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Some Properties of Dedekind Modules and Q-modules

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Abstract. A Q-module is a module in which every nonzero submodule of M is a finite product of primary submodules of M. This paper is devoted to study some properties of Dedekind modules and Q-modules.

1. Introduction

Throughout this paper all rings are considered commutative rings with identiry and all modules are considered unitary. Let R be a ring and M an R-module. A proper submodule N of M is a prime submodule if for each $r \in R$ and for each $m \in M$ with $rm \in N$, we have $m \in N$ or $r \in (N :_R M)$, where $(N :_R M) = Ann(M/N) = \{r \in R | rM \subseteq N\}$. Also N is called a primary submodule of M if for each $r \in R$ and for each $m \in M$ with $rm \in N$, we have $m \in N$ or $r^n \in (N :_R M)$ for a positive integer n. We say that a submodule N of M is a radical submodule of M if $N = \sqrt{N}$, where $\sqrt{N} = \sqrt{(N :_R M)}M$.

The R-module M is said to be a multiplication R-module if every submodule N of M has the form IM for some ideal I of R. If M be a multiplication R-module and N a submodule of M, then $N = IM$ for some ideal I of R. Hence $I \subseteq (N :_R M)$ and so $N = IM \subseteq (N :_R M)M \subseteq N$. Therefore $N = (N :_R M)M$ [8]. Let M be a multiplication R-module, $N = IM$ and $L = JM$ be submodules of M for ideals I and J of R. Then, the product of N and L is denoted by $N.L$ or NL and is defined by IJM [5]. An R-module M is called a cancellation module if $IM = JM$ for two ideals I and J of R implies $I = J$ [1]. By [13, Corollary 1 to Theorem 9], finitely generated faithful multiplication modules are cancellation modules. It

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follows that if M is a finitely generated faithful multiplication R-module, then $(IN :_R M) = I(N :_R M)$ for all ideals I of R and all submodules N of M. If R is an integral domain and M a faithful multiplication R-module, then M is a finitely generated R-module [9]. Let R be a ring, $Z(R)$ the set of zero-divisors of R and $S = R \setminus Z(R)$. Then $T(R)$ denotes the total quotient ring of R. A non-zero-divisor of a ring R is called a regular element and an ideal of R is said to be regular if it contains a regular element. For a non-zero ideal I of R, Let

$$I^{-1} = \{x \in T(R) : xI \subseteq R\}.$$

In this case $II^{-1} \subseteq R$. I is called an invertible ideal of R if $II^{-1} = R$. An integral domain R is called a Dedekind domain if every nonzero ideal of R is invertible.

Let M be an R-module. An element $r \in R$ is said to be a zero-divisor on M if $rm = 0$ for some nonzero element $m \in M$. We denote by $Z(M)$ the set of all zero-divisors of M. It is easy to see that $Z(M)$ is not necessarily an ideal of R, but it has the property that if $a, b \in R$ with $ab \in Z(M)$, then either $a \in Z(M)$ or $b \in Z(M)$. Let M be an R-module and set

$$T = \{t \in S : \text{ for all } m \in M, tm = 0 \text{ implies } m = 0\} = (R \setminus Z(M)) \cap (R \setminus Z(R)).$$

Then T is a multiplicatively closed subset of R with $T \subseteq S$, and if M is torsion-free then $T = S$. In particular, $T = S$ if M is a faithful multiplication R-module [9, Lemma 4.1]. Let N be a nonzero submodule of M. Then we write $N^{-1} = (M :_{R_T} N) = \{x \in R_T : xN \subseteq M\}$. Then N^{-1} is an R-submodule of R_T, $R \subseteq N^{-1}$ and $NN^{-1} \subseteq M$. We say that N is invertible in M if $NN^{-1} = M$. Clearly $0 \neq M$ is invertible in M. An R-module M is called a Dedekind module if every nonzero submodule of M is invertible. In Section 2, we investigate some properties of Dedekind modules. It is proved that if M is a faithful multiplication R-module over an integral domain R, then M is Dedekind R-module if and only if every proper submodule of M is a finite product of prime submodules of M. In Section 3 we prove some results on Q-modules. Let R be a ring and M a finitely generated faitful multiplication R-module. We show that if M is a Noetherian module with $dim(M) = 1$, then M is a Q-module. Finally we prove that if M a Noetherian finitely generated multiplication module over R, then M is a Q-module if and only if every prime submodule which is not a maximal submodule of M is a multiplication submodule.

Here we list some preliminaries and results used throughout the paper.

Lemma 1.1([9]). *Let M be multiplication module and let N be a submodule of M. Then $N = Ann(M/N)M$*

Lemma 1.2.([9, Theorem 2.5]) *Let M be a nonzero multiplication R-module. Then,*

(i) *every proper submodule of M is contained in a maximal submodule of M;*

(ii) *K is a maximal submodule of M if and only if there exists a maximal ideal P of R such that $K = PM \neq M$.*

Theorem 1.3.([9, Corollary 2.11]) *Let R be ring and M an R-module. The following statements are equivalent for a proper submodule N of M:*

(i) *N is a prime submodule of M;*

(ii) *$Ann(M/N)$ is a prime ideal of R;*

(iii) *$N = PM$ for some prime ideal P of R whit $Ann(M) \subseteq P$.*

Theorem 1.4.([9, Theorem 3.1]) *Let R be a ring and M a faithful multiplication R-module. Then the following statements are equivalent:*

(i) *is finitely generated;*

(ii) *$AM \subseteq BM$ if and only if $A \subseteq B$;*

(iii) *for each submodule N of M, there exists a unique ideal I of R such that $N = IM$;*

(iv) *$M \neq AM$ for any proper ideal A of R;*

(v) *$M \neq PM$ for any maximal ideal P of R.*

Definition 1.5. Let R be a ring and M be an R-module and let N be a submodule of M such that $N = IM$ for same ideal I of R. Then, we say that I is a *presentation ideal* of N.

Theorem 1.6.([5, Theorem 3.4]) *Let $N = IM$ and $K = JM$ be submodules of a multiplication R-module M. Then, the product of N and K is independent of presentations of N and K.*

Definition 1.7. Let R be a ring, M an R-module and N a submodule of M. Then N is called *decomposable* if it has a primary decomposition $N = Q_1 \cap ... \cap Q_n$ where for each $1 \leq i \leq n$, Q_i is P_i-primary. Such a primary decomposition of N is said to be a *minimal primary decomposition* if

(1) $P_1, ..., P_n$ are distinict prime ideal of R.

(2) $\bigcap_{i=1, i\neq j}^{n} \nsubseteq Q_j$ for all $j = 1, ..., n$.

It is proved that every decomposable submodule of M has a minimal primary decomposition.

Theorem 1.8([12]). *Let R be a ring and M a Noetherian R-module. Then every proper submodule of M is decomposable.*

A commutative ring R is called a Q-ring if every ideal in R is a finite product of primary ideals in R. First, the class of Noetherian Q-rings have been studied and characterized by D. D. Anderson in [6]. Then Anderson and Mahaney in [7] have studied Q-rings in general.

2. Dedekind Modules

Proposition 2.1. *Let R be a ring and M a multiplication R-module. If N, K, L are submodules of M such that $NK = NL$ and N is invertible, then $K = L$.*

Proof. Let N, K, L are submodules of M such that $NK = NL$ and N is invertible. Then $K = MK = N^{-1}NK = N^{-1}NL = ML = L$. □

Lemma 2.2. *Let R be a ring, M a multiplication R-module and $N_1, \cdots, N_n$ submodules of M. Then the submodule $N_1 \cdots N_n$ is invertible if and only if for each $1 \leq i \leq n$, N_i is invertible.*

Proof. Let $I_1, I_2, \cdots, I_n$ be ideals of R such that $N_1 = I_1M, N_2 = I_2M, \cdots, N_n = I_nM$. Suppose $N_1N_2 \cdots N_n$ is invertible submodule. If K is a fractional ideal of R such that $KN_1N_2 \cdots N_n = M$, then for each $i = 1, 2, \cdots, n$, we have,

$$(KI_1I_2 \cdots I_{i-1}I_{i+1} \cdots I_n)N_i = (KI_1I_2 \cdots I_{i-1}I_{i+1} \cdots I_n)I_iM$$

$$= (KI_1I_2 \cdots I_n)M = K(I_1I_2 \cdots I_n)M = KN_1N_2 \cdots N_n = M.$$

So N_i is invertible. Conversely, suppose for each $1 \leq i \leq n$, N_i is invertible. Then

$$(N_1^{-1}N_2^{-1} \cdots N_n^{-1})(N_1N_2 \cdots N_n) = (N_1^{-1}N_1)(N_2^{-1}N_2) \cdots (N_n^{-1}N_n) = M.$$

So $N_1N_2 \cdots N_n$ is invertible submodule of M. □

Lemma 2.3. *Let R be an integral domain and M a faithful multiplication R-module. If $K_1K_2 \cdots K_n = N = L_1L_2 \cdots L_m$ where K_i, L_i are prime submodules of M and K_i is invertible then $n = m$ and $K_i = L_i$ for each $i = 1, 2, \cdots, n$.*

Proof. The proof is by induction on n. Suppose $n = 1$ and $K_1 = N = L_1L_2 \cdots L_m$ and $J_1, J_2, \cdots, J_n, I_1, I_2, \cdots, I_m$ are prime ideals of R such that $K_j = J_jM$ and $L_i = I_iM$. So we have $JM = I_1I_2 \cdots I_mM$, since M is cancelative R-module $J = I_1I_2 \cdots I_m$. So after reindexing $J = I_1$, thus $K_1 = L_1$. If $n > 1$, choose one of the K_i, say K_1, such that K_1 does not properly contain K_i, for $i = 2, 3, \cdots, n$. Since

$$I_1I_2 \cdots I_mM = L_1L_2 \cdots L_m = K_1K_2 \cdots K_n = J_1J_2 \cdots J_nM \subset K_1$$

and M is cancelative, we have

$$I_1I_2 \cdots I_m = J_1J_2 \cdots J_n \subset J_1$$

and J_1 is prime so by prime avoidenc Theorem there exists some I_i, say I_1, is contained in J_1. Similarly since

$$J_1J_2 \cdots J_n = I_1I_2 \cdots I_m \subset I_1$$

so $J_i \subseteq I_1$. Hence $J_i \subseteq I_1 \subseteq J_1$ and so $K_i \subseteq L_1 \subseteq K_1$. By the minimality of K_1 we must have $K_i = L_1 = K_1$. Since $K_1 = L_1$ is invertible, Proposition 2.1 implies that $K_2K_3\cdots K_n = L_2L_3\cdots L_m$. Therefore by the induction hypothesis $n = m$ and after reindexing $K_i = L_i$ for $i = 1, 2, \cdots, n$. □

Proposition 2.4. *Let R be a ring and M be a finitely generated faithful multiplication R-module in which every proper submodule is a finite product of prime submodules. Then every proper ideal of R is a finite product of prime ideals of R.*

Proof. Let I be a proper ideal of R. Then IM is a proper submodule of M, so $IM = K_1K_2\cdots K_n$ where K_i, $i = 1, 2, \cdots, n$ is prime submodule of M, and there exist prime ideals $P_1, P_2, \cdots, P_n$ of R such that $K_i = P_iM$ for each $i = 1, 2, \cdots, n$. So we have

$$IM = K_1K_2\cdots K_n = (P_1M)(P_2M)\cdots(P_nM) = P_1P_2\cdots P_n.$$

Since M is a finitely generated faithful multiplication R-module, M is a cancelative module, hence we must have $I = P_1P_2\cdots P_n$. □

Theorem 2.5. *Let R be a ring and M be a finitely generated faithful multiplication R-module in which every proper submodule is the product of a finite number of prime submodules. Then every invertible prime submodule of M is maximal.*

Proof. Let N be an invertible prime submodule of M. So there exists prime ideal P of R such that $N = PM$. Since N is invertible, P is an invertible ideal of R. Hence, by [10, Theorem 6.5], P is a maximal ideal of R. Therefore N is maximal, because M is cancelation R-module. □

Proposition 2.6. *Every faithful multiplication module over an integral domain is a D_1 module.*

Proof. See [11, Remark 3.8]. □

Theorem 2.7. *Let R be an integral domain and M be a faithful multiplication R-module in which every proper submodule is the product of a finite number of prime submodules. Then every prime submodule of M is invertible.*

Proof. Suppose N is a nonzero prime submodule of M and $0 \neq a \in N$. Then $Ra = K_1K_2\cdots K_n$ where P_i is a prime submodule of M for all $i = 1, 2, \cdots, n$. There exist prime ideals $P, P_1, P_2, \cdots, P_n$ such that $N = PM$ and for each $1 \leq i \leq\leq n$, $K_i = P_iM$. Since

$$(P_1P_2\cdots P_n)M = (P_1M)(P_2M)\cdots(P_nM) = K_1K_2\cdots K_n = Ra \subseteq N = PM$$

and M is a cancelation R-module, $P_1P_2\cdots P_n \subseteq P$. Therefore for some k, $P_k \subseteq P$ and hence $K_k \subseteq N$. Since by Proposition 2.6, Ra is invertible, K_k is invertible,by Lemma 2.2. Hence K_k is invertible prime submodule. So K_k is maximal by Theorem 2.5, whence $N = K_k$. Therefore N is maximal and invertible. □

Theorem 2.8. *Let R be an integral domain and M be a faithful multiplication R-module. Then M is Dedekind R-module if and only if every proper submodule of M is a finite product of prime submodules of M.*

Proof. Let N be anonzero submodule of M. Choose maximal submodule K_N such that $N \subseteq K_N \subsetneq M$. If $N = M$, let $K_M = R$. Now we have

$$K_N^{-1}N \subseteq K_N^{-1}K_N \subseteq M$$

therefore $K_N^{-1}N$ is a submodule of M and contains N. If N is proper submodule of M, then $N \subsetneq K_N^{-1}N$, because, if not

$$M = RM = RMRM = (N^{-1}N)(K_N^{-1}K_N)$$

$$= N^{-1}(NK_N^{-1})K_N = N^{-1}NK_N = MK_N = K_N$$

is a contradiction. Let S be the set of all submodules of M and define a function $f : S \to S$ by $N \mapsto K_N^{-1}N$. Given a proper submodule N, there exists a function $\phi : N \to S$ such that $\phi(0) = N$ and $\phi(n+1) = f(\phi(n))$. If we denote $\phi(n)$ by N_n and K_{N_n} by K_n, then we have an ascending chain of submodules

$$N = N_0 \subset N_1 \subset N_2 \subset \cdots$$

such that $N_{n+1} = f(N_n) = K_n^{-1}N_n$. Since M is Dedekind, M is Notherian R-module and N is proper submodule of M, there is a least integer l such that

$$N = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_{l-1} \subsetneq N_l = N_{l+1}.$$

Thus $N_l = N_{l+1} = f(N_l) = K_l^{-1}N_l$. So we must have $N_l = M$. Consequently,

$$M = N_l = f(N_{l-1}) = K_{l-1}^{-1}N_{l-1}$$

whence

$$N_{l-1} = N_{l-1}M = N_{l-1}K_{l-1}^{-1}K_{l-1} = MK_{l-1} = K_{l-1}.$$

Since $K_{l-1} = N_{l-1} \subsetneq N_l = M$, K_{l-1} is a maximal submodule of M. The minimality of l insures that each of $K_0, \cdots, K_{l-2}$ is also maximal, because, if not we have $K_i = M$, whence

$$N_{i+1} = K_i^{-1}N_i = M^{-1}N_i = RN_i = N_i$$

is a contradiction. Now we have

$$K_{l-1} = N_{l-1} = K_{l-2}^{-1}N_{l-2} = K_{l-2}^{-1}K_{l-3}^{-1}N_{l-3} = \cdots = K_{l-2}^{-1} \cdots K_1^{-1}K_0^{-1}N.$$

Consequently, since each K_i is invertible,

$$(K_0K_1 \cdots K_{l-2})K_{l-1} = (K_0K_1 \cdots K_{l-2})K_{l-2}^{-1} \cdots K_1^{-1}K_0^{-1}N = N.$$

Conversely, by Lemma 2.2 and Theorem 2.7, M is a Dedekind R-module. □

3. Q-modules

Definition 3.1. Let R be a ring and M an R-module. Then M is called a Q-module if every submodule of M is a finite product of primary submodules of M.

It is clear that a Q-module is a Dedekind module.

Theorem 3.2. *Let R be a ring and M a finitely generated faithful multiplication R-module. If M is a Q-module, then*

(1) *M_S is a Q-module for multiplicative subset S of R.*

(2) *M/N is a Q-module for each submodule N of M.*

Proof. (1) Let $\jmath$ be a submodule of M_S. Then $\jmath \cap M$ is a submodule of M. So $\jmath \cap M = P_1...P_n$ where for each $1 \leq i \leq n$, P_i is a primary submodule of M. Hence $\jmath = S^{-1}(\jmath \cap M) = S^{-1}(P_1...P_n) = (S^{-1}P_1)...(S^{-1}P_n)$ which is a product of primary submodules of M_S. Therefore M_S is a Q-module.
(2) Let K/N be a submodule of M/N where K is a submodule of M. Then $K = P_1...P_n$ where for each $1 \leq i \leq n$, P_i is a primary submodule of M. Hence $K/N = P_1...P_n/N = (P_1/N)...(P_n/N)$ which is a product of primary submodules of M/N. Therefore M/N is a Q-module. □

Remark 3.3. Let R be a ring, M a multiplication R-module, I an ideal of R and N a submodule of M. Then $(N :_R M)M^n = (N :_R M)MM^{n-1} = NM^{n-1} = NMM^{n-2} = ... = NM = N$ and $IM^n = I(RM...RM) = IM$.

Lemma 3.4. *Let R be a ring, M a finitely generated multiplication R-module, I an ideal of R and N a submodule of M. Then*

(1) *N is a product of primary submodules of M if and only if $(N :_R M)$ is a product of primary ideals of R.*

(2) *I is a product of primary ideals of R if and only if IM is a product of primary submodules of M.*

Proof. (1) Let $N = P_1...P_n$ where for each $1 \leq i \leq n$, P_i is a primary submodule of M. Then $(N :_R M) = (P_1...P_n :_R M) = (P_1 :_R M)...(P_n :_R M)$ where for each $1 \leq i \leq n$, $(P_i :_R M)$ is a primary ideal of R, by [3, Lemma 4]. Conversely, let $(N :_R M) = P_1...P_n$ where for each $1 \leq i \leq n$, P_i is a primary ideals of R. Hence, by [3, Lemma 4], $N = (N :_R M)M^n = (P_1...P_n)M^n = (P_1M)...(P_nM)$ where for each $1 \leq i \leq n$, P_iM is a primary submodule of M.
(2) let $I = P_1...P_n$ where for each $1 \leq i \leq n$, P_i is a primary ideals of R. Hence, by [3, Lemma 4], $IM = IM^n = (P_1...P_n)M^n = (P_1M)...(P_nM)$ where for each $1 \leq i \leq n$, P_iM is a primary submodule of M. Conversely, let $IM = P_1...P_n$ where for each $1 \leq i \leq n$, P_i is a primary submodule of M. Then $I = (IM :_R M) = (P_1...P_n :_R M) = (P_1 :_R M)...(P_n :_R M)$ where for each $1 \leq i \leq n$, $(P_i :_R M)$ is a primary ideal of R, by [3, Lemma 4]. □

Now we have the following Corollary.

Corollary 3.5. *Let R be a ring and M be a finitely generated multiplication R-module. Then R is a Q-ring if and only if M is a Q-module.*

Theorem 3.6. *Let R be a ring and M be a finitely generated multiplication R-module. If a submodule N of M is a finite product of primary submodules, then there are only finitely many prime submodules of M which are minimal over N.*

Proof. Let N be a product of primary submodules of M. Then, by Lemma 3.4, $(N :_R M)$ is a product of primary ideals. Hence, by [6, Lemma 4], there are only finitely many minimal prime submodules over $(N :_R M)$.

Therefore, by [3, Lemma 4], there are only finitely many prime submodules of M which are minimal over N. □

Corollary 3.7. *Let R be a ring and M be a finitely generated multiplication R-module. If M is a Q-module, then there are only finitley many minimal prime submodules over any submodule of M.*

Lemma 3.8. *Let R be a ring, M a multiplication R-module and N, K submodules of M. If $\sqrt{N} + \sqrt{K} = M$, then $N + K = M$. Moreover, $NK = N \cap K$.*

Proof. Let $\sqrt{N}+\sqrt{K} = M$. Then $(\sqrt{(N :_R M)}+\sqrt{(K :_R M)})M = \sqrt{(N :_R M)}M+\sqrt{(K :_R M)}M = M$. So $\sqrt{(N :_R M)} + \sqrt{(K :_R M)} = R$. Hence $(N :_R M) + (K :_R M) = R$ and thus $(N :_R M)(K :_R M) = (N :_R M) \cap (K :_R M)$. Therefore $N + K = (N :_R M)M + (K :_R M)M = M$. Moreover, $KN = (N :_R M)M(K :_R M)M = (N :_R M)M \cap (K :_R M)M = N \cap K$. □

Theorem 3.9. *Let R be a ring and M a finitely generated faitful multiplication R-module. Let M be a Noetherian module with $dim(M) = 1$. Then M is a Q-module.*

Proof. Let N be a submodule of M. Then N has a minimal primary decomposition, say, $N = Q_1 \cap Q_2 ... \cap Q_n$ where for each $1 \leq i \leq n$, Q_i is a P_i-primary submodule of M. Since $dim(M) = 1$, each nonzero prime submodule of M is maximal. So for each $1 \leq i \leq n$, P_i is a maximal submodule of M. Hence, $P_i + P_j = M$ for all $i \neq j$.

Thus, by Lemma 3.8, $Q_i + Q_j = M$ for all $i \neq j$. Then $N = Q_1 \cap Q_2 ... \cap Q_n = Q_1Q_2...Q_n$. Therefore M is a Q-module. □

Note that if R is a ring, M is a multiplication R-module and N a submodule of M, then N is a multiplication R-submodule of M if and only if $(N :_R M)$ is a multiplication ideal of R. For this, let N be a multiplication R-submodule of M and I an ideal of R such that $I \subseteq (N :_R M)$. Then $IM \subseteq N$. So $IM = KN$ for an ideal K of R. Hence $I = (IM :_R M) = K(N :_R M)$. Therefore $(N :_R M)$ is a multiplication ideal of R. Conversely, let $(N :_R M)$ be a multiplication ideal of R and K a submodule of N. Then $(K :_R M) \subseteq (N :_R M)$. So $(K :_R M) = I(N :_R M)$ for an ideal I of R. Hence $K = IN$. Therefore N is a multiplication submodule of M.

Proposition 3.10. *Let R be a ring, M a multiplication R-module and N be a multiplication submodule of M. If P is a prime submodule of M with $P \subsetneq N$, then $P \subseteq \bigcap_{n=1}^{\infty} N^n$.*

Proof. Let N be a multiplication submodule of M. Then $(N :_R M)$ is a multiplication ideal of R. Let P be a prime submodule of M with $P \subsetneq N$. Then $(P :_R M)$ is a prime ideal of R with $(P :_R M) \subsetneq (N :_R M)$. Hence, by [6], $(P :_R M) \subseteq \bigcap_{n=1}^{\infty}(N :_R M)^n$. Therefore

$$P = (P :_R M)M^n \subseteq \bigcap_{n=1}^{\infty}(N :_R M)^n M^n = \bigcap_{n=1}^{\infty} N^n. \qquad \square$$

It is shown that if R is a ring and M is a finitely generated faithful multiplication R-module, then M is a Noetherian R-module if and only if R is a Noetherian ring.

Theorem 3.11. *Let R be a ring and M a Noetherian finitely generated multiplication R-module. Then M is a Q-module if and only if every prime submodule which is not a maximal submodule of M is a multiplication submodule.*

Proof. It is obvious that M is a Noetherian module if and only if R is a Noetherian ring. Then M is a Q-module if and only if R is a Q-ring by Theorem 3.5, if and only if every prime ideal which is not maximal in R is multiplication by [6, Theorem 10], if and only if every prime submodule which is not a maximal submodule in M is a multiplication submodule. □

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