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SOME SHADOWING PROPERTIES OF THE SHIFTS ON THE INVERSE LIMIT SPACES

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Abstract. Let $f : X \to X$ be a continuous surjection of a compact metric space X and let $\sigma_f : X_f \to X_f$ be the shift map on the inverse limit space X_f constructed by f. We show that if a continuous surjective map f has some shadowing properties: the asymptotic average shadowing property, the average shadowing property, the two side limit shadowing property, then σ_f also has the same properties.

1. Introduction and preliminaries

Inverse limit is a useful tool to study the dynamical properties of smooth systems; some dynamical properties of can be interpreted by the topological structures of the inverse limit dynamical system; for example, Chen and Li [4] proved that (X, f) has the shadowing property if and only if $(\lim_{\leftarrow}(X, f), \sigma_f)$ has so. Li [8] proved that some dynamical properties hold simultaneously for both f and σ_f. Also A. Barzanouni [2] show that the relationship between ergodic shadowing property and inverse shadowing property for a surjective continuous map on a compact metric space and shift map on the inverse limit space. M. Lee [7] show that f has the asymptotic average shadowing, the average shadowing, the ergodic shadowing property then σ_f is topologically transitive.

In this paper, we discuss (X, f) has the two-sided limit shadowing property, the average shadowing property and the asymptotic average shadowing property then shift map on the inverse limit space has so.

Let X be a compact metric space with metric d and $X^{\mathbb{Z}}$ denote the product topological space $X^{\mathbb{Z}} = \{(x_i) : x_i \in X, i \in \mathbb{Z}\}$. Then $X^{\mathbb{Z}}$ is compact. We define a compatible metric $\tilde{d}$ for $X^{\mathbb{Z}}$ by

$$\tilde{d}((x_i)(y_i)) = \sum_{i=-\infty}^{\infty} \frac{d(x_i, y_i)}{2^{|i|}}.$$

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A homeomorphism $\sigma_f : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$, which is defined by

$$\sigma_f((x_i)) = (y_i) \text{ and } y_i = x_{i+1} \text{ for all } i \in \mathbb{Z},$$

is called the shift map.

For $f : X \to X$ a continuous surjection, we let,

$$X_f = \{(x_i) : x_i \in X, \text{ and } f(x_i) = x_{i+1}, i \in \mathbb{Z}\}.$$

Then X_f is a closed subset of $X^{\mathbb{Z}}$. Moreover we have $\sigma_f((x_i)) = (f(x_i))$ for all $(x_i) \in X_f$, and so X_f is σ_f-invariant, i.e. $\sigma_f(X_f) = X_f$.

The space X_f is called the inverse limit space constructed by f. The restriction $\sigma_f = \sigma_f|_{X_f} : X_f \to X_f$ is called the shift map determined by f.

A sequence $(x_i)_{i\in\mathbb{Z}}$ of points X is a *two-sided limit pseudo orbit* for f if it satisfying

$$d(f(x_i), x_{i+1}) \to 0, \quad \text{as } |i| \to \infty.$$

A sequence $(x_i)_{i\in\mathbb{Z}}$ of points X is *two-sided limit shadowed* by $y \in X$ if it satisfying

$$d(f^i(y), x_i) \to 0, \quad \text{as } |i| \to \infty.$$

We say that f has the *two-sided limit shadowing property* if every two-sided limit pseudo-orbit is two-sided limit shadowed.

For $\delta > 0$, a sequence $(x_i)_{i\in\mathbb{Z}}$ of points in X is a δ-*average pseudo orbit* for f if there is an integer $N = N(\delta) > 0$ such that:

$$\frac{1}{n}\sum_{i=1}^{n} d(f(x_{i+k}), x_{i+k+1}) < \delta, \quad \text{for all}\, n \geq N, k \in \mathbb{Z}.$$

A sequence $(x_i)_{i\in\mathbb{Z}}$ of points in X is ε- *shadowed in average* by $y \in X$ if

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d(f^i(y), x_i) < \varepsilon.$$

We say that f has the *average shadowing property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-average pseudo orbit of f is ε-shadowed on average by some point in X.

A sequence $(x_i)_{i\in\mathbb{Z}}$ of points in X is a δ-*asymptotic average pseudo orbit* for f if there is an integer $N = N(\delta) > 0$ such that:

$$\frac{1}{n}\sum_{i=1}^{n} d(f(x_{i+k}), x_{i+k+1}) \to 0, \quad \text{for all}\, n \geq N, k \in \mathbb{Z}.$$

A sequence $(x_i)_{i\in\mathbb{Z}}$ of points X is ε-*asymptotically shadowed in average* by $y \in X$ if

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d(f^i(y), x_i) \to 0.$$

We say that f has the *asymptotic average shadowing property* provided that every asymptotic average pseudo-orbit of f is asymptotically shadowed in average by some point in X. We say that a finite δ-pseudo orbit $\{x_i\}_{i=0}^{k}$ of f is a δ-*chain* from x_0 to x_k with length $k+1$. A non-empty subset A of X is said to be *chain transitive* if for any $x, y \in A$ and any $\delta > 0$ there is a δ-chain of f from x to y. A map f is said to be *chain transitive* if X is a chain transitive set.

THEOREM 1.1. [5, Theorem 3.1] *Let X be a compact metric space and f be a continuous map from X onto itself. If f has the asymptotic average shadowing property, then f is chain transitive.*

2. Main results

THEOREM 2.1. *Let $f : X \to X$ be a continuous surjective map on a compact metric space X.*

1. *If f has average shadowing property, then the shift map σ_f on the inverse limit space X_f has average shadowing property.*
2. *If f has asymptotic average shadowing property, then the shift map σ_f on the inverse limit space X_f has asymptotic average shadowing property.*
3. *If f has two-side limit shadowing property, then the shift map σ_f on the inverse limit space X_f has two-side limit shadowing property.*

Proof. Proof of 1. Let $\varepsilon > 0$ and $D = \operatorname{diam} X$. Choose $N > 0$ with $D/2^{N-2} < \varepsilon$, and let $\gamma > 0$ be a number such that

$$d(x,y) \le \gamma \Rightarrow \max_{0 \le i \le 2N} d(f^i(x), f^i(y)) \le \frac{\varepsilon}{8}.$$

By average shadowing property of f there is $\delta_1 > 0$ such that any δ_1-average pseudo orbit of f is γ-average shadowed. Choose $\delta_2 > 0$ with $0 < 2^N \delta_2 < \delta_1$. Define a sequence $\{w_i^n\}_{n,i\in\mathbb{Z}}$ is a δ_2-average pseudo orbit of σ_f in X_f such that $\{w_i^n\}_{n\in\mathbb{Z}}$ is a periodic δ_1-average pseudo orbit of f for each $i \in \mathbb{Z}$.

Then we have,

$$\begin{aligned}\delta_2 > \frac{1}{n}\sum_{i=0}^{n-1} \tilde{d}\left(\sigma_f(w_j^{k+i}), w_j^{k+i+1}\right) &= \frac{1}{n}\sum_{i=0}^{n-1}\left(\sum_{j=-\infty}^{\infty} \frac{d\big(f(w_j^{k+i}), w_j^{k+i+1}\big)}{2^{|j|}}\right)\\ &\ge \frac{1}{n}\sum_{i=0}^{n-1} \frac{d\big(f(w_{-N}^{k+i}), w_{-N}^{k+i+1}\big)}{2^{|N|}}, \quad n \ge N.\end{aligned}$$

Since $\{w^n_{-N}\}_{n\in\mathbb{Z}}$ is a periodic δ_1-average pseudo orbit of f, $\{w_i^n\}_{n,i\in\mathbb{Z}}$ δ_2-average pseudo orbit of σ_f. Also we can find $z \in X$, such that $n \ge N$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), w^i{}_{-N}) < \gamma.$$

We put $z_{i-N} = f^i(z)$ for $i \geq 0$ and take $z_{i-N} \in f^{-1}(z_{i+1-N})$ for $i < 0$. Then $\tilde{z} = \{z_i\}_{i\in\mathbb{Z}} \in X_f$ and

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \tilde{d}\left(\sigma_f^i(z_j), w_j^i\right) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\left(\sum_{j=-\infty}^{\infty} \frac{d\big(f^i(z_j), w_j^i\big)}{2^{|j|}}\right)\\
&= \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\left(\sum_{j=-N}^{N} \frac{d\big(f^i(z_j), w_j^i\big)}{2^{|j|}} + \sum_{j=-\infty}^{-N-1} \frac{d\big(f^i(z_j), w_j^i\big)}{2^{|j|}} + \sum_{j=N+1}^{\infty} \frac{d\big(f^i(z_j), w_j^i\big)}{2^{|j|}}\right)\\
&\leq \frac{3\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} < \varepsilon.
\end{aligned}$$

Hence $\{w_i^n\}_{n,i\in\mathbb{Z}}$ is a δ_2- average pseudo orbit of σ_f is ε- shadowed in average by $\tilde{z} \in X_f$.

Proof of 2. Let $D = \operatorname{diam}(X)$ and $\varepsilon > 0$. Choose $N > 0$ with $D/2^{N-2} < \varepsilon$, and let $\gamma > 0$ be a number such that

$$d(x, y) < \gamma \to 0 \quad \Rightarrow \quad \max_{0\leq i\leq 2N} d(f^i(x), f^i(y)) \leq \frac{\varepsilon}{8} \to 0.$$

Since asymptotic average shadowing property of f, there is $\delta_1 > 0$ such that any δ_1-asymptotic average pseudo orbit of f is γ-asymptotically shadowed. Choose $\delta_2 > 0$ with $0 < 2^N\delta_2 < \delta_1$. Define a sequence $\{w_i^n\}_{n,i\in\mathbb{Z}}$ is a δ_2-asymptotic average pseudo orbit of σ_f in X_f such that $\{w_i^n\}_{n\in\mathbb{Z}}$ is δ_1-asymptotic average pseudo orbit of f for all $i \in \mathbb{Z}$. Then

$$\begin{aligned}
\delta_2 > \frac{1}{n}\sum_{i=0}^{n-1} \tilde{d}\left(\sigma_f(w_j^{k+i}), w_j^{k+i+1}\right) &= \frac{1}{n}\sum_{i=0}^{n-1}\left(\sum_{j=-\infty}^{\infty} \frac{d\big(f(w_j^{k+i}), w_j^{k+i+1}\big)}{2^{|j|}}\right)\\
&\geq \frac{1}{n}\sum_{i=0}^{n-1} \frac{d\big(f(w_{-N}^{k+i}), w_{-N}^{k+i+1}\big)}{2^{|N|}}.
\end{aligned}$$

$$\Rightarrow \delta_1 > 2^{|N|}\delta_2 \geq \frac{1}{n}\sum_{i=0}^{n-1} d\left(f(w_{-N}^{k+i}), w_{-N}^{k+i+1}\right) \to 0.$$

Since $\{w_{-N}^n\}_{n\in\mathbb{Z}}$ is a δ_1-asymptotic average pseudo orbit of f, $\{w_i^n\}_{n,i\in\mathbb{Z}}$ is a δ_2-asymptotic average pseudo orbit of σ_f. Thus we can find $z \in X$, such that $n \geq N$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), w^i{}_{-N}) \to 0.$$

We put $z_{i-N} = f^i(z)$ for $i \geq 0$ and take $z_{i-N} \in f^{-1}(z_{i+1-N})$ for $i < 0$. Then $\tilde{z} = \{z_i\}_{i\in\mathbb{Z}} \in X_f$ and

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \tilde{d}\left(\sigma_f^i(z_j), w_j^i\right) = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\left(\sum_{j=-\infty}^{\infty} \frac{d\big(f^i(z_j), w_j^i\big)}{2^{|j|}}\right)\\
&= \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}\left(\sum_{j=-N}^{N} \frac{d\big(f^i(z_j), w_j^i\big)}{2^{|j|}} + \sum_{j=-\infty}^{-N-1} \frac{d\big(f^i(z_j), w_j^i\big)}{2^{|j|}} + \sum_{j=N+1}^{\infty} \frac{d\big(f^i(z_j), w_j^i\big)}{2^{|j|}}\right)\\
&\leq \frac{3\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} \to 0.
\end{aligned}$$

Hence $\{w_i^n\}_{n,i\in\mathbb{Z}}$ is a δ_2-asymptotic average pseudo orbit of σ_f is ε- asymptotically shadowed in average by $\tilde{z} \in X_f$.

Proof of 3. Let $D = \mathrm{diam}(X)$ and $\varepsilon > 0$. let $|k| > N$ and $\gamma > 0$ be a number such that

$$d(x,y) < \gamma = \frac{D}{2^{|k|}} \quad \Rightarrow \quad \max_{0\le i\le 2N} d(f^i(x), f^i(y)) \le \frac{\varepsilon}{8} \to 0.$$

Define a sequence $\{w_i^n\}_{n,i\in\mathbb{Z}}$ is a two-sided limit pseudo-orbit of σ_f in X_f such that $\{w_i^n\}_{n\in\mathbb{Z}}$ is a two-sided limit pseudo-orbit of f for all $i \in \mathbb{Z}$. Then we have

$$\tilde{d}(\sigma_f(x_i^n, x_i^{n+1}) \ge \frac{d(f(x_{-N}^n), x_{-N}^{n+1})}{2^N}.$$

Since $\{x_{-N}^n\}_{n\in\mathbb{Z}}$ is a two-sided limit pseudo orbit of f, $\{w_i^n\}_{n\in\mathbb{Z}}$ is a two-sided limit pseudo-orbit of σ_f. Then we can find $z \in X$ such that $d(f^n(z), x_{-N}^n) = 0$ when $|n| \to \infty$.

We put $z_{i-N} = f^i(z)$ for $i \ge 0$ and take $z_{i-N} \in f^{-1}(z_{i+1-N})$ for $i < 0$. Then $\tilde{z} = \{z_i\}_{i\in\mathbb{Z}} \in X_f$ and

$$\begin{aligned}
\tilde{d}\left(\sigma_f^i(z_j), w_j^i\right) &= \sum_{j=-\infty}^{\infty} \frac{d\big(f^i(z_j), w_j^i\big)}{2^{|j|}} \\
&= \sum_{j=-N}^{N} \frac{d\big(f^i(z_j), w_j^i\big)}{2^{|j|}} + \sum_{j=-\infty}^{-N-1} \frac{d\big(f^i(z_j), w_j^i\big)}{2^{|j|}} + \sum_{j=N+1}^{\infty} \frac{d\big(f^i(z_j), w_j^i\big)}{2^{|j|}} \\
&\le \frac{3\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} \to 0.
\end{aligned}$$

Hence $\{w_i^n\}_{n,i\in\mathbb{Z}}$ two-sided limit pseudo-orbit of σ_f is two-sided limit shadowed by $\tilde{z} \in X_f$. □

Corollary 2.2. *If a continuous map f has asymptotic average shadowing property, then the shift map σ_f on the inverse limit space X_f is chain transitive.*

Proof. By 2. of Theorem 2.1, f has asymptotic average shadowing property, then σ_f has asymptotic average shadowing property, so by Theorem 1.1, σ_f is chain transitive. □

Corollary 2.3. [7, Remark 2.3] *If surjective continuous map f has average shadowing property, then the shift map σ_f on the inverse limit space X_f is chain transitive.*

Corollary 2.4. *If homeomorphism f of a compact metric space has two-sided limit shadowing property, then the shift map σ_f on the inverse limit space X_f has average shadowing and asymptotic average shadowing property. Moreover, σ_f is chain transitive.*

Proof. By [3, Theorem B], f has two-sided limit shadowing property, then f has average shadowing and asymptotic average shadowing property, so by Theorem 2.1, σ_f has average shadowing and asymptotic average shadowing property. Then by Corollary 2.2 and Corollary 2.3, σ_f is chain transitive. □

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