consequence of Proposition 1.4 that there exists $\sigma_0 \in \Sigma$ such that $N'_{\sigma_0} v_0 \neq 0$ (take $x = v_0$, y a unit vector in $\mathcal{H}$, and $\varepsilon = 1/2$).

The proof is completed by setting $v = v_0$ and $X = N'_{\sigma_0}$. □

Recall finally that for $T \in \mathcal{B}(\mathcal{H})$, then $\mathrm{Lat}(T)$ is by definition the lattice of all invariant subspaces of T and AlgLat T is the algebra of all operators A in $\mathcal{B}(\mathcal{H})$ such that $\mathrm{Lat}(T) \subset \mathrm{Lat}(A)$. An operator T in $\mathcal{B}(\mathcal{H})$ is said to be *reflexive* if $\mathrm{AlgLat}(T) = \mathcal{W}_T$, the smallest unital subalgebra of $\mathcal{B}(\mathcal{H})$ that contains T and is closed in the weak operator topology on $\mathcal{B}(\mathcal{H})$.

Corollary 2.3. *Suppose that A is an arbitrary nonreflexive contraction in $\mathcal{B}(\mathcal{H})$ such that the spectrum of A contains the unit circle. Let also B, C, and D be arbitrary operators in $\mathcal{B}(\mathcal{H})$. Then either* i) *or* ii) *of Theorem 1.2 is valid.*

Proof. One knows from [2, Corollary 7.3] that every such operator A has a n.h.s. □

3. Applications

In this section we show that in certain situations Theorem 1.2 can be applied to yield a n.h.s. for the operator T_C in (1.1).

Proposition 3.1. *Suppose $A \in \mathcal{B}(\mathcal{H})$ has a n.h.s., and B is an arbitrary operator in $\mathcal{B}(\mathcal{H})$ such that B commutes with A. Then the operator $Q \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ given matricially by*

$$Q = \begin{pmatrix} A & 1_{\mathcal{H}} \\ 0 & B \end{pmatrix}$$

has a n.h.s.

Proof. We know from Theorem 1.2 that either Q or

$$\widetilde{Q} = \begin{pmatrix} B & 1_{\mathcal{H}} \\ 0 & A \end{pmatrix}$$

has a n.h.s., so it suffices to prove that Q and $\widetilde{Q}$ are similar, which we now do. It is well-known that the operator $S \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ given by

$$S = \begin{pmatrix} 1_{\mathcal{H}} & 0 \\ X & 1_{\mathcal{H}} \end{pmatrix},$$

where X is arbitrary in $\mathcal{B}(\mathcal{H})$, is invertible, and

$$S^{-1} = \begin{pmatrix} 1_{\mathcal{H}} & 0 \\ -X & 1_{\mathcal{H}} \end{pmatrix}.$$

Therefore we calculate $S^{-1}QS$, where X is yet to be chosen. Thus

$$\begin{aligned} S^{-1}QS &= \begin{pmatrix} 1_{\mathcal{H}} & 0 \\ -X & 1_{\mathcal{H}} \end{pmatrix} \begin{pmatrix} A & 1_{\mathcal{H}} \\ 0 & B \end{pmatrix} \begin{pmatrix} 1_{\mathcal{H}} & 0 \\ X & 1_{\mathcal{H}} \end{pmatrix} \\ &= \begin{pmatrix} A + X & 1_{\mathcal{H}} \\ -XA + (-X + B)X & -X + B \end{pmatrix} \\ &= \begin{pmatrix} B & 1_{\mathcal{H}} \\ 0 & A \end{pmatrix} = \widetilde{Q} \end{aligned}$$

if X is chosen to be $X = B - A$. □

The following shows that Theorem 1.2 can be useful in obtaining a n.h.s. for certain classes of operators.

Theorem 3.2. *Suppose $A \in \mathcal{L}(\mathcal{H})$ has a n.h.s., and B and C are arbitrary operators in $\mathcal{L}(\mathcal{H})$ that commute with A, with C invertible. Then*

$$T_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

has a n.h.s.

Proof. We know from Theorem 3.1 that

$$Q = \begin{pmatrix} A & 1_{\mathcal{H}} \\ 0 & B \end{pmatrix}$$

has a n.h.s., and the calculation

$$\begin{pmatrix} C^{-1} & 0 \\ 0 & 1_{\mathcal{H}} \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & 1_{\mathcal{H}} \end{pmatrix} = \begin{pmatrix} C^{-1}AC & 1_{\mathcal{H}} \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 1_{\mathcal{H}} \\ 0 & B \end{pmatrix}$$

shows that T_C and Q are similar. □

Remark 3.3. It would be interesting to show that Theorem 3.2 remains valid without the commutativity assumptions made there.

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