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STABILITY OF AN n-DIMENSIONAL QUADRATIC FUNCTIONAL EQUATION

Sun-Sook Jin* and Yang-Hi Lee**

Abstract. In this paper, we investigate the generalized Hyers-Ulam stability of the functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} f(x_i - x_j) - n\sum_{i=1}^{n} f(x_i) = 0$$

for integer values of n such that $n \ge 2$, where f is a mapping from a vector space V to a Banach space Y.

1. Introduction

A stability problem of the functional equation was formulated by S. M. Ulam in 1940 [20]. In the following year, D. H. Hyers [5] gave a partial solution of Ulam's problem for the case of approximate additive functions. Subsequently, during the last seven decades, Hyers' theorem was generalized by several mathematicians worldwide [1, 2, 3, 4, 11, 12, 13, 14, 15, 18, 19].

Throughout this paper, assuming that $n \ge 2$ is an integer, V and W are real vector spaces, X is a normed space, and that Y is a Banach space, we consider the *n-dimensional quadratic functional equation*

$$f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} f(x_i - x_j) - n\sum_{i=1}^{n} f(x_i) = 0 \tag{1.1}$$

whose solutions are *quadratic mappings*.

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Correspondence should be addressed to Yang-Hi Lee, yanghi2@hanmail.net.
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In this paper, we investigate a general stability problem for the n-dimensional quadratic functional equation (1.1).

2. Stability of an n-dimensional quadratic functional equation (1.1)

For convenience, we use the following abbreviations for a given mapping $f : V \to W$:

$$Df(x_1, x_2, \ldots, x_n) := f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1 \le i < j \le n} f(x_i - x_j) - n \sum_{i=1}^{n} f(x_i),$$
$$Qf(x, y) := f(x + y) + f(x - y) - 2f(x) - 2f(y),$$
$$\bar{x} := \overbrace{x, x, \ldots, x}^{n-th}$$

for all $x, y, x_1, x_2, \ldots, x_n \in V$, where n is a fixed integer greater than 2.

If f is a solution of the functional equation $Qf(x, y) = 0$ for all $x, y \in V$, then f is called a quadratic mapping. The authors have shown several results about the stability problem of various kind of quadratic functional equations [6, 7, 8, 9, 10].

Lemma 2.1. *A mapping $f : V \to W$ is a solution of* (1.1) *if and only if f is a quadratic mapping.*

Proof. Let $f : V \to W$ satisfy $Df(x_1, x_2, \ldots, x_n) = 0$. Since $f(0) = \frac{2Df(0,0,\ldots,0)}{2-n^2-n} = 0$ and $f(-x) = Df(0, x, 0, \ldots, 0) + f(x) = f(x)$, we get

$$Qf(x, y) = Df(x, y, 0, \ldots, 0) = 0$$

for all $x, y \in V$, i.e., f is a quadratic mapping.

Conversely, assume that f is a quadratic mapping. We apply induction on $j \in \{2, \ldots, n\}$ to prove $Df(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in V$. For $j = 2$, we have

$$Df(x_1, x_2, 0, \ldots, 0) = Qf(x_1, x_2) = 0$$

for all $x_1, x_2 \in V$. If $n > 2$ and $Df(x_1, x_2, \ldots, x_j, 0, \ldots, 0) = 0$ for some integer j $(2 \le j < n)$ and for all $x_1, x_2, \ldots, x_j \in V$, then routine calculation yields

$$\begin{aligned}
&Df(x_1, x_2, \ldots, x_{j+1}, 0, \ldots, 0)\\
&= -\frac{1}{2}Qf(x_1 + \cdots + x_{j+1}, x_{j+1} - x_j)\\
&\quad + \frac{1}{2}Df(x_1, x_2, \ldots, x_{j-1}, 2x_j, 0, \ldots, 0)\\
&\quad + \frac{1}{2}Df(x_1, x_2, \ldots, x_{j-1}, 2x_{j+1}, 0, \ldots, 0) - \frac{1}{2}\sum_{i=1}^{j-1} Qf(x_i - x_j, x_j)\\
&\quad - \frac{1}{2}\sum_{i=1}^{j-1} Qf(x_i - x_{j+1}, x_{j+1}) + \frac{j}{2}Qf(x_j, x_j) + \frac{j}{2}Qf(x_{j+1}, x_{j+1})\\
&= 0
\end{aligned}$$

for all $x_1, x_2, \ldots, x_j, x_{j+1} \in V$. Hence, we get f is a solution of (1.1). □

In the following theorems, we will investigate the generalized Hyers-Ulam stability problems of the functional equation (1.1).

THEOREM 2.2. *Let $s = 1, -1$ and let $\varphi : V^n \to [0, \infty)$ be a function satisfying the conditions:*

$$\sum_{j=0}^{\infty} n^{-2sj} \varphi(n^{sj}x_1, n^{sj}x_2, \cdots, n^{sj}x_n) < \infty \tag{2.1}$$

for all $x_1, x_2, \cdots, x_n \in V$. Suppose $f : V \to Y$ is a mapping such that

$$\|Df(x_1, x_2, \ldots, x_n)\| \le \varphi(x_1, x_2, \ldots, x_n) \tag{2.2}$$

for all $x_1, x_2, \ldots, x_n \in V$ with $f(0) = 0$. Then there exists a quadratic mapping $F : V \to Y$ such that

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} n^{2\tau_{-s,i}} \varphi(\overline{n^{\tau_{s,i}} x}) \tag{2.3}$$

for all $x \in V$, where $\tau_{s,m}$ are the integers defined by

$$\tau_{s,m} = s\left(m + \frac{1}{2}\right) - \frac{1}{2}$$

for $s \in \{-1, 1\}$, $m \in \mathbb{N} \cup \{0\}$.

Proof. It follows from (2.2) that

$$\begin{aligned} &\|n^{-2sm} f(n^{sm}x) - n^{-2s(m+m')} f(n^{s(m+m')}x)\| \\ &\qquad \le \sum_{i=m}^{m+m'-1} \left\| -n^{2\tau_{-s,i}} Df(\overline{n^{\tau_{s,i}}x}) s \right\| \\ &\qquad \le \sum_{i=m}^{m+m'-1} n^{2\tau_{-s,i}} \varphi(\overline{n^{\tau_{s,i}}x}) \end{aligned} \tag{2.4}$$

for all $x_1, x_2, \dots, x_n \in V$ and $m + m' > m \ge 0$.

By (2.1) and (2.4), we get the sequence $\{n^{-2sm} f(n^{sm}x)\}$ is a Cauchy sequence for all $x \in V$. Since Y is complete, the sequence $\{n^{-2sm} f(n^{sm}x)\}$ converges in Y. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m\to\infty} n^{-2sm} f(n^{sm}x)$$

for all $x \in V$. Moreover, by putting $m = 0$ and letting $m' \to \infty$ in (2.4), we get (2.3). From the definition of F, we easily have

$$DF(x_1, x_2, \dots, x_n) = \lim_{i\to\infty} n^{-2si} Df\big(n^{si}x_1, \dots, n^{si}x_n\big) = 0$$

for all $x_1, x_2, \dots, x_n \in V$, which implies that F is a quadratic mapping by Lemma 2.1.

Now let $F' : V \to Y$ be another quadratic mapping satisfying the inequality (2.3). Because F' is a quadratic mapping, we can easily show that $F'(x) = n^{-2sm} F'(n^{sm}x)$ for all $x \in V$. Using this equality and (2.3), we obtain

$$\begin{aligned} \|F'(x) - n^{-2sm} f(n^{sm}x)\| &= \|n^{-2sm} F'(n^{sm}x) - n^{-2sm} f(n^{sm}x)\| \\ &\le \sum_{j=m}^{\infty} n^{2\tau_{-s,i}} \varphi(\overline{n^{\tau_{s,i}}x}) \\ &\to 0, \text{ as } m \to \infty, \end{aligned}$$

which implies that $F'(x) = \lim_{m\to\infty} n^{-2sm} f(n^{sm}x) = F(x)$ for all $x \in V$. This proves the uniqueness of F. □

Put $\varphi(x_1, x_2, \dots, x_n) := \theta\big(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p\big)$ in Theorem 2.2. Then we prove the following corollary.

Corollary 2.3. *Let $p \ne 2$ be a nonnegative real number. Suppose $f : X \to Y$ is a mapping such that*

$$\|Df(x_1, x_2, \dots, x_n)\| \le \theta\big(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p\big) \tag{2.5}$$

for all $x_1, x_2, \ldots, x_n \in X$ *and for some constant* $\theta \geq 0$. *Then there exists a unique quadratic mapping* F *such that*

$$\|f(x) - F(x)\| \leq \frac{n\theta\|x\|^p}{|n^p - n^2|}$$

for all $x \in X$.

In particular, we prove the stability of the functional equation (1.1) for the case $n = 3$. In other word, we prove the stability of the functional equation

$$f(x+y+z)+f(x-y)+f(y-z)+f(x-z)-3f(x)-3f(y)-3f(z)=0$$

for all $x, y, z \in V$.

Lemma 2.4. *If* $f : V \to W$ *is a mapping such that*

$$Df(x, y, z) = 0$$

for all $x, y, z \in V \backslash \{0\}$, *then*

$$Df(x, y, z) = 0$$

for all $x, y, z \in V$.

Proof. Since

$$f(x) = \frac{Df(x, -x, -x) - Df(x, x, -x)}{2} + f(-x) = f(-x)$$

for all $x \in V\backslash\{0\}$, we have

$$f(0) = \frac{4Df(x, x, x) - 2Df(2x, -x, -x) - 3Df(x, x, -x)}{5} = 0$$

and

$$f(2x) = \frac{Df(x, x, -x)}{2} + 4f(x) = 4f(x).$$

So we easily know that $Df(x, y, 0) = Df(x, y, -y) = 0$, $Df(x, 0, z) = Df(x, z, -z) = 0$, $Df(0, y, z) = Df(y, z, -z) = 0$, $Df(x, 0, 0) = 0$, $Df(0, 0, z) = 0$, $Df(0, y, 0) = 0$, $Df(0, 0, 0) = 0$ for all $x, y, z \in V\backslash\{0\}$ as we desired. □

By Lemma 2.4 and Theorem 2.2, we can easily obtain the following theorem.

Theorem 2.5. *Let* $s = 1, -1$ *and let* $\varphi : (V\backslash\{0\})^3 \to [0, \infty)$ *be a function satisfying the condition:*

$$\sum_{j=0}^{\infty} 3^{-2sj}\varphi(3^{sj}x, 3^{sj}y2, 3^{sj}z) < \infty$$

for all $x, y, z \in V\backslash\{0\}$. *Suppose* $f : V \to Y$ *is a mapping such that*

$$\|Df(x, y, z)\| \le \varphi(x, y, z)$$

for all $x, y, z \in V\backslash\{0\}$ *with* $f(0) = 0$. *Then there exists a unique quadratic mapping* $F : V \to Y$ *such that*

$$\|f(x) - F(x)\| \le \sum_{i=0}^{\infty} 3^{2\tau_{-s,i}}\varphi(\overline{3^{\tau_{s,i}}x})$$

for all $x \in V\backslash\{0\}$.

Corollary 2.6. *Let* p *be a real number such that* $p < 0$. *If* $f : X \to Y$ *is a mapping such that*

$$\|Df(x, y, z)\| \le \theta\big(\|x\|^p + \|y\|^p + \|z\|^p\big) \tag{2.6}$$

for all $x, y, z \in X\backslash\{0\}$ *and for some constant* $\theta \ge 0$, *then* f *is itself a quadratic mapping.*

Proof. Put $\varphi(x, y, z) := \theta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)$ for all $x, y, z \in X\backslash\{0\}$ in Theorem 2.5. Choose $x \in X\backslash\{0\}$. Then

$$\begin{aligned}\|10f(0)\| =&\|8Df(nx, nx, nx) - 4Df(2nx, -nx, -nx)\\ &- 27Df(nx, nx, -nx) + 21Df(nx, -nx, -nx)\|\\ \le&8\|Df(nx, nx, nx)\| + 4\|Df(2nx, -nx, -nx)\|\\ &+ 27\|Df(nx, nx, -nx)\| + 21\|Df(nx, -nx, -nx)\|\\ \le&(176 + 4 \cdot 2^p)n^p\|x\|^p\\ \to& 0, \text{ as } n \to \infty,\end{aligned}$$

which means that $f(0) = 0$. On the other hand, there exists a unique quadratic mapping F such that

$$\|f(x) - F(x)\| \le \frac{3\theta\|x\|^p}{9 - 3^p} \tag{2.7}$$

for all $x \in X\backslash\{0\}$ by Theorem 2.5. Since $2f(x) = Df((k+1)x, kx, kx) - f((3k + 1)x) + 3f((k + 1)x) + 6f(kx)$ and $DF((k + 1)x, kx, kx) = 0$ for all $x \in X\backslash\{0\}$, it follows from (2.7) that

$$\begin{aligned}&2\|f(x) - F(x)\|\\ &\le \|Df((k + 1)x, kx, kx)\| + \|(F - f)((3k + 1)x)\|\\ &\quad + 3\|(F - f)((k + 1)x)\| + 6\|(F - f)(-kx)\|\\ &\le \left((k + 1)^p + 2k^p + \frac{3((3k + 1)^p + 3(k + 1)^p + 6k^p)}{9 - 3^p}\right)\theta\|x\|^p\\ &\to 0, \text{ as } k \to \infty,\end{aligned}$$

i.e, $f(x) = F(x)$ for all $x \in X\backslash\{0\}$. Because $f(0) = 0 = F(0)$, we get the desired result. $\square$

3. Stability of the set-valued functional equation (1.1)

In this section, we present some related concepts and results which are mainly derived from [16, 17].

From now on, let V be a real vector space and Y a Banach space. The family of all nonempty closed convex subsets of Y will be denoted by $cc(Y)$.

Let A, B be nonempty subsets of a real vector space V and let λ and μ be real numbers. If we define

$$A + B := \{x \in V : x = a + b, \quad a \in V,\ b \in B\},$$
$$\lambda A := \{x \in V : x = \lambda a, \quad a \in V\},$$

then

$$\lambda(A + B) = \lambda A + \lambda B$$
$$(\lambda + \mu)A \subseteq \lambda A + \mu A.$$

Moreover, if A is a convex set and $\lambda\mu \geq 0$, then we have

$$(\lambda + \mu)A = \lambda A + \mu A.$$

In this paper, we get the stability result of the set-valued functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\leq i<j\leq n} f(x_i - x_j) \subseteq n\sum_{i=1}^{n} f(x_i) \tag{3.1}$$

for all $x, x_1, \ldots, x_n \in V$.

THEOREM 3.1. *Let $cc(Y)$ are the family of all nonempty closed convex subsets of Y. If $f : V \to cc(Y)$ is a set-valued mapping satisfying the inclusion* (3.1) *and*

$$\lim_{m\to\infty} \frac{diam(f(n^m x))}{n^{2m}} = 0 \tag{3.2}$$

for all $x, x_1, \ldots, x_n \in V$, then there exists a unique quadratic mapping $g : V \to Y$ such that $g(x) \in f(x) - \frac{n}{2n+2} f(0)$ for all $x \in V$.

Proof. Since $f(0) \in cc(Y)$, $f(0)$ has at least an element, say $p \in f(0)$. Putting $x_k = 0$ for $k \in \{1, 2, \ldots, n\}$ in (3.1), we have

$$\frac{n(n-1)+2}{2} p \in f(0) + \frac{n(n-1)}{2} f(0) \subseteq n^2 f(0),$$

which means that $\frac{n(n-1)+2}{2n^2} p \in f(0)$. So $\left(\frac{n(n-1)+2}{2n^2}\right)^m p \in f(0)$ for all $m \in \mathbb{N}$ and $0 = \lim_{m\to\infty} \left(\frac{n(n-1)+2}{2n^2}\right)^m p \in f(0)$. Putting $x_k = x$ for $k \in \{1, 2, \ldots, n\}$ in (3.1), we have

$$\begin{aligned} f(nx) &= f(nx) + \frac{n(n-1)}{2}\{0\} \\ &\subseteq f(nx) + \frac{n(n-1)}{2} f(0) \subseteq n^2 f(x), \end{aligned} \tag{3.3}$$

i.e.,

$$f(nx) \subseteq n^2 f(x). \tag{3.4}$$

Replacing x by $n^{m-1}x$ and dividing both sides by n^{2m} in (3.4), then we obtain

$$n^{-2m} f(n^m x) \subseteq n^{-2m+2} f(n^{m-1}x)$$

for all $x \in V$. Denoting $F_m(x) := n^{-2m} f(n^m x)$ for all $x \in V$ and $m \in \mathbb{N} \cup \{0\}$, it results that $\{F_m(x)\}_m$ is a decreasing sequence of closed subsets of the Banach space Y. By (3.2), we get $\lim_{n\to\infty} diam(F_m(x)) = diam\left(n^{-2m}\left(f(n^m x)\right)\right) = 0$ for all $x \in V$. For the sequence $\{F_m(x)\}_{m\geq 0}$, the intersection $\bigcap_{m\geq 0} F_m(x)$ has a single element and we denote this single element by $g(x)$ for all $x \in V$. Thus we obtain a mapping $g : V \to Y$ which is a selection of f because $g(x) \in F_0(x) = f(x)$ for all $x \in V$.

Now we show that g is quadratic. From the definition of $F_m(x)$, we know that

$$\begin{aligned} F_m\left(\sum_{i=1}^n x_i\right) + \sum_{1\leq i<j\leq n} F_m(x_i - x_j) &= n^{-2m} f\left(\sum_{i=1}^n n^m x_i\right) \\ &\quad + \sum_{1\leq i<j\leq n} n^{-2m} f(n^m x_i - n^m x_j) \\ &\subseteq n \sum_{i=1}^n n^{-2m} f(n^m x_i) \\ &= n \sum_{i=1}^n F_m(x_i) \end{aligned}$$

for all $x_1, \ldots, x_n \in V$. With the definition of g and the above property, we have

$$\begin{aligned} g\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} g(x_i - x_j) &\in F_m\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} F_m(x_i - x_j) \\ &\subseteq n\sum_{i=1}^{n} F_m(x_i) \end{aligned}$$

and

$$n\sum_{i=1}^{n} g(x_i) \in n\sum_{i=1}^{n} F_m(x_i)$$

for all $m \ge 0$ and $x_1, \cdots, x_n \in V$. Since

$$n\sum_{i=1}^{n} F_{m+1}(x_i) \subseteq n\sum_{i=1}^{n} F_m(x_i)$$

and

$$diam\left(n\sum_{i=1}^{n} F_m(x_i)\right) \le n\sum_{i=1}^{n} diam\,(F_m(x_i)) \to\ 0 \text{ as } m \to \infty,$$

for any $x_1, \ldots, x_n \in V$, it results that $\{n\sum_{i=1}^{n} F_m(x_i)\}_{\ge 0}$ is a decreasing sequence of closed subsets of the Banach space Y. For this sequence, the intersection $\bigcap_{m\ge 0}(n\sum_{i=1}^{n} F_m(x_i))$ has a single element and so we have

$$g\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} g(x_i - x_j) = n\sum_{i=1}^{n} g(x_i)$$

for all $x_1, \ldots, x_n \in V$. Therefore, we conclude that there exists a quadratic mapping $g : V \to Y$ such that $g(x) \in f(x)$ for all $x \in V$.

Next, we will finalize the proof by proving the uniqueness of g for the case $g(x) \in f(x)$. Suppose that $g' : V \to Y$ is another quadratic mapping such that $g'(x) \in f(x)$ for all $x \in V$. We have

$$g(x) = \tfrac{g(n^m x)}{n^{2m}} \in \tfrac{f(n^m x)}{n^{2m}} \text{ and}$$
$$g'(x) = \tfrac{g'(n^m x)}{n^{2m}} \in \tfrac{f(n^m x)}{n^{2m}}$$

for all $m \in \mathbb{N} \cup \{0\}$. Since the intersection $\bigcap_{m\ge 0} \frac{f(n^m x)}{n^{2m}}$ has a single element, we have $g(x) = g'(x)$ for all $x \in V$, as desired. □

The following corollary is a refined stability result of Theorem 3.1 in [17] if we take $n = 2$.

COROLLARY 3.2. *If $f : V \to cc(Y)$ is a set-valued mapping satisfying the conditions*

$$f(x+y) + f(x-y) \subseteq 2f(x) + 2f(y)$$

and

$$\sup\{diam(f(x)) : x \in V\} < +\infty$$

for all $x, y \in V$, then there exists a unique quadratic mapping $g : V \to Y$ such that $g(x) \in f(x)$ for all $x \in V$.

Proof. Since $\sup\{diam(f(x)) : x \in V\} < +\infty$, we get

$$\lim_{m\to\infty} diam\left(\frac{f(2^m x)}{4^m}\right) = 0$$

for all $x \in V$. By Theorem 3.1, we complete the proof, where $g(x) \in f(x)$. □

THEOREM 3.3. *If $f : V \to cc(Y)$ is a set-valued mapping satisfying*

$$n\sum_{i=1}^{n} f(x_i) \subseteq f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} f(x_i - x_j) \tag{3.5}$$

and

$$\lim_{m\to\infty} n^{2m} diam\left(f\left(\frac{x}{n^m}\right)\right) = 0 \tag{3.6}$$

for all $x, x_1, \ldots, x_n \in V$, then there exists a unique quadratic mapping $g : V \to Y$ such that $g(x) \in f(x) + (-1)f(0)$ for all $x \in V$.

Proof. Since $n > 1$ and $n^2 f(0) \subset \frac{n(n-1)+2}{2} f(0)$, we easily get $f(0)$ is a singleton set and $f(0) = \{0\}$. Taking $x_i = x$ for all $i = 1, 2, \ldots, n$ in (3.5), we obtain

$$n^2 f(x) \subseteq f(nx) + \frac{n(n-1)+2}{2}\{0\} = f(nx) \tag{3.7}$$

for all $x \in V$. Denoting $F_m(x) = n^{2m} f\left(\frac{x}{n^m}\right)$, $x \in V$, $m \in \mathbb{N} \cup \{0\}$, we obtain that $\{F_m(x)\}_{m\ge 0}$ is a decreasing sequence of closed subsets of the Banach space Y. We have also

$$diam(F_m(x)) = diam\left(n^{2m} f\left(\frac{x}{n^m}\right)\right) = n^{2m} diam\left(f\left(\frac{x}{n^m}\right)\right).$$

By (3.6), we get $\lim_{m\to\infty} diam(F_m(x)) = 0$ for all $x \in V$.

For the sequence $\{F_m(x)\}_{m\ge 0}$, we obtain that the intersection $\bigcap_{m\ge 0} F_m(x)$ has a single element and we denote this element by $g(x)$ for all

$x \in V$. Thus we obtain a mapping $g : V \to Y$ such that $g(x) \in F_0(x) = f(x)$ for all $x \in V$.

Now we show that g is quadratic. From the definition of $F_m(x)$, we know that

$$\begin{aligned} n\sum_{i=1}^{n} F_m(x_i) &= n\sum_{i=1}^{n} n^{2m} f\left(\frac{x_i}{n^m}\right) \\ &\subseteq n^{2m} f\left(\sum_{i=1}^{n} \frac{x_i}{n^m}\right) + \sum_{1\le i<j\le n} n^{2m} f\left(\frac{x_i - x_j}{n^m}\right) \\ &= F_m\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} F_m(x_i - x_j) \end{aligned}$$

for all $x_1, \ldots, x_n \in V$. With the definition of g and the above property, we have

$$n\sum_{i=1}^{n} g(x_i) \in n\sum_{i=1}^{n} F_m(x_i) \subseteq F_m\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} F_m(x_i - x_j)$$

and

$$g\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} g(x_i - x_j) \in F_m\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} F_m(x_i - x_j)$$

for all $m \ge 0$ and $x_1, \cdots, x_n \in V$. Since

$$\begin{aligned} &F_{m+1}\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} F_{m+1}(x_i - x_j) \\ &\qquad \subseteq F_m\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} F_m(x_i - x_j) \end{aligned}$$

and

$$\begin{aligned} &diam\left(F_m\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} F_m(x_i - x_j)\right) \\ &\le diam\left(F_m\left(\sum_{i=1}^{n} x_i\right)\right) + \sum_{1\le i<j\le n} diam\left(F_m(x_i - x_j)\right) \to\ 0 \text{ as } m \to \infty, \end{aligned}$$

for any $x_1, \ldots, x_n \in V$, it results that $\{F_m(\sum_{i=1}^{n} x_i) + \sum_{1\le i<j\le n} F_m(x_i - x_j)\}_{m\ge 0}$ is a decreasing sequence of closed subsets of the Banach space Y.

For this sequence, the intersection $\bigcap_{m\geq 0}(F_m(\sum_{i=1}^n x_i)+\sum_{1\leq i<j\leq n} F_m(x_i-x_j))$ has a single element and so we have

$$n\sum_{i=1}^{n} g(x_i) = g\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\leq i<j\leq n} g(x_i - x_j)$$

for all $x_1, \ldots, x_n \in V$.

Therefore, we conclude that there exists a quadratic mapping $g : V \to Y$ such that $g(x) \in f(x)$ for all $x \in V$.

Next, we will finalize the proof by proving the uniqueness of g for the case $g(x) \in f(x)$. Suppose that $g' : V \to Y$ is another quadratic mapping such that $g'(x) \in f(x)$ for all $x \in V$. We have

$$g(x) = n^{2m} g\left(\frac{x}{n^m}\right) \in n^{2m} f\left(\frac{x}{n^m}\right),$$
$$g'(x) = n^{2m} g'\left(\frac{x}{n^m}\right) \in n^{2m} f\left(\frac{x}{n^m}\right)$$

for all $m \in \mathbb{N} \cup \{0\}$. Since

$$diam\left(n^{2m} f\left(\frac{x}{n^m}\right)\right) \to 0 \text{ as } m \to \infty,$$

the intersection $\bigcap_{m\geq 0} n^{2m} f\left(\frac{x}{n^m}\right)$ has a single element and so we have $g(x) = g'(x)$ for all $x \in V$, as desired. □

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Department of Mathematics Education
Gongju National University of Education
Gongju 32588, Republic of Korea
E-mail: ssjin@ gjue.ac.kr

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Department of Mathematics Education
Gongju National University of Education
Gongju 32588, Republic of Korea
E-mail: yanghi2@hanmail.net