KYUNGPOOK Math. J. 58(2018), 453-462
https://doi.org/10.5666/KMJ.2018.58.3.453
pISSN 1225-6951 eISSN 0454-8124

On A Symbolic Method for Error Estimation of a Mixed Interpolation

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Abstract. In this paper, we present a symbolic formulation of the error obtained due to an approximation of a given function by the mixed-interpolating function. Using the proposed symbolic method, we compute the *error evaluation operator* as well as the error estimation at any arbitrary point. We also present an algorithm to compute an approximation of a function by the mixed interpolation technique in terms of projector operator. Certain examples are presented to illustrate the proposed algorithm. Maple implementation of the proposed algorithm is discussed with sample computations.

1. Introduction

In this paper, we focused on a mixed interpolation problem of the form

$$\widetilde{f}_s(x) = aU_1(x) + bU_2(x) + \sum_{i=0}^{s-2} c_i x^i, \quad s \geq 2, \tag{1.1}$$

such that

$$\theta \widetilde{f}_s = \theta f, \tag{1.2}$$

where $U_1(x)$ and $U_2(x)$ are given functions, $\Theta = \{\theta_0, \dots, \theta_s\}$ is a set of bounded functionals of a normed linear space $\mathcal{S} = C^\infty[a,b]$, $\Sigma = \{\theta f : \theta \in \Theta\} \subset \mathbb{R}$, and s is the order of interpolating function $\widetilde{f}_s(x)$. We call the tuple (M, Θ) an *interpolation problem*, where $M = \{U_1, U_2, 1, \dots, x^{s-2}\} \subset \mathcal{S}$ and $\Theta \subset \mathcal{S}^*$ are the finite dimensional basis and linearly independent set respectively. The interpolation problem (M, Θ) is

Received March 31, 2017; revised June 12, 2018; accepted June 27, 2018.
2010 Mathematics Subject Classification: 41A05, 47A57, 30E05.
Key words and phrases: mixed interpolation, error estimation, projector, Stieltjes functionals, symbolic method.

said to be *regular* if it has unique solution, otherwise it is called *singular*. Given any function $f(x)$ can be approximate by the mixed interpolation of the form (1.1) and the error induced due to an approximation of the given function can be formulated in symbolic manner.

De Meyer et al. (1990) derived a formula for the mixed interpolation of the form (1.1) with fixed $U_1(x) = \cos(kx)$ and $U_2(x) = \sin(kx)$ correspondence to finite difference formulae for polynomial interpolation with uniformly spaced nodes (general conditions) in [6] and the error term is discussed in [7]. Chakrabarti et al. (1996) revised and extended the idea of De Meyer by replacing $\cos(kx)$ and $\sin(kx)$ by $U_1(kx)$ and $U_2(kx)$ respectively with uniformly spaced general conditions. Chakrabarti presented a closed form of the error term by choosing an appropriate differential operator [1]. Coleman (1998) presented an algorithm for mixed interpolation of the form (1.1) and derived both Newtonian and Lagrangian formulae for the interpolant for arbitrarily chosen distinct nodes [2]. However, the authors considered the mixed interpolation of the form (1.1) and presented the error estimation with uniformly or arbitrary chosen general conditions. Thota and Kumar (2016) discussed in [11] about the formulation of the mixed interpolation of the form (1.1) with integral conditions at arbitrary nodes for a special case $U_1(x) = \cos kx$ and $U_2(x) = \sin kx$, $k > 0$ is a given parameter, and in [10, 12, 13], a symbolic method for polynomial interpolation with Stieltjes conditions is discussed. This paper presents a symbolic formulation of the error estimation due to an approximation of a given function by the mixed interpolation of the form (1.1) with Stieltjes conditions (the combination of general, differential and integral conditions). In literature survey, it is seen that there is no symbolic method available for the error estimation of a mixed interpolation problem with Stieltjes functionals. Therefore, we develop a symbolic algorithm for the error estimation in this paper.

The error due to an approximation of a given function $f(x)$ by the mixed interpolant $\widetilde{f}_s(x)$ is defined as

$$E_s(f, M, \Theta) = f(x) - \widetilde{f}_s(x). \tag{1.3}$$

The matrix [12], related Stieltjes conditions, given below

$$\mathcal{E} = \begin{pmatrix} \theta_0 U_1 & \theta_0 U_2 & \theta_0 1 & \cdots & \theta_0 x^{s-2} \\ \theta_1 U_1 & \theta_1 U_2 & \theta_1 1 & \cdots & \theta_1 x^{s-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_s U_1 & \theta_s U_2 & \theta_s 1 & \cdots & \theta_s x^{s-2} \end{pmatrix} \tag{1.4}$$

is called *evaluation matrix* of Θ and M. Denote the determinant of evaluation matrix by $\mathcal{D} = \det(\mathcal{E})$ for simplicity, and if

$$\mathcal{B} = \begin{vmatrix} f(x) & N \\ f^{\dagger}(x) & \mathcal{E} \end{vmatrix},$$

where $N = (U_1, U_2, 1, \dots, x^{s-2})$ and $f^{\dagger}(x) = (\theta_0 f, \dots, \theta_s f)^T$ are row and column vectors respectively, then the error formula in equation (1.3) can be expressed as a quotient of $\mathcal{B}$ and $\mathcal{D}$ [2] as follows

$$E_s(f, M, \Theta) = \frac{\mathcal{B}}{\mathcal{D}}.$$

In other words, as a particular case of a general remainder theorem [3, p. 75], this quotient represents the linear combination of $\{f(x), U_1, U_2, 1 \ldots, x^{s-2}\}$ in which the coefficient of $f(x)$ is unity.

The following section presents a symbolic formulation of the error estimation as defined in equation (1.3) over algebras.

2. Symbolic Formulation of Error Estimation

In order to formulate the error estimation, we need to express the error $E_s(f, M, \Theta)$ in a closed form. This can be achieved by selecting a differential operator T whose fundamental system is $M = \{U_1, U_2, 1, x, \ldots, x^{s-2}\}$ and such a differential operator is as follows [1, 2]:

$$T = \frac{\mathcal{D}_s}{\mathcal{D}_{s+1}} \frac{d^{s+1}}{dx^{s+1}} - \frac{\mathcal{D}'_s}{\mathcal{D}_{s+1}} \frac{d^s}{dx^s} + \frac{d^{s-1}}{dx^{s-1}}, \tag{2.1}$$

where $\mathcal{D}_s = \begin{vmatrix} \frac{d^s U_2}{dx^s} & \frac{d^s U_1}{dx^s} \\ \frac{d^{s-1} U_2}{dx^{s-1}} & \frac{d^{s-1} U_1}{dx^{s-1}} \end{vmatrix}$. Hence the function $\widetilde{f}_s(x)$ satisfies the differential equation

$$T\widetilde{f}_s = 0$$

and the error $E(f, M, \Theta)$ satisfies

$$\begin{aligned} TE(f, M, \Theta) &= Tf \quad \text{and} \\ \theta E(f, M, \Theta) &= 0. \end{aligned} \tag{2.2}$$

Now the goal is to find an operator $T^\dagger$ such that $TT^\dagger = T$ and $\Theta T^\dagger = 0$. The operator $T^\dagger$ is called an *error evaluation operator*. Using the error evaluatation operator, we can calculate the error term $E(f, \Theta, \Sigma)$ as $T^\dagger f$. The key to find such operator for (2.2) is the *Moore-Penrose inverse* (or just pseudoinverse for short) defined as follows. One of the advantages of Moore-Penrose inverse is that provides us a alternate inverse of a non-bijective linear operator in any linear space.

Definition 2.1.([4, 8]) Let $T : \mathcal{S} \to \mathcal{S}$ be a linear operator. We say T has a *Moore-Penrose inverse* if and only if there is an operator $G : \mathcal{S} \to \mathcal{S}$ such that

$$TGT = T, \tag{2.3}$$

$$GTG = G, \tag{2.4}$$

$$GT = 1 - P, \tag{2.5}$$

$$TG = Q, \tag{2.6}$$

where P and Q are the projectors to the kernel and range of T respectively. Furthermore, if $\mathcal{K}$ and $\mathcal{R}$ are the compliments of kernel and range respectively, then $\text{Ker}(G) = \mathcal{R}$ and $\text{Range}(G) = \mathcal{K}$.

If we set $Q = 1$ in Definition 2.1, i.e. G is right inverse of T such that $TG = 1$, then the equations (2.3) and (2.4) are clearly follow from equation (2.6). Now equation (2.5) is to be consider, in which P is a projector onto null space M of T involving the Stieltjes functionals. It is shown by Rosenkranz et al. in [9] over integro-differential algebras that, if $T^{\Diamond}$ is any right inverse (irrespective of Stieltjes functionals) of T then

$$G = (1 - P)T^{\Diamond}$$

is the *Green's operator* for a regular boundary problem

$$\begin{aligned} TE(f, M, \Theta) &= g \\ \theta E(f, M, \Theta) &= 0, \end{aligned}$$

where g is a forcing function. Since $T^{\Diamond}$ is a right inverse and from Definition 2.1, one can observed that $TT^{\Diamond} = 1$ and $T^{\Diamond}T = 1 - P$. Now

$$E(f, M, \Theta) = Gg = (1 - P)T^{\Diamond}g. \tag{2.7}$$

Replacing g with Tf in equation (2.7) and using the fact that $1 - P$ is projector, we get

$$E(f, M, \Theta) = (1 - P)f,$$

and the error evaluation operator is

$$T^{\dagger} = 1 - P.$$

To find the error evaluation operator $T^{\dagger}$ and the corresponding error $E(f, M, \Theta)$ of a given problem, we need a closed form of the projector operator P in terms of Stieltjes conditions/functionals. The following lemma gives a closed form of the projector operator with Stieltjes conditions in the language of linear algebras.

Lemma 2.2.([9]) *Let $\mathcal{S}$ be a vector space and $M = \{U_1, U_2, 1, \ldots, x^{s-2}\} \leq \mathcal{S}$, and $\Theta = \{\theta_0, \ldots, \theta_s\} \leq \mathcal{S}^*$ are orthogonally closed sets. Then $P : \mathcal{S} \to \mathcal{S}$ is a projector onto M along Θ given by*

$$P = U_1\widetilde{\theta}_0 + U_2\widetilde{\theta}_1 + \sum_{i=0}^{s-2} x^i \widetilde{\theta}_{i+2}, \tag{2.8}$$

where $(\widetilde{\theta}_0, \ldots, \widetilde{\theta}_s)^T = \mathcal{E}^{-1}(\theta_0, \ldots, \theta_s)^T$.

For sake of completeness, we include a sketch of the proof as follows.

Proof. Since the evaluation matrix $\mathcal{E}$ is invertible, $(\widetilde{\theta}_0, \ldots, \widetilde{\theta}_s)$ and $\{U_1, U_2, 1, \ldots, x^{s-2}\}$ are bi-orthogonal. Hence P is projector with $\mathrm{Range}(P) = M$ and $\mathrm{Ker}(P) = \Theta^{\perp}$. □

Using Lemma 2.2, we can compute the projector operator, and hence the evaluation operator $T^{\dagger}$. The following theorem gives the generalization of the above formulation.

Theorem 2.3. *Suppose $\widetilde{f}_s(x)$ is of the form* (1.1)*, an approximation of $f(x)$, then the error $E(f, M, \Theta)$ involved due to the approximation is given by*

$$E(f, M, \Theta) = (1 - P)f, \tag{2.9}$$

where P is a projector onto M along $\Theta^{\perp}$ computed as in Lemma 2.2.

To illustrate the symbolic formulation of the error $E(f, M, \Theta)$ given in Theorem 2.3, we provide the following example to find the error formula.

Example 2.4. We present a symbolic formulation of the error due the approximation of $f(x) = e^{0.2x}$ by a mixed interpolant $\widetilde{f}_3(x)$ of the form (1.1) with Stieltjes conditions $f(0) = 1, f'(0.2) = 0.208, f(0.3) + \int\limits_0^{0.3} f(x)dx = 1.371, f'(0.5) + \int\limits_0^{0.5} f(x)dx = 0.747$ for a special choice of $U_1(x) = \cos x$ and $U_2(x) = \sin x$.

In symbolic notations, we have $\Theta = \{\mathtt{E}_0, \mathtt{E}_{0.2}\mathtt{D}, \mathtt{E}_{0.3} + \mathtt{E}_{0.3}\mathtt{A}, \mathtt{E}_{0.5}\mathtt{D} + \mathtt{E}_{0.5}\mathtt{A}\}$ with associated values $\Sigma = \{1, 0.208, 1.371, 0.747\}$ and $M = \{\cos x, \sin x, 1, x\}$. The differential operator with fundamental system M is constructed similar to equation (2.1) as follows

$$T = \frac{d^4}{dx^4} + \frac{d^2}{dx^2}.$$

It is clear that

$$TE(f, M, \Theta) = Tf,$$
$$\Theta E(f, M, \Theta) = 0,$$

where $E(f, M, \Theta) = f(x) - \widetilde{f}_3(x)$ is the error to be find. Following the algorithm in Theorem 2.3, we have

$$E(f, M, \Theta) = (1 - P)f,$$

where P is the projector onto M along $\Theta^{\perp}$ computed as in Lemma 2.2 and is given by

$$\begin{aligned} P = {} & x(-273.596\mathtt{E}_0 - 85.180\mathtt{E}_{0.2}\mathtt{D} + 205.198\mathtt{E}_{0.3} + 205.198\mathtt{E}_{0.3}\mathtt{A} + 13.678\mathtt{E}_{0.5}\mathtt{D} + 13.678\mathtt{E}_{0.5}\mathtt{A}) \\ & + \sin(x)(259.976\mathtt{E}_0 + 82.550\mathtt{E}_{0.2}\mathtt{D} - 194.727\mathtt{E}_{0.3} - 194.727\mathtt{E}_{0.3}\mathtt{A} - 13.661\mathtt{E}_{0.5}\mathtt{D} - 13.661\mathtt{E}_{0.5}\mathtt{A}) \\ & + \cos(x)(-94.638\mathtt{E}_0 - 26.556\mathtt{E}_{0.2}\mathtt{D} + 72.240\mathtt{E}_{0.3} + 72.240\mathtt{E}_{0.3}\mathtt{A} + 1.451\mathtt{E}_{0.5}\mathtt{D} + 1.451\mathtt{E}_{0.5}\mathtt{A}) \\ & + 95.638\mathtt{E}_0 + 26.556\mathtt{E}_{0.2}\mathtt{D} - 72.240\mathtt{E}_{0.3} - 72.240\mathtt{E}_{0.3}\mathtt{A} - 1.451\mathtt{E}_{0.5}\mathtt{D} - 1.451\mathtt{E}_{0.5}\mathtt{A}. \end{aligned}$$

Now the error $E(f, M, \Theta) = (1 - P)f$ is given by

$$E(f, M, \Theta) = e^{0.2x} - 1.039 + 0.039\cos(x) + 0.0184\sin(x) - 0.218x.$$

If we select $U_1(x) = x^3$ and $U_2(x) = x^2$, then the corresponding differential operator is $T = \frac{d^4}{dx^4}$ and the error $\widetilde{E}(f, M, \Theta)$ is given by

$$\widetilde{E}(f, M, \Theta) = e^{0.2x} - 1 - 0.200x - 0.020x^2 - 0.0014x^3.$$

2.1. Approximation of a Function by Mixed Interpolation

Comparing equations (1.3) and (2.9), one can observe that the mixed interpolation $\widetilde{f}_s(x)$ can be computed with a given set Θ of Steiltjes conditions using the following formula:

$$\widetilde{f}_s = Pf, \tag{2.10}$$

where P is a projector onto M along $\Theta^\perp$ and P is computed as in Lemma 2.2. From Example 2.4, the mixed interpolant, $\widetilde{f}_3(x)$ for $U_1(x) = \cos x$ and $U_2(x) = \sin x$ with given Stieltjes conditions, is obtained as

$$\widetilde{f}_3(x) = 1.039 + 0.218x - 0.039\cos x - 0.0184\sin x,$$

for $U_1(x) = x^3$ and $U_2(x) = x^2$, we have

$$\widetilde{f}_3(x) = 1 + 0.200x + 0.020x^2 + 0.0014x^3,$$

and it satisfies $\Theta\widetilde{f}_3 = \Sigma$, i.e. $\theta\widetilde{f}_3 = \theta f$.

We generalize the above observation in the following theorem.

Theorem 2.5. *The approximation $\widetilde{f}_s(x)$ of a function $f(x)$, with given linearly independent sets $\Theta = \{\theta_0, \theta_1, \ldots, \theta_s\}$ and $M = \{U_1, U_2, 1, \ldots, x^{s-2}\}$, is computed by*

$$\widetilde{f}_s(x) = Pf(x), \tag{2.11}$$

where P is a projector operator onto M along $\Theta^\perp$ as given in Lemma 2.2.

The following section gives an implementation of the proposed algorithm in Maple with sample computations.

3. Maple Implementation

We present the implementation of the proposed algorithm by creating different data types in Maple with help of the Maple package `IntDiffOp` implemented by Korporal et al. [5]. To display the operators, we have `D` for the differential operator, `A` for the integral operator and `E` for the evaluate operator as defined in `IntDiffOp` package. The data type `StieltjesCondition(evp,a,n,b)`, given below, is created to represent the Stieltjes conditions, where `evp` is the evaluation point, `a` is the coefficient of evaluation operator, `n` is the order of differential operator and `b` is the coefficient of integral operator.

```
with(IntDiffOp):
StieltjesCondition:=proc(evp,a,n,b)
   local diffpower;
   if n=0 then
     return BOUNDOP(EVOP(evp,EVDIFFOP(a),
            EVINTOP(EVINTTERM(b,1))));
   else
     diffpower:=seq(0,i=1..n-1);
     return BOUNDOP(EVOP(evp,EVDIFFOP(a,diffpower,1),
            EVINTOP(EVINTTERM(b,1))));
   end if;
end proc:
```

The procedure `EvaluationMat(U1, U2, SC)`, given below, calculates the evaluation matrix, where `SC` is the column matrix of given Stieltjes conditions and `U1,U2` are the known functions.

```
with(IntDiffOp):
EvaluationMat:=proc(U1,U2,SC::Matrix)
   local boundlist,r,c,elts,fs,C,S;
   r,c:=LinearAlgebra[Dimension](SC);
   C(x):=U1;S(x):=U2;
   fs:=Matrix(1,r,[C(x),S(x),seq(x^ (i-1),i=1..r-2)]);
   elts:=seq(seq(ApplyOperator(SC[k,1],fs[1,j]),j=1..r),k=1..r);
   return Matrix(r,r,[elts]);
end proc:
```

An approximation of a given function $f(x)$ with known `U1,U2` is calculated using the data type `ApproxedMixedInterpolation(U1,U2,f,SC)` as given below.

```
with(IntDiffOp):
ApproxedMixedInterpolation:= proc(U1,U2,f,SC::Matrix)
   local r,c,fs,evm,invevm,cm,approx,C,S;
   r,c:=LinearAlgebra[Dimension](SC);
   C(x):=U1;S(x):=U2; fun(x):=f;
   fs:=Matrix(1,r,[C(x),S(x),seq(x^ (i-1),i=1..r-2)]);
   cm:=ApplyOperator (SC,fun);
   evm:=EvaluationMat(U1,U2,SC);
   invevm:=1/evm;
```

```
   approx:=fs.invevm.cm;
   return simplify(approx[1,1]);
end proc:
```

The error estimation is calculated using the following procedure `ErrorEstimation(U1,U2,f,SC)`.

```
with(IntDiffOp):
ErrorEstimation:= proc(U1,U2,f,SC::Matrix)
   local err,fun,approx;
   fun(x):=f;
   approx(x):=ApproxedMixedInterpolation(U1,U2,f,SC);
   err:=simplify(fun(x)-approx(x));
   return err;
end proc:
```

Example 3.1. For sample computations using Maple implementation, recall the Example 2.4 presented in Section 2.

```
> with(IntDiffOp):
> f:= exp(0.2x);
```

$$e^{0.2x}$$

```
> SC1:=StieltjesCondition(0.0,1,0,0);
  SC2:=StieltjesCondition(0.2,0,1,0);
  SC3:=StieltjesCondition(0.3,1,0,1);
  SC4:=StieltjesCondition(0.5,0,1,1);
```

$$\begin{aligned} SC1 &:= \mathtt{E}[0.0] \\ SC2 &:= \mathtt{E}[0.2].\mathtt{D} \\ SC3 &:= \mathtt{E}[0.3] + \mathtt{E}[0.3].\mathtt{A} \\ SC4 &:= \mathtt{E}[0.5].\mathtt{D} + \mathtt{E}[0.5].\mathtt{A} \end{aligned}$$

```
> SC:=Matrix([[SC1],[SC2],[SC3],[SC4]]);
```

$$SC := \begin{bmatrix} \mathtt{E}[0.0] \\ \mathtt{E}[0.2].\mathtt{D} \\ \mathtt{E}[0.3] + \mathtt{E}[0.3].\mathtt{A} \\ \mathtt{E}[0.5].\mathtt{D} + \mathtt{E}[0.5].\mathtt{A} \end{bmatrix}$$

```
> U11:=cos(x):U12:=sin(x):
> U11:=x^3:U12:=x^2:
> EvaluationMat(U11,U12,SC);
```

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -0.19867 & 0.98007 & 0 & 1 \\ 1.25086 & 0.34018 & 1.3 & 0.345 \\ 0 & 1 & 0.5 & 1.125 \end{bmatrix}$$

```
> ApproxedMixedInterpolation(U11,U12,f,SC);
```

$-0.03911007 * \cos(x) - 0.01845023 * \sin(x) + 1.039110069 + 0.21847466 * x$

```
> f3(x):= -0.03911007*cos(x)-0.01845023*sin(x)+1.039110069+0.21847466*x:

> ErrorEstimation(U11,U12,f,SC);
```

$$exp(0.2 * x) + 0.03911007 * \cos(x) + 0.01845023 * \sin(x) \\ - 1.039110069 - 0.2184746600 * x$$

```
> er(x):=exp(0.2*x)+0.03911007*cos(x)+0.01845023*sin(x)
         -1.039110069-0.2184746600*x:
> er(0.15);
```

$$0.00000134000$$

```
> er(0.46);
```

$$-0.0000079696$$

```
> EvaluationMat(U21,U22,SC);
```

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0.12 & 0.4 & 0 & 1 \\ 0.029025 & 0.099 & 1.3 & 0.345 \\ 0.765625 & 1.0416667 & 0.5 & 1.125 \end{bmatrix}$$

```
> ApproxedMixedInterpolation(U21,U22,f,SC);
```

$$0.001400394 * x^3 + 0.019982904 * x^2 + 1 + 0.2000009373 * x$$

```
> ErrorEstimation(U21,U22,f,SC);
```

$exp(0.2 * x) - 0.001400394 * x^3 - 0.019982904 * x^2 - 1 - 0.2000009373 * x$

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