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On the Toroidal Comaximal Graph of Lattices

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Abstract. In this paper, we study the toroidality of the comaximal graphs of a finite lattice.

1. Introduction

The concept of the comaximal graph of a commutative ring R was first defined in [9]. In [9], Sharma and Bhatwadekar defined the comaximal graph of R, denoted by $\Gamma(R)$, with all elements of R being the vertices, and two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. In [5] and [10], the authors considered a subgraph $\Gamma_2(R)$ of $\Gamma(R)$ consisting of non-unit elements of R, and studied several properties of the comaximal graph. Also the comaximal graph of a non-commutative ring was defined and studied in [11]. Recently, in [1], the comaximal graph of a lattice was defined and studied. The comaximal graph of a lattice $L = (L, \wedge, \vee)$, denoted by $\Gamma(L)$, is an undirected graph with all elements of L being the vertices, and two distinct vertices a and b are adjacent if and only if $a \vee b = 1$.

First we recall some definitions and notation on lattices and graphs.

Recall that a *lattice* is an algebra $L = (L, \wedge, \vee)$ satisfying the following conditions: for all $a, b, c \in L$,

1. $a \wedge a = a,\ a \vee a = a$,
2. $a \wedge b = b \wedge a,\ a \vee b = b \vee a$,
3. $(a \wedge b) \wedge c = a \wedge (b \wedge c),\ a \vee (b \vee c) = (a \vee b) \vee c$, and
4. $a \vee (a \wedge b) = a \wedge (a \vee b) = a$.

Note that in every lattice the equality $a \wedge b = a$ always implies that $a \vee b = b$. Also, by [7, Theorem 2.1], one can define an order $\leqslant$ on L as follows: For any

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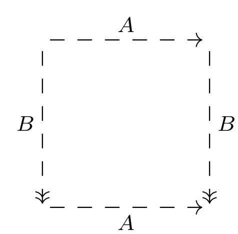
$a, b \in L$, we set $a \leqslant b$ if and only if $a \wedge b = a$. Then $(L, \leqslant)$ is an ordered set in which every pair of elements has a greatest lower bound ($g.l.b.$) and a least upper bound ($l.u.b.$). Conversely, let L be an ordered set such that, for every pair $a, b \in L$, $g.l.b.(a, b), l.u.b.(a, b) \in L$. For each a and b in L, we define $a \wedge b := g.l.b.(a, b)$ and $a \vee b := l.u.b.(a, b)$. Then $(L, \wedge, \vee)$ is a lattice. A lattice L is said to be *bounded* if there are elements 0 and 1 in L such that $0 \wedge a = 0$ and $a \vee 1 = 1$, for all $a \in L$. Note that every finite lattice is bounded. Recall that for two elements a and b in a partially ordered set $(P, \leqslant)$, we say that a covers b or b is covered by a, in notation $b \prec a$, if and only if $b < a$ and there is no element p in P such that $b < p < a$. An element a in L is called a *co-atom* if $a \prec 1$. We denote the sets of all co-atoms in a lattice L by $C(L)$. Also, for an element $a \in L$, we set $[a]^l = \{b \in L \mid b \leq a\}$.

For a positive integer r, an *r-partite graph* is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. The *complete bipartite graph* (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. An *elementary contraction* consists of the deletion of a vertex or an edge or the identification of two adjacent vertices. A graph G is said to *contract to* a graph H if there exists a sequence of elementary contractions which transforms G into H. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges with paths. A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [2, p.153]).

By a *surface*, we mean a connected compact 2-dimensional real manifold without boundary, that is a connected topological space such that each point has a neighbourhood homeomorphic to an open disc. The sphere is designated to be the surface S_0; the surface formed by adding g handles to the sphere is denoted S_g. It is well-known that every compact surface is homeomorphic to a sphere, or to a connected sum of g tori (S_g), or to a connected sum of k projective planes (N_k) (see [6, Theorem 5.1]). This number g is called the *genus* of the surface. The torus can be thought of as 1 tori (S_1) or as a sphere with 1 handle.



The canonical representation of a torus

A graph G is embeddable in a surface S if the vertices of G are assigned to distinct points in S such that every edge of G is a simple arc in S connecting the two vertices which are joined in G. If G can not be embedded in S, then G has at least two edges intersecting at a point which is not a vertex of G. We say a graph G is *irreducible* for a surface S if G does not embed in S, but any proper subgraph of G embeds in S. The least non-negative integer g that the graph G can be embedded in S_g is called the *genus* of G. A *toroidal graph* is a graph that can be embedded without crossings on the torus. Hence, toroidal graphs have genus 1. Note that a planar graph is a graph that can be embedded on S_0. And so a planar graph is not toroidal. Also a complete graph K_n is toroidal if $n = 5$, 6 or 7, and the only toroidal complete bipartite graphs are $K_{4,4}$, $K_{3,3}$, $K_{3,4}$, $K_{3,5}$ and $K_{3,6}$ (see [3] or [8]).

2. Toroidal comaximal graph of a lattice

In this paper, we assume that L is a finite lattice. The comaximal graph of a lattice L, denoted by $\Gamma(L)$, is an undirected graph with all elements of L being the vertices, and two distinct vertices a and b are adjacent if and only if $a \vee b = 1$. We denote the induced subgraph of $\Gamma(L)$ with vertex set $L \setminus (J(L) \cup \{1\})$, by $\Gamma_2(L)$, where $J(L)$ is the set $\bigcap_{m\in C(L)}[m]^l$ (see [1]). Consider the *l.u.b.* of two vertices in $J(L)$ to see that they can not be connected. Therefore, the vertices in $J(L)$ are isolated vertices.

In this section, we explore the toroidality of the graph $\Gamma_2(L)$.

By [1, Lemma 4.1.], if $\Gamma_2(L)$ is planar, then $|C(L)| \leq 4$. If $|C(L)| = 1$, then $\Gamma_2(L)$ is an empty graph. Note that when $|C(L)| = 2$, we observe that $\Gamma_2(L)$ is a complete bipartite graph (see [1, Corollary 3.5.]). So $\Gamma_2(L)$ is planar if and only if either $|[m_1]^l \setminus [m_2]^l| \leq 2$ or $|[m_2]^l \setminus [m_1]^l| \leq 2$, where $C(L) = \{m_1, m_2\}$. Also one can easily see that $\Gamma_2(L)$ is toroidal if and only if either $|[m_1]^l \setminus [m_2]^l| = |[m_2]^l \setminus [m_1]^l| = 4$ or $|[m_1]^l \setminus [m_2]^l| = 3$, $|[m_2]^l \setminus [m_1]^l| \in \{3, 4, 5, 6\}$, where $C(L) = \{m_1, m_2\}$. We begin this section with the following lemma.

Lemma 2.1. *If* $\Gamma_2(L)$ *is toroidal, then the size of* $C(L)$ *is at most* 7.

Proof. Assume to the contrary that $|C(L)| \geq 8$. Then the induced subgraph of $\Gamma_2(L)$ with vertex set $C(L)$ is isomorphic to K_8, which is a contradiction. □

By Lemma 2.1., it is sufficient for us to probe the toroidality of the graph $\Gamma_2(L)$ in the cases in which the size of $C(L)$ is $3, 4, 5, 6$ or 7. In this paper, we discuss on the case that $|C(L)| = 3$. First we begin with the following notation.

Notation 2.2. Let $|C(L)| = n$, where $n > 1$. To simplify notation, we denote the maximal ideal $[m]^l$, where $m \in C(L)$, by $\mathfrak{m}$. We set $S_t := \mathfrak{m}_t \setminus \bigcup_{i\notin\{t\}} \mathfrak{m}_i$, where $1 \leq i, t \leq n$. Also $S_{t_1 t_2 \ldots t_k} := (\mathfrak{m}_{t_1} \cap \mathfrak{m}_{t_2} \cap \cdots \cap \mathfrak{m}_{t_k}) \setminus \bigcup_{i\notin\{t_1,t_2,\ldots,t_k\}} \mathfrak{m}_i$, where $1 \leq t_1, t_2, \ldots, t_k \leq n$.

Note that each element in S_i is adjacent to each element in S_j, for $1 \leq i \neq j \leq n$, and also it is adjacent to each element in $S_{t_1 t_2 \ldots t_k}$, where $t_1, \ldots, t_k \notin \{i\}$.

Remark 2.3. In [1, Theorem 4.3.], Afkhami and Khashyarmanesh completely determined those lattices with 3 co-atoms whose graph $\Gamma_2(L)$ is planar.

Lemma 2.4. *Assume that* $|\bigcup_{t=1}^{3} S_t| \geq 10$. *Then* $\Gamma_2(L)$ *is not a toroidal graph.*

Proof. Set $n := |\bigcup_{t=1}^{3} S_t|$. Then we have the following cases:
Case 1. $|S_i| \neq n-2$, for $i = 1,2,3$. Since the contraction of $\Gamma_2(L)$ contains a subgraph isomorphic to $K_{3,7}$ or $K_{4,5}$, one can conclude that the graph $\Gamma_2(L)$ is not toroidal.
Case 2. There exists $1 \leq i \leq 3$, such that $|S_i| = n-2$. If S_{jk} is an empty set, for $j,k \notin \{i\}$ with $1 \leq i,j,k \leq 3$, then $\Gamma_2(L)$ is planar, which is not a toroidal graph. If $S_{jk} \neq \varnothing$, for $j,k \notin \{i\}$ with $1 \leq i,j,k \leq 3$, then we can find a copy of $K_{3,8}$ in the contraction of $\Gamma_2(L)$, and thus the graph $\Gamma_2(L)$ is not toroidal. □

Now, by Lemma 2.4., we state necessary and sufficient conditions for toroidality of the graph $\Gamma_2(L)$, when $|C(L)| = 3$. It should be noted that in the proof of following theorem, according to Remark 2.3., the cases where the graph $\Gamma_2(L)$ is planar is ignored.

Theorem 2.5. *Suppose that* $|C(L)| = 3$. *Then* $\Gamma_2(L)$ *is a toroidal graph if and only if one of the following conditions holds:*

(i) $|\bigcup_{t=1}^{3} S_t| = 5$ *and one of the following conditions is satisfied:*

- (a) *There is some* S_i *with* $|S_i| = 3$, *for* $1 \leq i \leq 3$ *and* $|S_{i_1 i_2}| \in \{1,2,3,4\}$, *for* $i_1, i_2 \notin \{i\}$.
- (b) *There is a unique* S_i *with* $|S_i| = 1$, *for* $1 \leq i \leq 3$, *and* $S_{ii_1} \neq \varnothing$, *for* $i_1 \notin \{i\}$.

(ii) $|\bigcup_{t=1}^{3} S_t| = 6$ *and one of the following conditions is satisfied:*

- (a) *There exists some* i *with* $1 \leq i \leq 3$ *such that* $|S_i| = 4$, *and* $|S_{i_1 i_2}| \in \{1,2\}$, *for* $i_1, i_2 \notin \{i\}$.
- (b) *There exist unique* i *and* j *with* $1 \leq i,j \leq 3$ *such that* $|S_i| = 3$ *and* $|S_j| = 2$, *also if* $|S_{ji_1}| = 2$, *then* $|S_{ii_1}| \geq 0$, *and if* $|S_{ji_1}| = 3$, *then* $S_{ii_1} = \varnothing$, *for* $i_1 \notin \{i,j\}$.
- (c) $|S_i| = 2$, *for all* i *with* $1 \leq i \leq 3$, *and* $S_{i_1 i_2} \neq \varnothing$, *for* $1 \leq i_1, i_2 \leq 3$.

(iii) $|\bigcup_{t=1}^{3} S_t| = 7$ *and one of the following conditions is satisfied:*

- (a) *There is some* i *with* $1 \leq i \leq 3$ *such that* $|S_i| = 5$, *and* $|S_{i_1 i_2}| = 1$, *for* $i_1, i_2 \notin \{i\}$.
- (b) *There is some* i *with* $1 \leq i \leq 3$ *such that* $|S_i| = 4$, *and* $|S_{i_1 i_2}| = 1$, *for* $i_1, i_2 \notin \{i\}$.
- (c) $|S_i| = |S_j| = 3$ *for some* i *and* j *with* $1 \leq i,j \leq 3$, *and* $|S_{ii_1}| = |S_{ji_1}| = 1$, *for* $i_1 \notin \{i,j\}$. *Also if* $|S_{i_1 i_2}| = 2$, *for some* $i_1 \in \{i,j\}$, $i_2 \notin \{i,j\}$, *then* $S_{i_2 i_3} = \varnothing$, *for* $i_3 \in \{i,j\} \setminus \{i_1\}$.
- (d) *There is a unique* i *with* $1 \leq i \leq 3$ *such that* $|S_i| = 3$, *also* $|S_{i_1 i_2}| = 1$, *and if* $|S_{i_1 i_2}| = 2$, *then* S_{ii_1}, S_{ii_2} *are empty sets, for* $i_1, i_2 \notin \{i\}$.

(iv) $|\bigcup_{t=1}^{3} S_t| = 8$ *and one of the following conditions is satisfied:*

(a) *There exists some i with $1 \leq i \leq 3$ such that $|S_i| = 6$, and $|S_{i_1 i_2}| = 1$, for $i_1, i_2 \notin \{i\}$.*

(b) *There exists some i with $1 \leq i \leq 3$ such that $|S_i| = 5$, and $S_{i_1 i_2} = \varnothing$, for $i_1, i_2 \notin \{i\}$.*

(c) *There exist unique i and j with $1 \leq i, j \leq 3$ such that $|S_i| = 4$ and $|S_j| = 3$ and $S_{ii_1} = S_{ji_1} = \varnothing$, for $i_1 \notin \{i, j\}$.*

(d) *$|S_i| = |S_j| = 3$ for some i and j with $1 \leq i, j \leq 3$, and if $S_{i_1 i_2} = \varnothing$, for all $i_1 \in \{i, j\}$, $i_2 \notin \{i, j\}$, then $|S_{ij}| \geq 0$, also if $|S_{i_1 i_2}| = 1$, for some $i_1 \in \{i, j\}$, $i_2 \notin \{i, j\}$, then $S_{ij} = \varnothing$.*

(e) *$|S_i| = |S_j| = 2$ for some i and j with $1 \leq i, j \leq 3$, and $S_{ij} = \varnothing$.*

(v) $|\bigcup_{t=1}^{3} S_t| = 9$ *and one of the following conditions is satisfied:*

(a) *There is some i with $1 \leq i \leq 3$ such that $|S_i| = 6$, and $S_{i_1 i_2} = \varnothing$, for $i_1, i_2 \notin \{i\}$.*

(b) *$|S_i| = 3$ for all i with $1 \leq i \leq 3$, and $S_{i_1 i_2} = \varnothing$, for $i_1, i_2 \in \{1, 2, 3\}$.*

Proof. If one of the above statements holds, then one can easily check that $\Gamma_2(L)$ is a toroidal graph.

Conversely, let $\Gamma_2(L)$ be toroidal. By Lemma 2.4., $5 \leq |\bigcup_{t=1}^{3} S_t| \leq 9$. Thus we have the following situations:

(i) $|\bigcup_{t=1}^{3} S_t| = 5$.

Assume that there is some i, say $i = 1$, such that $|S_i| = 3$. If S_{23} is non-empty, then the contraction of $\Gamma_2(L)$ contains a subgraph isomorphic to $K_{3,3}$, and so it is not planar. Also when $|S_{23}| \geq 5$, we have a copy of $K_{3,7}$ in the contraction of $\Gamma_2(L)$ with vertex set $\{a_1, a_2, a_3\} \cup \{b, c, s_1, s_2, \ldots, s_5\}$, where $a_1, a_2, a_3 \in S_1$, $b \in S_2$, $c \in S_3$ and $s_1, s_2, \ldots, s_5 \in S_{23}$. It is clear that the graph $\Gamma_2(L)$ is not toroidal. Therefore, we may assume that $1 \leq |S_{23}| \leq 4$. In this situation, the complement of $\Gamma_2(L)$ contains $C603$, one of the listed graphs in [4], which is pictured in Figure 1. In Figure 1, we replace $x_1, x_2, \ldots, x_9$ by $a_1, a_2, a_3, b, s_1, s_2, s_3, s_4, c$, respectively, which $a_1, a_2, a_3 \in S_1$, $b \in S_2$, $c \in S_3$ and $s_1, s_2, s_3, s_4 \in S_{23}$. Therefore, $\Gamma_2(L)$ is a toroidal graph (see Figure 2).

Now, if there is a unique i, say $i = 1$, such that $|S_i| = 1$, and the sets S_{12} and S_{13} are non-empty, then we can find a subdivision of K_5 in the structure of the contraction of $\Gamma_2(L)$ as it is shown in Figure 3, where $a \in S_1$, $b_1, b_2 \in S_2$, $c_1, c_2 \in S_3$, $s_{12} \in S_{12}$ and $s_{13} \in S_{13}$. Thus $\Gamma_2(L)$ is toroidal.

(ii) $|\bigcup_{t=1}^{3} S_t| = 6$.

Suppose that there exists only one i, say $i = 1$, such that $|S_i| = 4$. If the size of S_{23} is at least 3, then one can easily observe that the contraction of $\Gamma_2(L)$ contains a subgraph isomorphic to $K_{4,5}$. Hence the graph $\Gamma_2(L)$ is not toroidal. So, for toroidality, the size of S_{23} is necessarily 1 or 2. In this case, $\Gamma_2(L)$ is contained in

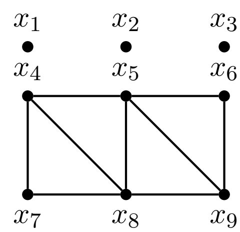


Figure 1: $C603$

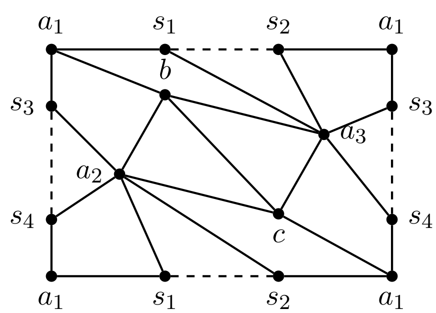


Figure 2:

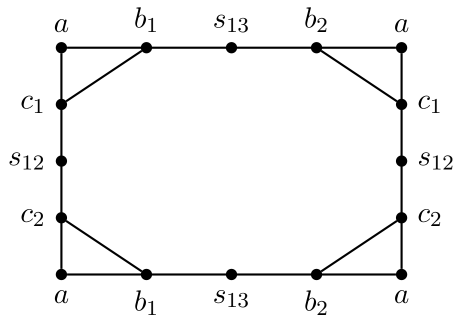


Figure 3:

$K_8 \setminus (K_3 \cup K_2)$ (cf. [4, p.55]). Hence $\Gamma_2(L)$ is a toroidal graph (see Figure 4). In Figure 4, we assume that $a_1, a_2, a_3, a_4 \in S_1$, $b \in S_2$, $c \in S_3$ and $s_{23}, s'_{23} \in S_{23}$.

Now, assume that there exists a unique i, say $i = 1$, such that $|S_i| = 3$. If S_{23} has at least 4 elements, then the contraction of $\Gamma_2(L)$ contains a copy of $K_{3,7}$, and hence $\Gamma_2(L)$ is not a toroidal graph. Also if $|S_{23}| = 3$ and S_{13} is non-empty, then the complement of the contraction of $\Gamma_2(L)$ is contained in $U6.6b$, one of the listed

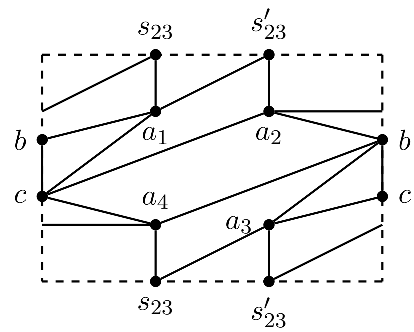


Figure 4:

graphs in [4]. So the graph $\Gamma_2(L)$ is not toroidal (see Figure 5). In Figure 5, we have the vertices $a_1, a_2, a_3 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $s_{13} \in S_{13}$ and $s_{23}, s'_{23}, s''_{23} \in S_{23}$.

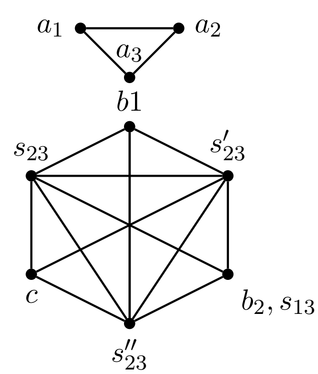


Figure 5: $U6.6b$

Therefore, for toroidality of $\Gamma_2(L)$, when $|S_{23}| = 3$, necessarily, $S_{13} = \varnothing$. In this situation, the complement of $\Gamma_2(L)$ contains $C610$, one of the listed graphs in [4] (see Figure 6). In Figure 6, we have the vertices $a_1, a_2, a_3 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$ and $s_{23}, s'_{23}, s''_{23} \in S_{23}$. In this situation, the embedding of the graph $\Gamma_2(L)$ in the torus is pictured in Figure 7.

In addition, if $|S_{23}| \leq 2$, then the contraction of $\Gamma_2(L)$ contains a copy of $K_{3,3}$, and so $\Gamma_2(L)$ is not planar. In this situation, $\Gamma_2(L)$ is contained in $K_8 \setminus (K_3 \cup K_2)$, (cf. [4, p.55]). Therefore, $\Gamma_2(L)$ is a toroidal graph (see Figure 8). In Figure 8, we have the vertices $a_1, a_2, a_3 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$, $s_{13} \in S_{13}$ and $s_{23}, s'_{23} \in S_{23}$.

Finally, suppose that $|S_i| = 2$, for all $1 \leq i \leq 3$ and only one of the sets S_{12}, S_{13} or S_{23} is non-empty. Then it is easy to check that the contraction of $\Gamma_2(L)$ contains $K_{3,3}$ and so it is not planar. Now we assume that at least one of the sets S_{12}, S_{13} or

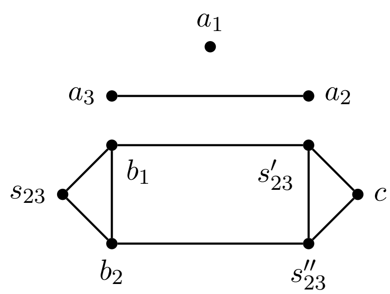


Figure 6: $C610$

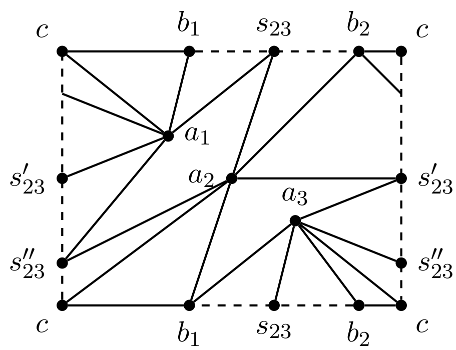


Figure 7:

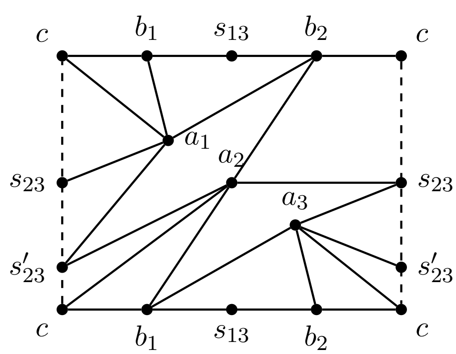


Figure 8:

S_{23} are non-empty sets. Obviously, we can find a subdivision of K_6 in $\Gamma_2(L)$ as it is shown in Figure 9, where $a_1, a_2 \in S_1$, $b_1, b_2 \in S_2$, $c_1, c_2 \in S_3$, $s_{12} \in S_{12}$, $s_{13} \in S_{13}$

and $s_{23} \in S_{23}$. Thus $\Gamma_2(L)$ is a toroidal graph.

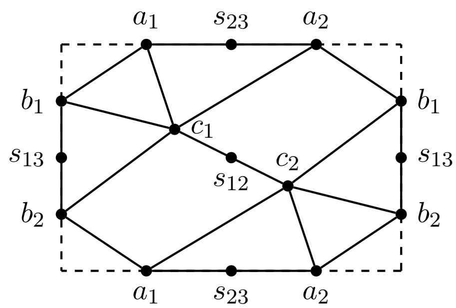


Figure 9:

(iii) $|\bigcup_{t=1}^{3} S_t| = 7$.

First, suppose that $|S_i| = 5$, for some $1 \leq i \leq 3$, without loss of generality, we may assume that $i = 1$. If S_{23} has exactly 1 element, then $\Gamma_2(L)$ is contained in $K_8 \setminus (K_3 \cup K_2)$, (cf. [4, p.55]). Hence the graph $\Gamma_2(L)$ is toroidal (see Figure 10). In Figure 10, we have the vertices $a_1, a_2, a_3, a_4, a_5 \in S_1$, $b \in S_2$, $c \in S_3$ and $s_{23} \in S_{23}$.

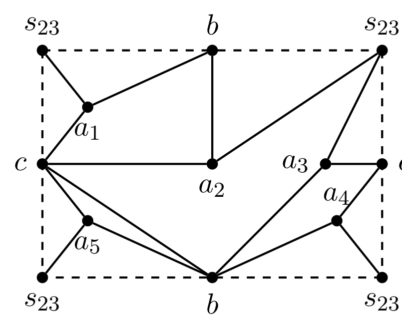


Figure 10:

We may assume that S_{23} has at least 2 elements. In this case, the vertices of the set $\{a_1, a_2, \ldots, a_5\} \cup \{b, c, s_{23}, s'_{23}\}$ form a subgraph isomorphic to $K_{4,5}$ in the contraction of $\Gamma_2(L)$, where $a_1, a_2, \ldots, a_5 \in S_1$, $b \in S_2$, $c \in S_3$ and $s_{23}, s'_{23} \in S_{23}$. Therefore, $\Gamma_2(L)$ is not a toroidal graph.

Suppose that there is i with $1 \leq i \leq 3$, say 1, such that $|S_i| = 4$. When S_{23} is a singleton set, the graph $\Gamma_2(L)$ is contained in $K_8 \setminus (K_3 \cup K_2)$ (cf. [4, p.55]). So the graph $\Gamma_2(L)$ is toroidal (see Figure 11). In Figure 11, we have the vertices $a_1, a_2, a_3, a_4 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$ and $s_{23} \in S_{23}$.

As $|S_{23}| \geq 2$, the contraction of $\Gamma_2(L)$ contains a copy of $K_{4,5}$, which implies that $\Gamma_2(L)$ is not a toroidal graph. If S_1 and S_2 have exactly 3 elements, and

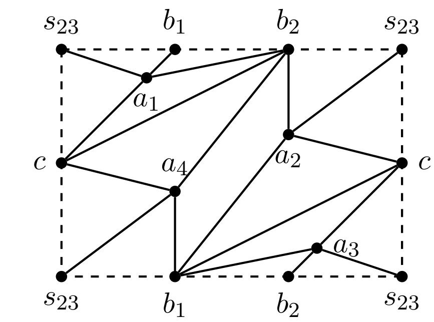


Figure 11:

S_{13} or S_{23} has at least 3 elements, then we have a subgraph isomorphic to $K_{3,7}$ in the contraction of $\Gamma_2(L)$, and so $\Gamma_2(L)$ is not a toroidal graph. Therefore, we assume that $|S_{13}| \leq 2$, $S_{23} = \varnothing$ or $|S_{23}| \leq 2$, $S_{13} = \varnothing$. Then the complement of $\Gamma_2(L)$ contains $C603$, one of the listed graphs in [4] (see Figure 1). In Figure 1, we replace vertices $x_1, x_2, \ldots, x_9$ by $b_1, b_2, b_3, a_1, s_{13}, c, a_2, a_3, s'_{13}$, respectively, where $a_1, a_2, a_3 \in S_1$, $b_1, b_2, b_3 \in S_2$, $c \in S_3$ and $s_{13}, s'_{13} \in S_{13}$. Hence the graph $\Gamma_2(L)$ is toroidal (see Figure 12).

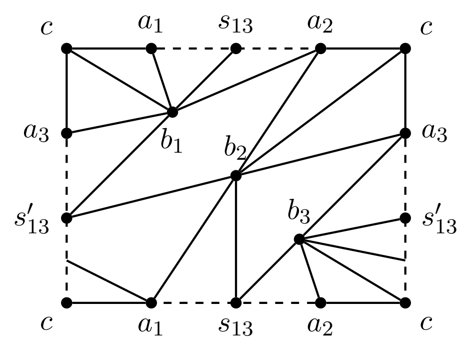


Figure 12:

In addition, we may assume S_{13} and S_{23} have 1 element, exactly. Then the complement of $\Gamma_2(L)$ contains $C402$, one of the listed graphs in [4] (see Figure 13). In Figure 13, we have the vertices $a_1, a_2, a_3 \in S_1$, $b_1, b_2, b_3 \in S_2$, $c \in S_3$, $s_{13} \in S_{13}$ and $s_{23} \in S_{23}$. So $\Gamma_2(L)$ is a toroidal graph, which is pictured in Figure 14.

If one of the sets S_{13} or S_{23} has exactly 2 elements and the other one has only 1 element, then $\Gamma_2(L)$ contains a subgraph isomorphic to G_3, one of the listed graphs in [12]. So $\Gamma_2(L)$ is not a toroidal graph. To do this, in Figure 15, we have the vertices $a_1, a_2, a_3 \in S_1$, $b_1, b_2, b_3 \in S_2$, $c \in S_3$, $s_{13}, s'_{13} \in S_{13}$ and $s_{23} \in S_{23}$.

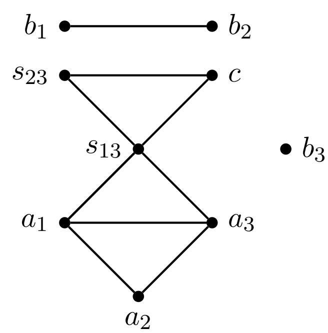


Figure 13: $C402$

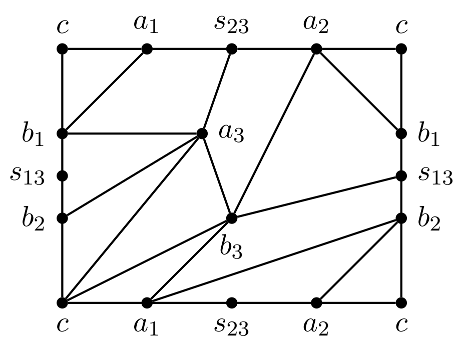


Figure 14:

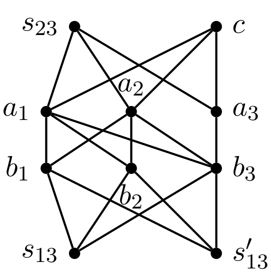


Figure 15: G_3

Now, consider the case that there is a unique i, say 1, such that $|S_i| = 3$. If S_{23} has at least 3 elements, then the contraction of $\Gamma_2(L)$ contains a copy of $K_{3,7}$, and so the graph $\Gamma_2(L)$ is not toroidal. When $|S_{23}| = 2$, and also S_{12} or S_{13} is non-empty, the complement of the contraction of $\Gamma_2(L)$ is contained in $U6.6b$, one of the listed graphs in [4] (see Figure 16). To do this, in Figure 16, we have the vertices $a_1, a_2, a_3 \in S_1$, $b_1, b_2 \in S_2$, $c_1, c_2 \in S_3$, $s_{23}, s'_{23} \in S_{23}$ and $s_{12} \in S_{12}$. So the graph $\Gamma_2(L)$ is not toroidal.

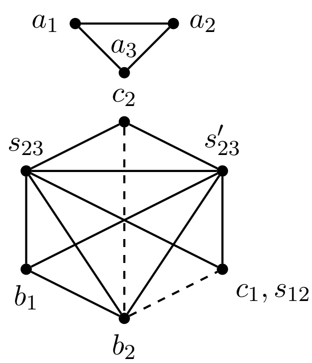


Figure 16: $U6.6b$

Hence we assume that $|S_{23}| = 2$ and $S_{12} = S_{13} = \varnothing$. Then the complement of $\Gamma_2(L)$ contains $C603$, one of the listed graphs in [4] (see Figure 1). In Figure 1, we replace the vertices $x_1, x_2, \ldots, x_9$ by $a_1, a_2, a_3, b_1, s_{23}, c_1, b_2, s'_{23}, c_2$, respectively, where $a_1, a_2, a_3 \in S_1$, $b_1, b_2 \in S_2$, $c_1, c_2 \in S_3$ and $s_{23}, s'_{23} \in S_{23}$. Hence $\Gamma_2(L)$ is a toroidal graph, which is pictured in Figure 17.

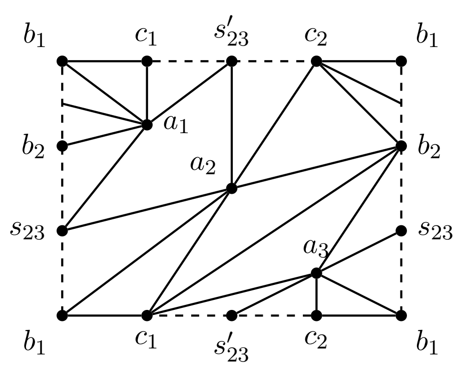


Figure 17:

When S_{23} is a singleton set, $\Gamma_2(L)$ is contained in $K_8 \setminus (K_3 \cup K_2)$, (cf. [4, p.55]). Thus the graph $\Gamma_2(L)$ is toroidal (see Figure 18). In Figure 18, we have the vertices $a_1, a_2, a_3 \in S_1$, $b_1, b_2 \in S_2$, $c_1, c_2 \in S_3$, $s_{12} \in S_{12}$, $s_{13} \in S_{13}$ and $s_{23} \in S_{23}$.

(iv) $|\bigcup_{t=1}^{3} S_t| = 8$.

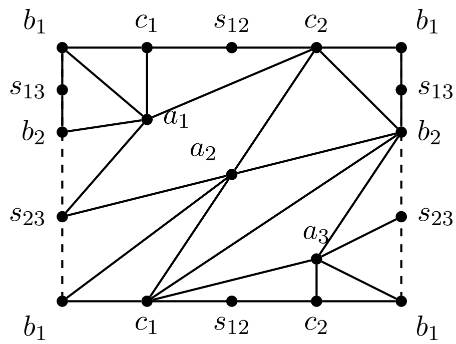


Figure 18:

Suppose that there exists only one i, say $i = 1$, such that $|S_i| = 6$. If the size of S_{23} is at least 2, then one can easily see that the contraction of $\Gamma_2(L)$ contains a subgraph isomorphic to $K_{4,6}$. Hence $\Gamma_2(L)$ is not a toroidal graph. So the size of S_{23} is necessarily 1. In this situation, the complement of $\Gamma_2(L)$ contains $C603$, one of the listed graphs in [4] (see Figure 1). To do this, in Figure 1, we replace the vertices $x_1, x_2, \ldots, x_9$ by $b, s_{23}, c, a_1, a_3, a_5, a_2, a_4, a_6$, respectively, where $a_1, a_2, \ldots, a_6 \in S_1$, $b \in S_2$, $c \in S_3$ and $s_{23} \in S_{23}$. Thus $\Gamma_2(L)$, which is pictured in Figure 19, is a toroidal graph.

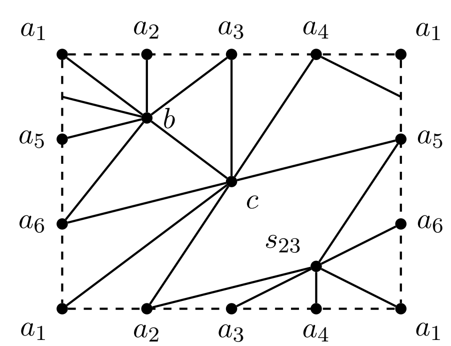


Figure 19:

Now, suppose that there exists some i, say $i = 1$, such that $|S_i| = 5$. If S_{23} is non-empty, then the contraction of $\Gamma_2(L)$ contains a copy of $K_{4,5}$, and so $\Gamma_2(L)$ is not a toroidal graph. Otherwise, $S_{23} = \varnothing$, and so $\Gamma_2(L)$ is contained in $K_8 \setminus (K_3 \cup K_2)$, (cf. [4, p.55]). Therefore, $\Gamma_2(L)$ is a toroidal graph (see Figure 20). In Figure 20, we have the vertices $a_1, a_2, \ldots, a_5 \in S_1$, $b_1, b_2 \in S_2$ and $c \in S_3$.

Suppose that there exist unique i and j with $1 \leq i, j \leq 3$, say $i = 1$ and $j = 2$,

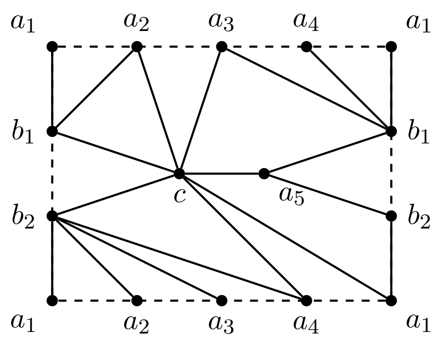


Figure 20:

such that $|S_i| = 4$, $|S_j| = 3$. If S_{13} is non-empty, then the complement of $\Gamma_2(L)$ is contained in $S5.5$, one of the listed graphs in [4] (see Figure 21). In fact, in Figure 21, we have the vertices $a_1, a_2, a_3, a_4 \in S_1$, $b_1, b_2, b_3 \in S_2$, $c \in S_3$ and $s_{13} \in S_{13}$. And so the graph $\Gamma_2(L)$ is not toroidal.

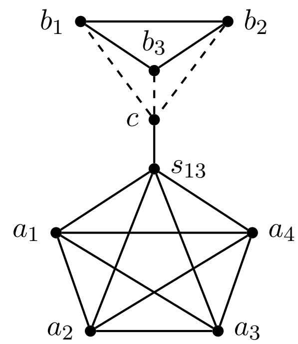


Figure 21: $S5.5$

Also if S_{23} is non-empty, then one can easily see that the contraction of $\Gamma_2(L)$ contains a copy of $K_{4,5}$. Hence $\Gamma_2(L)$ is not toroidal. So the size of the sets S_{13} and S_{23} is 0. In this case, $\Gamma_2(L)$ is contained in $K_8 \setminus (K_3 \cup K_2)$ (cf. [4, p.55]), which is a toroidal graph (see Figure 22). In Figure 22, we have the vertices $a_1, a_2, a_3, a_4 \in S_1$, $b_1, b_2, b_3 \in S_2$ and $c \in S_3$.

Suppose that S_1 and S_2 have exactly 3 elements. If $|S_{13}| \geq 2$ or $|S_{23}| \geq 2$, then it is easy to see that the contraction of $\Gamma_2(L)$ contains a copy of $K_{3,7}$. Hence $\Gamma_2(L)$ is not a toroidal graph. Also, if S_{13} and S_{23} have only 1 element, then $\Gamma_2(L)$ contains a subgraph isomorphic to G_3, one of the listed graphs in [12]. So the graph $\Gamma_2(L)$ not toroidal. To do this, in Figure 23, we have the vertices $a_1, a_2, a_3 \in S_1$,

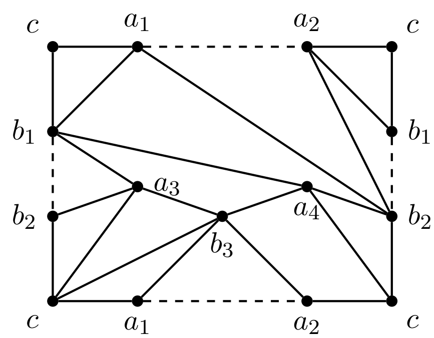


Figure 22:

$b_1, b_2, b_3 \in S_2$, $c_1, c_2 \in S_3$, $s_{13} \in S_{13}$ and $s_{23} \in S_{23}$.

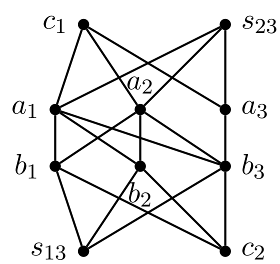


Figure 23: G_3

So we may assume that S_{12} is non-empty and also S_{13} or S_{23} has only 1 element. Then the complement of the contraction of $\Gamma_2(L)$ is contained in $S5.6$, one of the listed graphs in [4] (see Figure 24). Hence the graph $\Gamma_2(L)$ is not toroidal. In Figure 24, we have the vertices $a_1, a_2, a_3 \in S_1$, $b_1, b_2, b_3 \in S_2$, $c_1, c_2 \in S_3$, $s_{12} \in S_{12}$ and $s_{13} \in S_{13}$.

Therefore if the size of the set S_{13} or S_{23} is 1, then necessarily S_{12} must be an empty set. Since the complement of $\Gamma_2(L)$ contains $C402$, one of the listed graphs in [4]. To do this, in Figure 25, we have the vertices $a_1, a_2, a_3 \in S_1$, $b_1, b_2, b_3 \in S_2$, $c_1, c_2 \in S_3$, $s_{13} \in S_{13}$. Hence $\Gamma_2(L)$ is a toroidal graph (see Figure 26).

Consequently, two sets S_{13} and S_{23} are both empty. In this situation, $\Gamma_2(L)$ is contained in $K_8 \setminus (K_3 \cup K_2)$ (cf. [4, p.55]), which is a toroidal graph. To do this, in Figure 27, we have the vertices $a_1, a_2, a_3 \in S_1$, $b_1, b_2, b_3 \in S_2$, $c_1, c_2 \in S_3$ and $s_{12} \in S_{12}$.

Now, suppose that S_1 and S_2 have 2 elements. If S_{12} is non-empty, then the contraction of $\Gamma_2(L)$ contains a copy of $K_{4,5}$, and thus the graph $\Gamma_2(L)$ is not toroidal. So the set S_{12} is necessarily empty. In this case, $\Gamma_2(L)$ is contained in

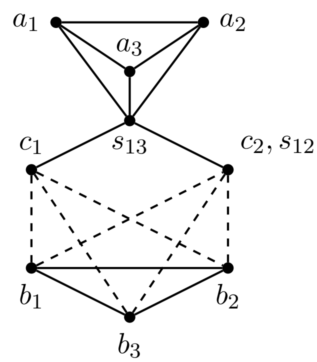


Figure 24: $S5.6$

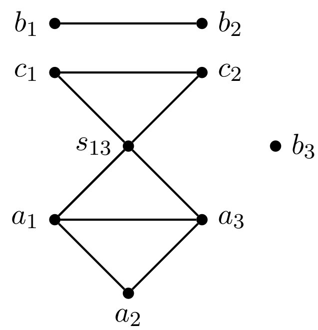


Figure 25: $C402$

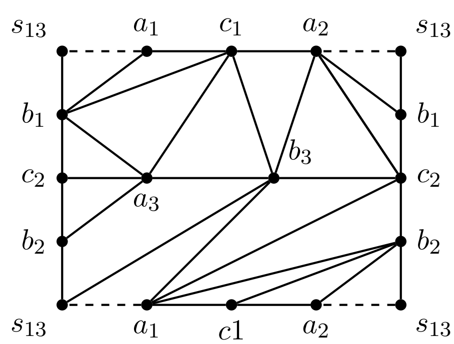


Figure 26:

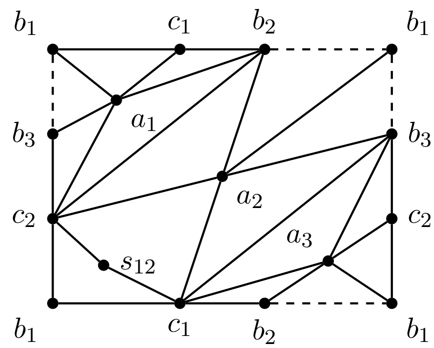


Figure 27:

$K_8 \setminus (K_3 \cup K_2)$, (cf. [4, p.55]), which is a toroidal graph (see Figure 28). In Figure 28, we have the vertices $a_1, a_2 \in S_1$, $b_1, b_2 \in S_2$ and $c_1, c_2, c_3, c_4 \in S_3$.

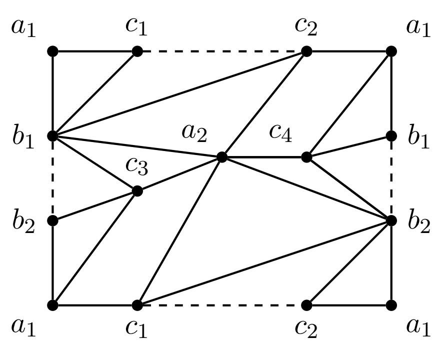


Figure 28:

(v) $|\bigcup_{t=1}^{3} S_t| = 9$.

First, suppose that $|S_i| = 7$, for some $1 \leq i \leq 3$. Without loss of generality, we may assume that $i = 1$. If S_{23} is non-empty, then the contraction of $\Gamma_2(L)$ contains a copy of $K_{3,7}$. Hence the graph $\Gamma_2(L)$ is not toroidal.

Also suppose that there is some i with $1 \leq i \leq 3$, say 1, such that $|S_i| = 6$ and S_{23} is non-empty. Then we can find a copy of $K_{4,6}$ in the contraction of $\Gamma_2(L)$. Therefore, $\Gamma_2(L)$ is not a toroidal graph. So, for toroidality of $\Gamma_2(L)$, it is sufficient $S_{23} = \varnothing$, because in this situation the complement of $\Gamma_2(L)$ contains $C603$, one of the listed graphs in [4]. To do this, in Figure 1, we replace vertices $x_1, x_2, \ldots, x_9$ by $b_1, b_2, c, a_1, a_3, a_5, a_2, a_4, a_6$, respectively, where $a_1, a_2, \ldots, a_6 \in S_1$, $b_1, b_2 \in S_2$, $c \in S_3$. The embedding of $\Gamma_2(L)$ in the torus is pictured in Figure 29.

Now, suppose that all of the sets S_1, S_2 and S_3 have 3 elements, exactly. If

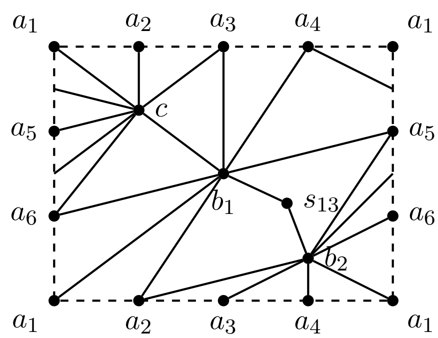


Figure 29:

S_{12}, S_{13} and S_{23} are empty, then the complement of $\Gamma_2(L)$ contains $C315$, one of the listed graphs in [4] (see Figure 30). In Figure 30, we have the vertices $a_1, a_2, a_3 \in S_1$, $b_1, b_2, b_3 \in S_2$, $c_1, c_2, c_3 \in S_3$. Therefore, the graph $\Gamma_2(L)$, which is pictured in Figure 31, is toroidal.

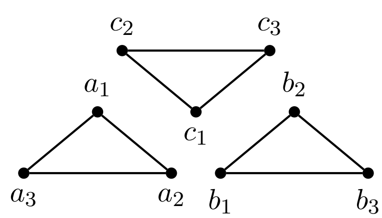


Figure 30: $C315$

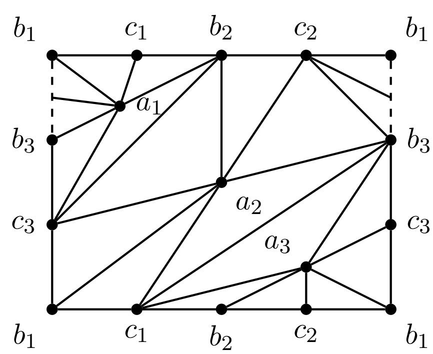


Figure 31:

Otherwise, we may assume that at least one of the sets S_{12}, S_{13} or S_{23} is non-empty. Then the contraction of $\Gamma_2(L)$ contains a copy of $K_{3,7}$. Thus $\Gamma_2(L)$ is not a toroidal graph. Otherwise, there exists some i, with $1 \le i \le 3$, such that $|S_i| = 4$ or $|S_i| = 5$. In these situations, the contraction of $\Gamma_2(L)$ contains a copy of $K_{4,5}$. Therefore, $\Gamma_2(L)$ is not a toroidal graph. □

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