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Notes on Chain Rings and Radicals

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Abstract. We investigate connections in the classes of rings with chain property and the lattice of strongly hereditary radicals.

1. Introduction

In this paper we will study associative rings, not necessarily with identity. The notation $I \trianglelefteq A$ means that I is an ideal of a ring A. Recall that a (Kurosh-Amitsur) radical γ is a class of rings which

(i) is closed under homomorphic images,

(ii) is closed under extensions (for I an ideal of the ring A, if I and A/I are in γ, then also $A \in \gamma$),

(iii) has the inductive property (if $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_\lambda \ldots$ is a chain of ideals in the ring $A = \cup I_\lambda$ and each $I_\lambda \in \gamma$, then $A \in \gamma$).

We denote by $\mathcal{L}(\mathcal{M})$, the lower radical class generated by a class $\mathcal{M}$ of rings. It is well known that the collection L of all radical classes forms a complete lattice with respect to inclusion of radical classes, where the meet and the join of a family of

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radical classes γ_λ, $\lambda \in \Lambda$ are defined by

$$\underset{\lambda\in\Lambda}{\wedge} \gamma_\lambda = \underset{\lambda\in\Lambda}{\cap} \gamma_\lambda \quad \text{and} \quad \underset{\lambda\in\Lambda}{\vee} \gamma_\lambda = \mathcal{L}\left(\underset{\lambda\in\Lambda}{\cup} \gamma_\lambda\right),$$

respectively. A radical class will always mean a Kurosh-Amitsur radical class. Sometimes we say only radical for a radical class. For the basic facts and terminology of radical theory we refer to [1]. Although collections of radicals do not form a set, it is custommary to talk about lattices of radicals. We denote by Ass the class of all associative rings. We recall, a radical γ is **small** [2, 6] if and only if

$$\gamma \vee \gamma' \neq Ass$$

for each proper radical γ'. Dually, call a non zero radical γ **large** if and only if

$$\gamma \cap \gamma' \neq 0$$

for each proper radical γ'.

Let $\mathcal{M}$ be a class of rings. We recall that $\mathcal{M}$ is an universal class of rings, if $\mathcal{M}$ is closed under homomorphic images and ideals. From [3], recall a relation σ on the class of rings is called an H relation if σ satisfies the following properties:

(i) $B\sigma A$ implies B is subring of A,

(ii) if $B\sigma A$ and f is a homomorphism of A, then $f(B)\sigma f(A)$,

(iii) if $B\sigma A$ and $I \trianglelefteq A$ then $(B \cap I)\sigma I$.

In this paper we assume that the H relation σ satisfies also the following additional condition:

(iv) if f is a homomorphism of A and $f(B)\sigma f(A)$ then also $B\sigma A$.

We also recall, a class $\mathcal{M}$ of rings is said to be σ-hereditary if $B\sigma A \in \mathcal{M}$ implies $B \in \mathcal{M}$.

There exist many such H relations with property (iv) (see [3]).

Proposition 1.1.([3, Theorem 4]) *Let σ be an H relation. If $\mathcal{M}$ is a class of rings which is closed under homomorphic images and is σ-hereditary, then $\mathcal{L}(\mathcal{M})$ is also $\sigma-$hereditary.*

2. Chain Rings

Definition 2.1. A ring A is said to *have the chain property* if either

$$S \subseteq S_1 \text{ or } S_1 \subseteq S$$

for any subrings S and S_1 of A.

We denote by $\langle a\rangle$ the subring of A generated by the element $a \in A$.

Proposition 2.2. *A ring A has the chain property if and only if either*

$$\langle a\rangle \subseteq \langle b\rangle \quad or \quad \langle b\rangle \subseteq \langle a\rangle$$

for any $a, b \in A$.

Proof. $(\Rightarrow)$ clear.
$(\Leftarrow)$ Suppose that the subrings S, S_1 of A fulfills $S \nsubseteq S_1$ and $S_1 \nsubseteq S$. Then there exist elements a, b of A such that $a \notin S, a \in S_1$ and $b \notin S_1, b \in S$. By the assumption, we have either $\langle a\rangle \subseteq \langle b\rangle$ or $\langle b\rangle \subseteq \langle a\rangle$. If $\langle a\rangle \subseteq \langle b\rangle$ then $\langle a\rangle \subseteq \langle b\rangle \subseteq S$. Hence $a \in S$. This is a contradiction. Therefore, $\langle b\rangle \subseteq \langle a\rangle \subseteq S_1$. Thus $b \in S_1$, again a contradiction. Hence we have either $S \subseteq S_1$ or $S_1 \subseteq S$. It shows that A is a ring with the chain property. □

Corollary 2.3. *Let A be a ring with the chain property. Then A is commutative.*

Proof. We consider elements $a, b \in A$. Then either $\langle a\rangle \subseteq \langle b\rangle$ or $\langle b\rangle \subseteq \langle a\rangle$ by Proposition 2.2. Suppose that $\langle a\rangle \subseteq \langle b\rangle$. Then $\langle b\rangle$ is a commutative ring we have $[a, b] = 0$. □

Let CH be the class of rings defined by

$$CH = \{A \mid A \text{ is a ring with the chain property}\}$$

A class $\mathcal{M}$ of rings said to be *matrix-extensible* if $A \in \mathcal{M}$ if and only if the matrix ring $M_n(A) \in \mathcal{M}$ for any natural number n.

Corollary 2.4. *CH is not matrix extensible.*

Proof. It is easy to see that $\mathbb{Z}_p \in CH$, where p is a prime number. If $M_n(\mathbb{Z}_p) \in CH$, where $n \geq 2$, then by Corollary 2.3, $M_n(\mathbb{Z}_p)$ is a commutative ring. But $M_n(\mathbb{Z}_p)$ is not commutative. Thus $M_n(\mathbb{Z}_p) \notin CH$. □

We recall that a class $\mathcal{M}$ of rings said to be strongly hereditary if it satisfies: If A is a ring in $\mathcal{M}$, then every subring S of A is in $\mathcal{M}$.

Proposition 2.5. *CH is a strongly hereditary universal class of rings.*

Proof. We shall show that CH is strongly hereditary. Let $A \in CH$ and S is a subring of A. Since A has the chain property, for any $a, b \in S \subseteq A$, we have either $\langle a\rangle \subseteq \langle b\rangle$ or $\langle b\rangle \subseteq \langle a\rangle$. Thus, by Proposition 2.2, S has the chain property. This shows that CH is a strongly hereditary. In particular, CH is hereditary class of rings. Now we claim that CH is closed class under homomorphic images. Let $\overline{A} = A/H$ be a homomorphic image of $A \in CH$. We consider any subrings $\overline{S}, \overline{S_1}$ of $\overline{A}$. Then there exist subrings S, S_1 of A such that $\overline{S} = \dfrac{S}{H}$, $\overline{S_1} = \dfrac{S_1}{H}$, where $H \subseteq S \cap S_1$. Since A is in CH, we have either $S \subseteq S_1$ or $S_1 \subseteq S$. If $S \subseteq S_1$, then $\overline{S} = \dfrac{S}{H} \subseteq \dfrac{S_1}{H} = \overline{S_1}$. Therefore $\overline{S} \subseteq \overline{S_1}$. The other case gives $\overline{S_1} \subseteq \overline{S}$. □

Proposition 2.6. *CH has the inductive property.*

Proof. Let $A = \cup I_\alpha$ be a ring, where

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_\alpha \subseteq \ldots$$

with each $I_\alpha \trianglelefteq A$ and $I_\alpha \in CH$. We consider any elements $a, b \in A$. Then there exists I_α, such that $a, b \in I_\alpha$. Since $I_\alpha \in CH$, we have either $\langle a \rangle \subseteq \langle b \rangle$ or $\langle b \rangle \subseteq \langle a \rangle$. Therefore, by Proposition 2.2, $A \in CH$. □

Theorem 2.7. *$\mathcal{L}(CH)$ is strongly hereditary and large in the lattice of all strongly hereditary radicals. Moreover it contains all atoms of the lattice of all strongly hereditary radicals.*

Proof. We shall show that $\mathcal{L}(CH)$ is strongly hereditary and by Proposition 1.1, $\mathcal{L}(CH)$ is strongly hereditary in the special case $\sigma =$"subring of". Now we claim that $\mathcal{L}(CH)$ is a large radical in the lattice of all strongly hereditary radicals.

First of all, we will be see that every non zero strongly hereditary radical γ contains a prime field $\mathbb{Z}_p$ or a simple zero ring $\mathbb{Z}_p^0$ with prime order. Let us consider a ring $A \in \gamma$ and a nonzero element $a \in A$. Since γ is strongly hereditary, the subring $\langle a \rangle \in \gamma$. Using Zorn's lemma, there exists an ideal I of $\langle a \rangle$ which is maximal respect to $a \notin I$. Then the factor ring $\overline{\langle a \rangle} = \langle a \rangle / I$ is a simple ring and $\overline{\langle a \rangle} \in \gamma$.

If $\overline{\langle a \rangle}^2 = \overline{0}$, then by the simplicity of $\overline{\langle a \rangle}$, $\overline{\langle a \rangle}$ is a zero ring of prime order. If $\overline{\langle a \rangle}^2 \neq \overline{0}$ then by the commutativity of $\overline{\langle a \rangle}$, $\overline{\langle a \rangle}$ is a field. Thus the subring of $\overline{\langle a \rangle}$ generated by the unit element of $\overline{\langle a \rangle}$ is isomorphic to the ring $\mathbb{Z}$ of integers or to the prime field $\mathbb{Z}_p$ of p elements. By the strong hereditariness of γ the relation $\overline{\langle a \rangle} \in \gamma$ implies $\mathbb{Z} \in \gamma$ or $\mathbb{Z}_p \in \gamma$ holds and in both cases $\mathbb{Z}_p \in \gamma$. Thus every strongly hereditary radical γ contains either a finite prime field or a simple zero-ring with prime order. But it is clear that CH contains all finite prime fields and all simple zero-rings with prime order. Thus $\mathcal{L}(CH) \cap \gamma \neq 0$, for every strongly hereditary radical $0 \neq \gamma$. Hence $\mathcal{L}(CH)$ is a large radical in the lattice of all strongly hereditary radicals.

From the above, every atom γ_0 in the lattice of all strongly hereditary radicals is generated by either a finite prime field or a simple zero-ring of prime order. Thus $\gamma_0 \subseteq \mathcal{L}(CH)$. □

We denote by $\mathbb{L}_s$ the collection of all strongly hereditary and large radicals.

Proposition 2.8. *$\mathbb{L}_s$ is a complete sublattice in the lattice of all strongly hereditary radicals. $\mathbb{L}_s$ is atomic and not coatomic.*

Proof. We consider radicals $\gamma_1, \ldots, \gamma_\alpha \ldots$ such that $\gamma_\alpha \in \mathbb{L}_s$. Since $\gamma_\alpha \in \mathbb{L}_s$ and each γ_s is large in the lattice of all strongly hereditary radicals, each γ_α contains all simple zero-rings with prime order and all prime fields. By Proposition 1.1, $\mathcal{L}(\cup\gamma_\alpha)$ is strongly hereditary. It is clear that $\cap\gamma_\alpha$ is strongly hereditary. Hence $\mathcal{L}(\cup\gamma_\alpha)$ and $\cap\gamma_\alpha$ contain all simple zero-rings with prime order and all prime fields. Therefore $\mathcal{L}(\cup\gamma_\alpha)$ and $\cap\gamma_\alpha$ are large radicals in the class of all strongly hereditary radicals.

We denote by γ_0 the lower radical generated by all simple zero-rings with prime order and all prime fields. Then it is clear that γ_0 is an atom in $\mathbb{L}_s$.

Let $X = \{x_1, \ldots, x_\lambda, \ldots\}$ be an infinite set of symbols. Then by Proposition 2.8 in [2], the lower radical $\mathcal{L}(F[X])$ determined by the free ring $F[X]$ is strongly hereditary. It is also σ-hereditary and small in the lattice of all radicals. Moreover, $\mathcal{L}(F[X])$ is large in the lattice of all strongly hereditary radicals. Suppose that γ^0 is a coatom in $\mathbb{L}_s$. Then there exists a free ring $F[X]$ such that $F[X] \notin \gamma^0$. Since $\mathcal{L}(F[X])$ is small in the lattice of all radicals, we have

$$\mathcal{L}(\gamma^0 \cup \mathcal{L}(F[X])) \neq Ass.$$

Thus, $\mathbb{L}_s$ is not coatomic. □

We denote by $\mathbb{L}$, the collection of all radicals γ such that $\gamma \cap \gamma_\alpha \neq 0$ for every $\gamma_\alpha \in \mathbb{L}_s$.

Proposition 2.9. *$\mathbb{L}$ is a complete sublattice in the lattice of all radicals.*

Proof. Let A be a simple zero-ring with prime order or a prime field. Then $\mathcal{L}(A)$ is strongly hereditary and an atom in the lattice of all hereditary radicals. Let us consider $\gamma_1, \ldots, \gamma_\alpha, \ldots \in \mathbb{L}$. Then $\mathcal{L}(A) \cap \gamma_\alpha \neq 0$ and also $\mathcal{L}(A) \subseteq \gamma_\alpha$. Hence $\cap\gamma_\alpha$ contains all simple zero-rings with prime order and all prime fields. Thus $(\cap\gamma_\alpha) \cap \gamma_\beta \neq 0$ and also $\mathcal{L}(\cup\gamma_\alpha) \cap \gamma_\beta \neq 0$ for every $0 \neq \gamma_\beta \in \mathbb{L}_s$. □

Corollary 2.10. *$\mathbb{L}$ is atomic and not coatomic.*

Proof. This can be proved in a similar way as the proof of Proposition 2.8. □

We recall from [4] the definition of an (hereditary) Amitsur ring and the definition of the radicals $\mathcal{T}$ and $\mathcal{T}_s$. A ring A is said to be an *(hereditary) Amitsur ring* if $\gamma(A[x]) = (\gamma(A[x]) \cap A)[x]$, for all (hereditary) radicals γ, respectively. Let us recall $\mathcal{T}$ and $\mathcal{T}_s$ as follows:

$\mathcal{T} = \{A \mid$ *every prime homomorphic image of the ring A is not a hereditary Amitsur ring*$\}$

and

$\mathcal{T}_s = \{A \mid$ *every prime homomorphic image of the ring A has no nonzero ideal which is a hereditary Amitsur ring*$\}$.

Remark 2.11. $\mathcal{T}$ and $\mathcal{T}_s \in \mathbb{L}$.

A radical γ said to be *prime-like* if for every prime ring A, the polynomial ring $A[x]$ is γ-semisimple. Let as consider the following condition (h) and the class *ch*.

(h): If A is a ring with the chain property, then $\overline{A} \cong S \subseteq A$ for every homomorphic image $\overline{A}$ of A, where S is a subring of A.

$$ch = \{A \mid A \text{ is a ring with condition (h)}\}$$

Lemma 2.12. *Let $A \in ch$ and suppose A is without zero-divisors. Then A is a field with* $\operatorname{char}(A) = p$ *where p is a prime number.*

Proof. We shall show that $a \in aA$ for every element $a \in A$. By Corollary 2.3, we have $aA \trianglelefteq A$. Let $a \notin aA$, for an element $a \in A$. Then $\overline{A} = A/aA \neq \overline{0}$. It is clear that $(a + aA)^2 \subseteq aA$. Therefore $\overline{A}$ has a nonzero nilpotent element. By condition (h), A has a nonzero nilpotent element,which is a contradiction. Thus $a \in Aa$ for every $a \in A$. There exists an element e such that $ae = ea = a$. It is clear that $a \in a^2A$. Thus there exists $x \in A$ such that $ax = e$. Hence A is a field. Suppose that $\operatorname{char} A = 0$ and let e be the unit element of A. Then there exists a subring S of A which is isomorphic to $\mathbb{Z}$. Therefore S does not have the chain property which is a contradiction. □

Lemma 2.13. *Let $A \in ch$. If A has a nonzero zero-divisor, then A is a nil ring.*

Proof. By Proposition 2.2, A has a nonzero nilpotent element. Put

$$I = \{a \in A | a^n = 0, \text{ for a natural number } n\}.$$

It is clear that $I = \mathcal{N}(A)$, where $\mathcal{N}$ is the nil radical. Moreover,

$$\overline{A} = A/\mathcal{N}(A) \cong S \subset A$$

and S has a nonzero nilpotent element. Therefore, since A is commutative ring $\mathcal{N}(A/\mathcal{N}(A)) \neq 0$, which is a contradiction. Thus $A = \mathcal{N}(A)$. □

Proposition 2.14. *Let $A \in ch$. If A has a nonzero zero-divisor then $\beta(A) = A$, where β is the Baer radical.*

Proof. First of all, we claim that $0 \neq \beta(A)$ for any ring $A \in ch$ which has a nonzero zero-divisor. Note that by Lemma 2.13, A is a nil ring. Let us consider the case $\beta(A) \neq A$. Then there exists an element $a \in A$ such that $a^n = 0$ and $aA \neq 0$. If $aA = A$, then $0 \neq A = aA = a^2A = \ldots = a^nA = 0$. This is impossible. Hence $aA \subsetneq A$. Therefore there exists a non-zero element $b \in A$ and $b \notin aA$ with $b^m = 0$ for some natural m. It is clear that $aA \subsetneq \langle b \rangle$. Thus aA is a nilpotent ideal of A. Therefore $0 \neq \beta(A)$, for any ring $A \in ch$ which has a nonzero zero-divisor. Since $\beta(A) \neq A$ there exists $c \in A$ such that $\beta(A) \subsetneq \mathbb{Z}c + cA \trianglelefteq A$. It is clear that $\mathbb{Z}c + cA$ is a nilpotent ideal of A. Thus $\beta(A/\beta(A)) \neq 0$. It is a contradiction. □

Corollary 2.15. *Let $A \in ch$. Then either A is a field or $A = \beta(A)$.*

Proof. It follows from Lemma 2.12, Proposition 2.14. □

A radical γ has the *Amitsur property* if

$$\gamma(A[x]) = (\gamma(A[x]) \cap A)[x], \text{ for all rings} A.$$

Theorem 2.16.([4]) *Every β-radical ring A is a hereditary Amitsur ring.*

Proposition 2.17. *Let $\gamma \subseteq \beta$ be a radical. Then γ is a prime-like radical.*

Proof. Clear. □

For a radical γ, let $\gamma_x = \{A \mid A[x] \in \gamma\}$.

Proposition 2.18.([5, Corollary 13]) *Let γ be a radical with $\beta \subseteq \gamma$. Then γ is prime-like if and only if $\gamma_x = \beta$ and γ has he Amitsur property.*

Theorem 2.19. *$\gamma = \mathcal{L}(\beta \cup \mathcal{L}(ch))$ has the Amitsur property and $\gamma_x = \beta$.*

Proof. By Corollary 2.15, $ch = C \cup D$ and $C \cap D = \emptyset$, where C is the class of Baer radical rings with condition (h) and D is the class of fields with the chain property. By Proposition 2.17 $\mathcal{L}(C)$ is prime-like and it is not hard to check that $\mathcal{L}(C)(A[x]) = 0$, for all prime rings A. Hence $\mathcal{L}(C \cup D) = \mathcal{L}(\mathcal{L}(C) \cup \mathcal{L}(D))$. Thus $\mathcal{L}(ch)$ is prime-like and also γ is prime-like. Hence by Proposition 2.18 we have $\beta = \gamma_x$ and γ has the Amitsur property. □

We put $\mathcal{F} = \{\text{all fields}\}$. Let $\mathcal{U}(\mathcal{M})$ denote the upper radical class generated by a class $\mathcal{M}$ of rings.

Corollary 2.20. *If $A \in \mathcal{U}(\mathcal{F}) \cap \mathcal{L}(\beta \cup \mathcal{L}(ch))$ then A is a hereditary Amitsur ring.*

Proof. Let $A \in \mathcal{U}(\mathcal{F}) \cap \mathcal{L}(\beta \cup \mathcal{L}(ch))$ be a nonzero semiprime ring. Then A has a nonzero accessible subring $B \in D$, where D is the class of fields with chain property. Since B is a field, we have $B^2 = B \trianglelefteq A$ and also B is direct sumand of A. Then there exists a ring B' such that $A = B \oplus B'$. Therefore we have $B \in \mathcal{U}(\mathcal{F}) \cap \mathcal{L}(\beta \cup \mathcal{L}(ch))$. Since B is a field we have

$$B \in \mathcal{F} \cap \mathcal{U}(\mathcal{F}) = 0$$

which is a contradiction. Hence $\mathcal{U}(\mathcal{F}) \cap \mathcal{L}(\beta \cup \mathcal{L}(ch)) = \beta$. Therefore, Theorem 2.16 implies that A is a hereditary Amitsur ring. □

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