JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **31**, No. 4, November 2018
http://dx.doi.org/10.14403/jcms.2018.31.1.421

PROLONGATIONS OF G-STRUCTURES IMMERSED IN GENERALIZED ALMOST r-CONTACT STRUCTURE TO TANGENT BUNDLE OF ORDER 2

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Abstract. The aim of this study is to investigate the prolongations of G-structures immersed in the generalized almost r-contact structure on a manifold M to its tangent bundle $T(M)$ of order 2. Moreover, theorems on Hsu structure, integrability and $(\overset{o}{F}, \overset{o}{\xi}\overset{o}{\omega}_p, a, \epsilon)$-structure have been established.

1. Introduction

The study was made based on general theory of prolongations, the geometric properties of the prolongations of pseudogroup structures and G-structures to tangent bundles [8]. The previous study investigated the prolongation of G-structures to tangent bundles of first and higher orders and showed that the integrability of G-structures is equivalent to the integrability of its prolongations [7]. Prolongation of different structures like as F-structure, G-structure and connections to the tangent bundle have been studied in [1, 2, 10]. Das at el [3] have studied submanifolds immersed in a Hsu-quarternion manifold. Das and the author [4] have introduced and obtained almost product structure by means of the complete, vertical and horizontal lifts of almost r-contact structures on almost r-contact structures. The author [5, 6] has studied lifts with connections to tangent bundles and Kaehler manifold.

Earlier investigators studied prolongation of some classical G-structure defined by tensor fields, almost complex and almost product structures [9]. The purpose of the present work is to study the prolongations of G-structure immersed in generalized almost r-contact structure on a

Received March 06, 2018; Accepted October 11, 2018.
2010 Mathematics Subject Classification: 53C10, 53C15, 58A20.
Key words and phrases: prolongation, G-structure, Hsu-structure, integrability, complete lift, vertical lift, invariant.
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manifold M to its tangent bundle $T(M)$, G being a Lie subgroup of $GL(n, R)$.

The paper is structured as follows: In Section 2, we recall definition of Hsu-structure, generalized almost r-contact structure, tangent bundle of order 2. Section 3 is devoted to the study of prolongation of tensor fields and G-structure to the tangent bundle and the integrability of the prolongation of a G-structure. Finally, In Section 4, we study some classical G-structures defined by tensor fields immersed in generalized almost r-contact structure to tangent bundle of order 2.

2. Preliminaries

Hsu-structure

The base space M is said to possess a Hsu-structure if there exists on M a tensor field F of type (1,1) such that

$$F^2 = a^r I, \tag{2.1}$$

where I is the unit tensor field and a is a real or imaginary number [3].

Generlized almost r-contact structure

If on manifold M, there exists a tensor field F of type (1,1), $r(C^\infty)$ vector fields U_p and $r(C^\infty)$ 1-forms ω_p satisfying the conditions [3]

$$F^2 = a^r I + \epsilon \sum_{p=1}^{r} \omega_p \otimes U_p, \tag{2.2}$$

such that

$$\text{(i) } FU_p = 0, \quad \text{(ii) } \omega_p \circ F = 0, \quad \text{(iii) } \omega_p(U_p) = -\frac{a^r}{\epsilon}\delta_q^p, \tag{2.3}$$

where $p, q = 1, 2, \cdots, r$ and δ_q^p denote the Kronecker delta while a and ϵ are non-zero complex numbers. The manifold M is called a *generalized almost r-contact manifold* and manifold with a generalized almost r-contact structure or in short with an $(F, U_p, \omega_p, a, \epsilon)$-structure. The structure is said to be *normal* if the tensor $S = [f, f] + \epsilon \sum_{p=1}^{r} \omega_p \otimes U_p$ vanishes.

Tangent Bundle of order 2

Let us introduce an equivalence relation $\sim$ in the set of all differentiable mappings $F : R \to M$, where R is the real line. Let $r \geq 1$ be a fixed integer. If two mappings $F : R \to M$ and $G : R \to M$ satisfy the conditions $F^h(0) = G^h(0), \frac{dF^h(0)}{dt} = \frac{dG^h(0)}{dt}, ..., \frac{dF^r(0)}{dt} = \frac{dG^r(0)}{dt}$ the mapping F and G being represented respectively by $x^h = F^h(t)$ and $x^h = G^h(t)$, where $t \in R$ with respect to local coordinates x^h in a coordinate neighborhood (U, x^h) containing the point $P = F(0) = G(0)$, then we say that the mapping F is equivalent to G. Each equivalence class determined by the equivalence relation $\sim$ is called an *r-jet* of M and denoted by $J^p_q(F)$. The set of all r-jets of M is called the tangent bundle of order r and denoted by $T_r(M)$ [9].

3. Prolongation

The prolongation of tensor fields and G-structure to the tangent bundle of order 2

Let M be an n-dimensional manifold and G a Lie subgroup of $GL(n, R)$. A G-structure on M is a G-subbundle $P(M, \pi, G)$ of the frame bundle FM over M. That is, a G-structure on M is a reduction of the structure group $GL(n, R)$ of the tangent bundle $T(M)$ to the subgroup G of $GL(n, R)$.

Definition 3.1. Let G be a Lie subgroup of $GL(n)$. Then the Lie subgroup of $GL(2n)$ is sometimes identified with $T(G)$ and called the tangent group of G.

The tangent bundle $T_2(M)$ of order 2 admits a $T_2(G)$-structure with adapted $3n$-frame $\left\{X^{II}_{(i)}, X^{I}_{(i)}, X^{0}_{(i)}\right\}$, where $X_{(i)}$ is an n-frame adapted to the G-structure P. The $T_2(G)$-structure introduced thus in $T_2(M)$ is called *prolongation of the G-structure* P on M to $T(M)$ and denoted by $\widetilde{P}$.

The Integrability of the prolongation of G-structure

The integrability of the prolongation of a G-structure P is defined as that for each point on M, if there is a coordinate neighborhood $\{U, X^h\}$ containing this point such that the natural frame $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^n}\right\}$ is adapted to the G-structure P, then the G-structure P is said to be integrable (or flat) [3].

Yano and Ishihara (1973) stated the following proposition:

Proposition 3.2. *The prolongation $\widetilde{P}$ of a G-structure P given in M is integrable in the tangent bundle $T(M)$ if and only if the G-structure P is integrable in M* [9].

4. Main Results

Prolongations of G-structure immersed in the generalized almost r-contact structure to tangent bundle of order 2

Let there be given a Lie subgroup G of $GL(n,R)$ and a tensor field $\overset{o}{F}$ of type (1,1) in R^n, which is invariant by G. An n-dimensional manifold M is assumed to admit a G-structure P. We take a coordinate neighborhood $\{U, X^h\}$ of M and an n-frame $\{X_{(i)}\}$ in U, which is adapted to the G-structure P. Thus, if we put

$$\overset{o}{F} = \overset{o}{F}{}_i^h \, X_{(h)} \theta^{(i)} \tag{4.1}$$

in U, $\{\theta^{(i)}\}$ being the co-frame dual to $\{X_{(i)}\}$ in U and $\overset{o}{F}{}_i^h$ being components of $\overset{o}{F}$ in R^n. The local tensor field F, defined by equation 4.1 in each coordinate neighborhood U is determined independently of the choice of the adapted frame $\{X_{(i)}\}$ and hence defines globally a tensor field in M denoted by F, which is called the tensor field induced in M from $(\overset{o}{F}, P)$ [3].

Some classical G-structures are defined by tensor fields immersed in the generalized almost r-contact structure to tangent bundle.

(I) $GL(n,C)$.

Let $\overset{o}{F}$ be a tensor of type (1,1) in R^{2n} such that $\overset{o}{F}{}^2 = a^r I$ and denoted by $GL(n,C)$ the group of all elements of $G = GL(2n,C)$ which leave $\overset{o}{F}$ invariant. Then the second lift $\overset{o}{F}{}^{II}$ of $\overset{o}{F}$ to $T_2(R^{2n})$ is a tensor of type (1,1) satisfying $(\overset{o}{F}{}^{II})^2 = a^r I$ and the tangent group $T_2(G)$ leaves $\overset{o}{F}{}^{II}$ invariant. Thus we obtain $T_2(G) = GL(3n,C)$. Therefore, we have the following theorem.

Theorem 4.1. *If a manifold M admits Hsu-structure P (as a G-structure) determined by a tensor field F of type (1,1) such that $F^2 = a^r I$, then on the tangent bundle $T_2(M)$ of order 2, the prolongation $\widetilde{P}$ of P is the Hsu-structure which is determined by the second lift $\overset{o}{F}{}^{II}$ of $\overset{o}{F}$ to*

$T_2(M)$. *When and only when the Hsu-structure P is the Hsu-structure, the prolongation $\widetilde{P}$ of P to $T_2(M)$ is also the Hsu-structure.*

(II) $G = GL(s, C) \times GL(m, R)$.

Let $\overset{o}{F}$ be a tensor of type (1,1) and of rank $2s$ in $R^n(n = 2s + m)$ such that $\overset{o}{F}{}^3 - a^r \overset{o}{F} = 0$. If we denote by G the group of all the elements of $GL(n, R)$, which leave $\overset{o}{F}$ invariant, then we easily obtain $T(G) \subset GL(s, C) \times GL(m, R) \subset GL(2n, R)$. Then second lift F^{II} of $\overset{o}{F}$ to satisfies $F^3 - a^r F = 0$ and is of $2s$ rank. Thus we obtain

$$T(G) \subset GL(2s, C) \times GL(2m, R) \subset GL(2n, R).$$

Therefore, we have the following theorem.

THEOREM 4.2. *If a manifold M admits Hsu-structure P (as a G-structure) determined by a tensor field F of type (1,1) and of rank s everywhere such that $F^3 - a^r F = 0$, then on the tangent bundle $T_2(M)$, the prolongation $\widetilde{P}$ of P to (TM) is the Hsu-structure determined by second lift $\overset{o}{F}{}^{II}$ of $\overset{o}{F}$ to $T_2(M)$, where $F^{II}C$ is of rank $3s$. When and only when the Hsu-structure P is integrable in M, the prolongation $\tilde{P}$ of P to $T_2(M)$ is integrable.*

(III) $G = GL(n, C) \times I$.

Let $\overset{o}{F}$ be a tensor of type (1,1) and of rank $2s$, contravariant vector fields $\overset{o}{U}_p$ and 1-forms $\overset{o}{\omega}_p, p = 1, 2, \cdots, r$ in R^{2n+r} such that

$$\overset{o}{F}{}^2 = a^r I + \epsilon \sum_{p=1}^{r} \overset{o}{\omega}_p \otimes \overset{o}{\xi}_p, \tag{4.2}$$

where

$$\text{(i) } \overset{o}{F}_p \overset{o}{\xi}_p = 0, \quad \text{(ii) } \overset{o}{\omega}_p \circ \overset{o}{F}_p = 0, \quad \text{(iii) } \overset{o}{\omega}_p (\overset{o}{\xi}_p) = -\frac{a^r}{\epsilon} \delta^p_q. \tag{4.3}$$

If we denote by G the group of all the elements of $GL(2n + r, R)$, which leave $\overset{o}{F}_p, \overset{o}{\xi}_p, \overset{o}{\omega}_p, p = 1, 2, \cdots, r$ invariant, then we easily obtain

$$G = GL(n, C) \times I \subset GL(2n + r, R),$$

where I denotes the trivial group.

If we put

$$\overset{o}{J} = \overset{o}{f}{}^{II} + \frac{\epsilon}{a\sqrt{r}} \sum_{p=1}^{r} \left\{ \overset{o}{\xi}{}^0_p \otimes \overset{o}{\omega}{}^C_p + \overset{o}{\xi}{}^{II}_p \otimes \overset{o}{\omega}{}^{II}_p \right\}, \overset{o}{v} = \overset{o}{\xi}{}^I_p, \overset{o}{\eta} = \overset{o}{\omega}{}^I_p, \tag{4.4}$$

we can easily shown that $(\overset{o}{F}, \overset{o}{\xi}\overset{o}{\omega}_p, a, \epsilon)$ is the generalized almost r-contact structure in $T_2(R^{2n+r})$. Hence $T_2(R^G)$ leaves that $\overset{o}{j}, \overset{o}{\xi}_p$ *and* $\overset{o}{\omega}_p$ invariant.

Thus we obtain

$$T(G) \subset GL(3n+r, C) \times I \subset GL(6n+3r, R).$$

Therefore, we have the following theorem.

THEOREM 4.3. *If a manifold M of $(2n+r)$-dimensions admits the generalized almost r-contact structure P (as a G-structure) determined by $(\overset{o}{F}, \overset{o}{\xi}\overset{o}{\omega}_p, a, \epsilon)$, then on the tangent bundle $T_2(M)$, the prolongation $\widetilde{P}$ of P is the generalized almost r-contact structure is defined by $(\overset{o}{F}, \overset{o}{\xi}\overset{o}{\omega}_p$ $, a, \epsilon)$, where*

$$(4.5) \qquad \overset{o}{J} = \overset{o}{F}^{II} + \frac{\epsilon}{a\sqrt{r}} \sum_{p=1}^{r} \left\{ \overset{o}{\xi}_p^{0} \otimes \overset{o}{\omega}_p^{C} + \overset{o}{\xi}_p^{II} \otimes \overset{o}{\omega}_p^{II} \right\}, \widetilde{\xi}_p = \overset{o}{\xi}_p^{I}, \widetilde{\omega}_p = \overset{o}{\omega}_p^{I}.$$

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