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Examples of Central Semicommutative Rings

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Abstract. An example of a strongly central semicommutative ring which is not semicommutative is constructed. This answers a question of Bhattachafjee and Chakraborty negatively.

1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. For a ring R, the symbol $N(R)$ denotes the set of nilpotent elements of R, $Z(R)$ its center, $N_*(R)$ its prime radical, $J(R)$ its Jacobson radical, and $R[x]$ the polynomial ring over R. The symbol E_{ij} stands for the usual $n \times n$ matrix units, and I_n the $n \times n$ identity matrix over R. For any $n \times n$ matrix A, we write $RA = \{rA|r \in R\}$, $V = \sum_{i=1}^{n-1} E_{i,i+1}$, and $V_n(R) = RI_n + RV + \cdots + RV^{n-1}$ whenever $n \geq 2$. The symbol $\mathbb{Z}_n$ denotes the ring of integers modulo a positive integer n, and S_n the permutation group on $\{1, 2, \cdots, n\}$.

Let R be a ring and $a, b, c \in R$. Usually R is called reduced if it has no nonzero nilpotent elements. According to Lambek [9], R is called symmetric if $abc = 0$ implies $acb = 0$, Cohn [3] called R reversible if $ab = 0$ implies $ba = 0$, and Narbonne [12] named R semicommutative if $ab = 0$ implies $aRb = 0$. It is known that reduced ⇒ symmetric ⇒ reversible ⇒ semicommutative, and no reversal holds (see [11]). Recently years, various generalized conditions of symmetric rings, reversible rings and semicommutative rings have been studied by many authors, and the results obtained are applied to many sorts of problems in noncommutative ring theory. Ungor et al. [14] called R central reduced if $N(R) \subseteq Z(R)$, Kose et al. [7] called R central reversible if $ab = 0$ implies $ba \in Z(R)$, and Kafkas et al. [6] called R central symmetric if $abc = 0$ implies $bac \in Z(R)$. As noted by Jung et

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al. [5], this notion is not left-right symmetric. Thus they called such a ring left central symmetric, and defined a right central symmetric ring similarly. Özen et al. [13] called R central semicommutative if $ab = 0$ implies $aRb \subseteq Z(R)$. By the existing literature, central reduced $\Rightarrow$ left and right central symmetric, left or right central symmetric $\Rightarrow$ central reversible and central semicommutative, and each converse implication does not hold. But whether a central reversible ring is central semicommutative is an open question in [5]. Recently, Bhattachafjee and Chakraborty [2] defined a ring R to be strongly central semicommutative (reversible) if $R[x]$ is central semicommutative (reversible). However, they do not know whether a strongly central semicommutative rings R is strongly semicommutative, i.e., $R[x]$ is semicommutative (see Yang and Du [15]). The main objective of this paper is to answer the question in the negative. Moreover, some new results of central reduced rings and central symmetric rings are included.

2. Main Results

We start this section with the observation.

Lemma 2.1. *A ring R is a strongly central semicommutative ring if and only if $f(x)g(x) = 0$ implies $f(x)Rg(x) \subseteq Z(R[x])$ for any $f(x), g(x) \in R[x]$.*

Proof. Clearly, a strongly central semicommutative ring R satisfies the condition stated in the lemma. Conversely, let $f(x), g(x) \in R[x]$ with $f(x)g(x) = 0$. For any $h(x) = c_0 + c_1x + \cdots + c_lx^l \in R[x]$, then we have $f(x)h(x)g(x) = f(x)c_0g(x) + f(x)c_1g(x)x + \cdots + f(x)c_lg(x)x^l$. By hypothesis, $f(x)c_kg(x) \in Z(R[x])$ for all $k = 0, 1, \cdots, l$. It follows that $f(x)h(x)g(x) \in Z(R[x])$. □

Theorem 2.2. *There exists a strongly central semicommutative ring which is not strongly semicommutative.*

Proof. Let $A = F[a, b, c]$ be the free algebra of polynomials with zero constant terms in noncommuting identerminates a, b, c over $\mathbb{Z}_2$. Let I be an ideal of $\mathbb{Z}_2 + A$, generated by $ab, r_1r_2r_3r_4$ where $r_1, r_2, r_3, r_4 \in A$, and let $R = (\mathbb{Z}_2 + A)/I$. Clearly, R is a local ring with $J(R) = A/I$ nilpotent. We show that R is a strongly central semicommutative ring but not a semicommutative ring. For simplicity, we identify elements of $\mathbb{Z}_2 + A$ with their images in R. Let $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ satisfying $f(x)g(x) = 0$. With the help of Lemma 2.1, it suffices to prove $f(x)Rg(x) \in Z(R[x])$. For convenience of the calculation, we rewrite $f(x) = \sum_{i=0}^{m+n} a_ix^i$ and $g(x) = \sum_{j=0}^{m+n} b_jx^j$ where $a_{m+1} = \cdots = a_{m+n} = b_{n+1} = \cdots = b_{m+n} = 0$. In what follows the assumption $f(x)g(x) = 0$ should be remembered.

Claim 1. Either $f(x) \in A[x]$ or $g(x) \in A[x]$.

Assume on the contrary, there exist the least coefficients a_i, b_j of $f(x), g(x)$, respectively such that a_i and b_j are units in R, since R is a local ring. Considering the coefficient of x^{i+j} in $f(x)g(x) = 0$, we have $a_0b_{i+j} + \cdots + a_{i-1}b_{j+1} + a_ib_j + a_{i+1}b_{j-1} + \cdots + a_{i+j}b_0 = 0$. Observing $a_0, \cdots, a_{i-1}, b_0, \cdots, b_{j-1} \in A$, $a_ib_j \in A$ is not a unit. This is a contradiction.

Claim 2. If $f(x) \notin A[x]$ and $g(x) \in A[x]$, then $g(x) \in A^2[x]$.

Since $f(x) \notin A$, there exists the least coefficient a_i of $f(x)$ such that a_i is a unit in R. Hence $f(x)g(x) = 0$ implies $a_0b_i + a_1b_{i-1} + \cdots + a_{i-1}b_1 + a_ib_0 = 0$. This gives $a_ib_0 \in A^2$, and so $b_0 \in A^2$. Similarly $a_0b_{i+1} + a_1b_i + \cdots + a_ib_1 + a_{i+1}b_0 = 0$ will give $b_1 \in A^2$. Continuing this process, finally we have all $b_j \in A^2$ and thus $g(x) \in A^2[x]$.

The validity of the next claim can be checked similarly to the proof of Claim 2.

Claim 3. If $f(x) \in A[x]$ and $g(x) \notin A[x]$, then $f(x) \in A^2[x]$.

Now we show $f(x)Rg(x) \subseteq Z(R[x])$. For any $r \in R$, then $r = k + c$ for some $k \in \mathbb{Z}_2$ and $c \in A$. This means $f(x)rg(x) = kf(x)g(x) + f(x)cg(x) = f(x)cg(x)$ since $f(x)g(x) = 0$.

Case (1). If $f(x) \in A[x]$ and $g(x) \in A[x]$, then we have $f(x)rg(x) = f(x)cg(x) \in A^3[x] \subseteq Z(R)[x] \subseteq Z(R[x])$ by Claim 1 and the fact $A^3 \subseteq Z(R)$.

Case (2). If $f(x) \notin A[x]$ and $g(x) \in A[x]$, then all $b_j \in A^2$ by Claim 2. It follows that $f(x)rg(x) = f(x)cg(x) \in A^3[x] \subseteq Z(R)[x] \subseteq Z(R[x])$ since $A^3 \subseteq Z(R)$.

Case (3). If $f(x) \in A[x]$ and $g(x) \notin A[x]$, then all $a_i \in A^2$ by Claim 3. Similarly to the proof of Case 2, we also have $f(x)rg(x) = f(x)cg(x) \subseteq Z(R[x])$.

From the above argument, we conclude that $R[x]$ is a strongly central semicommutative ring by Lemma 2.1. On the other hand, since $ab = 0$ but $acb \neq 0$, R is not a semicommutative ring and so R is not a strongly semicommutative ring. □

Theorem 2.2 gives a negative answer to [2, Question 3.4].

3. Related Results

Let R be a ring, $a_1, a_2, \cdots, a_n \in R$, and $\sigma \in S_n$ where $n \geq 2$ is any positive integer. Following Anderson and Camillo [1], a ring R is said to satisfy ZC_n if $a_1a_2 \cdots a_n = 0$ implies $a_{\sigma(1)}a_{\sigma(2)} \cdots a_{\sigma(n)} = 0$. A reduced ring satisfies ZC_n by [8, Lemma 2.1], and a ring is symmetric if and only if it satisfies ZC_n for all $n \geq 3$ by [9, Proposition 1]. Analogically, we call R to satisfy GZC_n if $a_1a_2 \cdots a_n = 0$ implies $a_{\sigma(1)}a_{\sigma(2)} \cdots a_{\sigma(n)} \in Z(R)$ for any $n \geq 2$.

Theorem 3.1. *If R is a central reduced ring, then $V_n(R)$ satisfies GZC_n for all $n \geq 2$.*

Proof. Combining Theorems 2.15 and 2.16 in [14], we conclude that R is a central reduced ring if and only if $N_*(R) \subseteq Z(R)$, and R is 2-primal (i.e., a ring with $N_*(R) = N(R)$). Now since R is a central reduced ring, R is a 2-primal ring and so $\overline{R} = R/N_*(R)$ is a reduced ring. For any $n \times n$ matrix $M = (m_{ij})$ over R, denote $\overline{M} = (\overline{m_{ij}})$. If $A_1, A_2, \cdots, A_n \in V_n(R)$ with $A_1A_2 \cdots A_n = 0$, then $\overline{A_1}\ \overline{A_2} \cdots \overline{A_n} = \bar{0}$ in $V_n(\overline{R})$. Since $\overline{R}$ is a reduced ring, $V_n(\overline{R})$ is a symmetric ring by [4, Theorem 2.1]. Applying [9, Proposition 1], we have $\overline{A_{\sigma(1)}}\ \overline{A_{\sigma(2)}} \cdots \overline{A_{\sigma(n)}} = \bar{0}$. It follows that $A_{\sigma(1)}A_{\sigma(2)} \cdots A_{\sigma(n)} \in V_n(N_*(R)) \subseteq V_n(Z(R)) \subseteq Z(V_n(R))$. □

Corollary 3.2. *A central reduced ring R satisfies GZC_n for any $n \geq 2$.*

Proof. It follows by the ring isomorphism $R \cong RI_n$ which is a subring of $V_n(R)$. □

It can be inferred from [14, Theorem 2.24] that a ring R is central reduced if and only if $R[x]$ is central reduced, thus the following corollary is immediately.

Corollary 3.3. *If R is a central reduced ring, then $V_n(R)[x]$ satisfies GZC_n for all $n \geq 2$. In particular, $V_n(R)[x]$ is a central symmetric ring.*

Proof. This is a direct consequence of Theorem 3.1 and the ring isomorphism $V_n(R)[x] \cong V_n(R[x])$. □

Since a left or right central symmetric ring is central reversible and central semicommutative, the next two corollaries are direct consequences of Corollary 3.3.

Corollary 3.4.([2, Proposition 2.13]) *If R is a central reduced ring, then $V_2(R)$ is a strongly central reversible ring.*

Corollary 3.5.([2, Proposition 3.14]) *If R is a central reduced ring, then $V_2(R)$ is a strongly central semicommutative ring.*

In view of the fact that a ring R is symmetric if and only if R satisfies ZC_n for any $n \geq 3$. One may naturally ask whether the parallel conclusion is true for a left and right central symmetric ring. The answer to this question is negative.

Example 3.6. A left and right central symmetric ring R need not satisfy GZC_3.

Proof. Let $A = F[a, b, c]$ be the free algebra of polynomials with zero constant terms in noncommuting identerminates a, b, c over $\mathbb{Z}_2$. Then $\mathbb{Z}_2 + A = \bigoplus_{i=0}^{\infty} A_i$ is a graded ring where A_i consists of homogeneous polynomials of degree i. Let I be the ideal of $\mathbb{Z}_2 + A$ generated by $abc, cabr_1 + r_1cab, bcar_2 + r_2bca, bacr_3 + r_3bac, acbr_4 + r_4acb, r_5r_6r_7r_8r_9$ where $r_i \in A$, and let $R = (\mathbb{Z}_2 + A)/I$. It is easy to check that I is a homogeneous ideal, and $R = \bigoplus_{i=0}^{\infty}(A_i + I)/I$ is a graded ring with its natural grading. Clearly, we have $R = \bigoplus_{i=0}^{4}(A_i + I)/I$ since $A^5 \subseteq I$. We identify the elements of $\mathbb{Z}_2 + A$ with their images for simplicity. Thus, R is a local ring with $J(R) = A$ nilpotent. To complete the proof, we need the following claims.

Claim 4. If $f_1, g_1, h_1 \in A_1$ with $f_1g_1h_1 = 0$, then $g_1f_1h_1, f_1h_1g_1 \in Z(R)$.

To see this, let $f_1 = k_1a + k_2b + k_3c, g_1 = l_1a + l_2b + l_3c, h_1 = m_1a + m_2b + m_3c$ where $k_i, l_i, m_i \in \mathbb{Z}_2$. The condition $f_1g_1h_1 = 0$ implies $k_il_jm_s = 0$ in case $(i, j, s) \neq (1, 2, 3)$. Thus we have $g_1f_1h_1 = l_2k_1m_3bac \in Z(R)$, and $f_1h_1g_1 = k_1m_3l_2acb \in Z(R)$ by the definition of I.

Claim 5. If $f_1 \in A_1, g_2 \in A_2$ with $f_1g_2 = 0$, then $g_2f_1 \in Z(R)$.

In fact, let $f_1 = k_1a + k_2b + k_3c$, and $g_2 = l_{11}a^2 + l_{12}ab + l_{13}ac + l_{21}ba + l_{22}b^2 + l_{23}bc + l_{31}ca + l_{32}cb + l_{33}c^2$ where $k_i, l_{js} \in \mathbb{Z}_2$. Now $f_1g_2 = 0$ implies $k_il_{js} = 0$ in case $(i, j, s) \neq (1, 2, 3)$. This means $g_2f_1 = l_{23}k_1bca \in Z(R)$ by the definition of I.

Claim 6. If $g_1 \in A_1, f_2 \in A_2$ with $f_2g_1 = 0$, then $g_1f_2 \in Z(R)$.

The validity of this claim can be easily checked as the proof of Claim 5.

Now we show that R is a left and right central symmetric ring. Let $\alpha, \beta, \gamma \in R$ with $\alpha\beta\gamma = 0$. If one of α, β, γ equals zero, then clearly $\beta\alpha\gamma \in Z(R)$ and $\alpha\gamma\beta \in Z(R)$. Therefore we may assume that $\alpha \neq 0, \beta \neq 0, \gamma \neq 0$.

Case (1). If $\alpha, \beta, \gamma \in A$, then $\alpha = f_1 + f_2 + f_3 + f_4$, $\beta = g_1 + g_2 + g_3 + g_4$ and $\gamma = h_1 + h_2 + h_3 + h_4$ where $f_i, g_i, h_i \in A_i$. This means $\alpha\beta\gamma = f_1g_1h_1 + \xi$ for some $\xi \in A_4$, and so $f_1g_1h_1 = 0$. Thus we have $g_1f_1h_1 \in Z(R)$ and $f_1h_1g_1 \in Z(R)$ by Claim 4. It follows that $\beta\alpha\gamma = g_1f_1h_1 + \eta$ for some $\eta \in A_4$, and hence $\beta\alpha\gamma \in Z(R)$. Similarly, we have $\alpha\gamma\beta \in Z(R)$.
Case (2). If $\alpha, \gamma \in A$ and $\beta \notin A$, then $\beta = 1 + \beta'$ for some $\beta' \in A$. Let α, γ be the same as in Case (1), and $\beta' = g_1 + g_2 + g_3 + g_4$. Then we have $\alpha\beta\gamma = \alpha\gamma + \alpha\beta'\gamma = f_1h_1 + (f_1h_2 + f_2h_1 + f_1g_1h_1) + \zeta$ for some $\zeta \in A_4$. This gives $f_1g_1 = 0$ and $f_1h_2 + f_2h_1 + f_1g_1h_1 = 0$, and so $f_1 = 0$ or $h_1 = 0$. If $f_1 = 0$, then $f_2h_1 = 0$ by the above argument. This means $\alpha\gamma = f_2h_1 + \rho = \rho$ for some $\rho \in A_4$, and so $\alpha\gamma \in Z(R)$. It is easy to see $\beta\alpha\gamma = \alpha\gamma + \beta'\alpha\gamma \in Z(R) + A^4 = Z(R)$, and similarly $\alpha\gamma\beta \in Z(R)$. If $h_1 = 0$, then $f_1h_2 = 0$. Now $\alpha\gamma = f_1h_2 + \varrho = \varrho$ for some $\varrho \in A_4$, clearly we have $\alpha\gamma\beta, \beta\alpha\gamma \in Z(R)$.
Case (3). If $\alpha, \beta \in A$ and $\gamma \notin A$, then $\alpha\beta = 0$ since R is a local ring. Let α, β be the same as in Case (1), and $\gamma = 1 + \gamma'$ for some $\gamma' \in A$. Since $\alpha\beta = 0$, we have $f_1g_1 = 0$ and $f_1g_2 + f_2g_1 = 0$. If $f_1 = 0$, then $f_2g_1 = 0$, and so $g_1f_2 \in Z(R)$ by Claim 6. This means $\beta\alpha = g_1f_2 + \upsilon$ for some $\upsilon \in A_4$. Thus $\beta\alpha\gamma = \beta\alpha + \beta\alpha\gamma' = \beta\alpha + \upsilon'$ for some $\upsilon' \in A_4$, this gives $\beta\alpha\gamma \in Z(R)$. Also, we have $\alpha\gamma\beta = \alpha\beta + \alpha\gamma'\beta = \alpha\gamma'\beta \in A_4 \subseteq Z(R)$. If $g_1 = 0$, then $f_1g_2 = 0$, and so $g_2f_1 \in Z(R)$ by Claim 5. This means $\beta\alpha = g_2f_1 + \tau$ for some $\tau \in A_4$. Now it is easy to check $\beta\alpha\gamma \in Z(R)$, and $\alpha\gamma\beta = \alpha\beta + \alpha\gamma'\beta = \alpha\gamma'\beta \in Z(R)$.
Case (4). If $\alpha \notin A$ and $\beta, \gamma \in A$, then we also have $\beta\alpha\gamma, \alpha\gamma\beta \in Z(R)$. We omit the proof, since it is very similarly to that of Case (3).

From the above argument, we conclude that R is a left and right central symmetric ring. On the other hand, since $abc = 0$ but $cba \notin Z(R)$, R does not satisfy GZC_3. □

We emphasize that a ring R satisfying GZC_n need not be central reduced.

Example 3.7. Let $A = \mathbb{Z}_2[a, b, c]$ be the same as in Example 3.6, I be the ideal of $\mathbb{Z}_2 + A$ generated by $r_1r_2r_3r_4$ for any $r_1, r_2, r_3, r_4 \in A$, i.e., $I = A^4$, and $R = (\mathbb{Z}_2 + A)/I$. Then $R = \bigoplus_{i=0}^{3}(A_i + I)/I$ where the meaning of A_i is similar to that in Example 3.6. We identify the elements of $\mathbb{Z}_2 + A$ with their images for simplicity. Thus, R is a local ring with $J(R) = A$ nilpotent. Similarly to the proof of Example 3.6, there is no difficulty to check that R satisfies GZC_n, but R is not a central reduced ring since $a^4 = 0$ and $ab \neq ba$.

We conclude this paper with a remarkable example.

Example 3.8. A symmetric ring R without identity need not be central reversible.

Proof. We show this with the help of [10, Example 1]. Let $S = \{a, b\}$ be a semigroup with multiplication $a^2 = ab = a, b^2 = ba = b$, then $R = \mathbb{Z}_2S = \{0, a, b, a + b\}$ is a four-element semigroup ring without identity. It is well known that R is a symmetric ring and not reversible. Observing that $a(a + b)a = 0$, $(a + b)aa = a + b$, and $a(a + b) = 0, (a + b)a = a + b$, R is not central reversible. Moreover, this example shows that the notion of symmetric rings is not left-right symmetric for rings without identity. Thus R does not satisfy ZC_3. □

References

[1] D. D. Anderson, V. Camillo, *Semigroups and rings whose zero products commute*, Comm. Algebra, **27**(1999), 2847–2852.

[2] A. Bhattacharjee and U. S. Chakraborty, *On some generalizations of reversible and semicommutative rings*, Int. Electron. J. Algebra, **22**(2017), 11–27.

[3] P. M. Cohn, *Reversible rings*, Bull. London Math. Soc., **31**(1999), 641–648.

[4] C. Huh, H. K. Kim, N. K. Kim and Y. Lee, *Basic examples and extensions of symmetric rings*, J. Pure Appl. Algebra, **202**(2005), 154–167.

[5] D. W. Jung, N. K. Kim, Y. Lee and S. J. Ryu, *On properties related to reversible rings*, Bull. Korean Math. Soc., **52**(2015), 247–261.

[6] G. Kafkas, B. Ungor, S. Halicioglu and A. Harmanci, *Generalized symmetric rings*, Algebra Discrete Math., **12**(2011), 72–84.

[7] H. Kose, B. Ungor, S. Halicioglu and A. Harmanci, *A generalization of reversible rings*, Iran. J. Sci. Technol. Trans. A Sci., **38(1)**(2014), 43–48.

[8] J. Krempa, *Some examples of reduced rings*, Algebra Colloq., **3**(1996), 289–300.

[9] J. Lambek, *On the representation of modules by sheaves of factor modules*, Canad. Math. Bull., **14**(1971), 359–368.

[10] G. Marks, *Reversible rings and symmetric rings*, J. Pure Appl. Algebra, **174**(2002), 311–318.

[11] G. Marks, *A taxonomy of* 2*-primal rings*, J. Algebra, **266**(2003), 494–520.

[12] L. Motais de Narbonne, *Anneaux semi-commutatifs et unisériels; anneaux dont les idéaux principaux sont idempotents*, Proceedings of the 106th National Congress of Learned Societies (Perpignan, 1981), Bib. Nat., Paris, 1982, 71–73.

[13] T. Özen, N. Agayev and A. Harmanci, *On a class of semicommutative rings*, Kyungpook Math. J., **51**(2011), 283–291.

[14] B. Ungor, S. Halicioglu, H. Kose and A. Harmanci, *Rings in which every nilpotent is central*, Algebra Groups Geom., **30**(2013), 1–18.

[15] G. Yang and R. Du, *Rings over which polynomial rings are semi-commutative*, Vietnam J. Math., **37**(2009), 527–535.