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COMPARISON OF TOPOLOGIES ON THE FAMILY OF ALL TOPOLOGIES ON X

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Abstract. Topology may described a pattern of existence of elements of a given set X. The family $\tau(X)$ of all topologies given on a set X form a complete lattice. We will give some topologies on this lattice $\tau(X)$ using a fixed topology on X and we will regard $\tau(X)$ a topological space. Our purpose of this study is to comparison new topologies on the family $\tau(X)$ of all topologies induced old one.

1. Introduction.

Let X be a set. The family $\tau(X)$ would consist of all topologies on a given *fixed* set X. Here we want to give topologies on the family $\tau(X)$ of all the topologies using the given a topology τ on X and compare the topologies induced from the fixed old one.

The family $\tau(X)$ of all topologies on X form a complete lattice, that is, given any collection of topologies on X, there is a smallest (respectively largest) topology on X containing (contained in) each member of the collection. Of course, the partial order $\leq$ on $\tau(X)$ is defined by inclusion $\subseteq$ naturally.

In the sequel, the *closure* and *interior* of A are denoted by $\bar{A}$ and $int(A)$ in a topological space (X, τ). The θ-closure of a subset G of a topological space (X, τ) is defined [8] to be the set of all point $x \in X$ such that every closed neighborhood of x intersect G non-emptily and is denoted by $\bar{G}_\theta$ (cf. [1],[5]). Of course for any subset G in X, $G \subset \bar{G} \subset \bar{G}_\theta$ and $\bar{G}_\theta$ is closed in X. The subset G is called θ-closed if $\bar{G}_\theta = G$. If G is open, the $\bar{G} = \bar{G}_\theta$.

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Similarly, the θ-interior of a subset G of a topological space (X, τ) is defined to be the set of all point $x \in X$ for which there exists a closed neighborhood of x contained in G. The θ-interior of G is denoted by $int_\theta G$. Naturally, for any subset G in X, $int_\theta(G) \subset G$. An open set U in (X, τ) is called θ-open if $U = int_\theta(U)$. By the definition of this θ-open, the collection of all θ-open in a topological space (X, τ) form a topology τ_θ on X which will called the θ topology induced by τ which is related to the semi-regular topology on (X, τ) (cf. [7] [3]).

The semi-regular topology τ_s is the topology having as its base the set of all regular open sets. A subset A of a topological space X is called *regular open* [7] if $A = int\bar{A}$. For any subset A of X, $int(\bar{A})$ is always regular open. The collection of all regular open subsets of a topological space (X, τ) form a base for a topology τ_s on X coarser than τ, (X, τ_s) is called the semiregularization of (X, τ).

We should recall the definitions of almost-continuity and θ-continuity: A function $f : X \to Y$ is *almost-continuous*(θ-continuous) if for each $x \in X$ and each regular-open V(open V) containing $f(x)$, there exists a open set U containing x such that $f(U) \subset V$ $(f(\bar{U}) \subset \bar{V})$ ([3]).

THEOREM 1.1. [3] *Let $f : X \to Y$ be continous map. If $V \subset X$ is θ-open, then $f^{-1}(V)$ is θ-open.*

THEOREM 1.2. [3] *Let $f : X \to Y$ be a function from X onto Y that is both open and closed. Then f preserves θ-open sets.*

2. Topology on the family $\tau(X)$ related to the θ topologies on X.

Let (X, τ) be a topological space, and $G \in \tau$. Let $i(G)=\{\zeta\in\tau(X) \,|G\in\zeta\}$ and denote $\epsilon=\{i(G)|G\in\tau\}$, a family of subset of $\tau(X)$. Then there is exactly one topology In_τ on $\tau(X)$ with ϵ as a subbasis. We will call this topology *the inner topology* induced by the topology τ ([4]).

If $\zeta \leq \eta$, then $\forall\ G \in \zeta,\ G \in \eta$. That is, if $i(G) \ni \zeta$, then $i(G) \cap \{\eta\} \neq \emptyset$. This implies $\zeta \in \overline{\{\eta\}}$. Conversely $\zeta \in \overline{\{\eta\}}$ implies $\zeta \leq \eta$. If this relation holds we say that ζ is a *specialization* of η [6]. For any $\eta \in \tau(X)$ we will denote the subset $\{\zeta \in \tau(X) | \zeta \geq \eta\}$ by $\uparrow(\eta)$. We shall also use later the notation $\downarrow(\eta)$ for $\{\zeta \in \tau(X) | \zeta \leq \eta\}$. Then since $i(G) = \{\ \zeta \in \tau\ (X) \mid G \in \zeta\ \}$, $i(G) = \uparrow(\{\emptyset, X, G\})$. Hence $\zeta \in \overline{\{\eta\}}$ if and only if $\zeta \leq \eta$. Since *Alexandrov topology* Υ on $\tau(X)$ is the collection of all *upper sets* in $\tau(X)$ (i.e. sets U such that $\eta \in U$ and $\eta \leq \zeta$ imply $\zeta \in U$)

[6], $i(G) \in \Upsilon$. Hence we have the following result [2]. If $\tau \leq \zeta \leq 1$, then $In_\tau \leq In_\zeta \leq In_1 \leq \Upsilon$.

Now we will define a different topology on $\tau(X)$ the θ topology induced by given topology τ. Let (X, τ) be a space, and $G \in \tau$. Let $\theta(G)= \{\zeta \in \tau(X) \mid G : \theta - \text{open in } \zeta \}$. And denote $\beta = \{\theta(G) | G \in \tau\}$, a family of subset of $\tau(X)$. Then there is exactly one topology θ_τ on $\tau(X)$ with β as a subbasis. We will call also this topology θ_τ on $\tau(X)$ *the θ topology* induced by the topology τ.

If we consider θ as a map from $\tau(X)$ to $\tau(X)$ defined by $\theta(\eta) = \eta_\theta$, then we have next result ([3]):

THEOREM 2.1. *Let (X, τ) be a topological space. Then the induced map*

$$\theta : (\tau(X), \theta_\tau) \to (\tau(X), \theta_\tau)$$

is continuous.

Such map $\theta : (\tau(X), \theta_\tau) \to (\tau(X), \theta_\tau)$ will be called θ-operator. Moreover this map satisfies that

COROLLARY 2.2. *$\theta(\zeta \wedge \eta) \leq \theta(\zeta) \wedge \theta(\eta)$ and $\theta(\zeta) \vee \theta(\eta) \leq \theta(\zeta \vee \eta)$.*

For a topological space (X, τ), the collection of all open neighborhoods of p and empty set, that is, $\{V \in \tau | p \in V\} \cup \{\emptyset\}$ becomes a topology on X for any point $p \in X$. We will denote such a topology by τ_p and call *localized topology* of τ at p. We will denote the localized topology of the discrete topology $\mathcal{P}(X)$ on X at p by 1_p.

Denote $\tau_p(X) = \{\eta_p \mid \eta \in \tau(X) \}$ for a point $p \in X$. Since $\tau(X)$ is a complete lattice, we can easily find that $\tau_p(X)$ is a sublattice of $\tau(X)$. The smallest element of this sublattice $\tau_p(X)$ is 0_p=0, the largest element is $\mathcal{P}(X)_p$=$1_p \neq 1$. We will call this sublattice $\tau_p(X)$ as *sublattice of all localized topologies* at p in X.

Now we will regard any member τ of $\tau(X)$ as a map from X to $\cup_p \tau_p(X) \subset \tau(X)$ defined by $\tau(p) = \tau_p$. Hence this map τ acts like a vector field on X. Such a map $f : X \to \tau(X)$ defined by $f(p) \in \tau_p(X)$ will be called *topology field* on X [4].

THEOREM 2.3. [4] *Topology field ζ:$(X$,$\tau) \to (\tau(X)$, $In_\tau)$ is continuous.*

THEOREM 2.4. [3] *If (X, ζ) is a θ topological space, then the topology field ζ:$(X$,$\tau) \to (\tau(X)$, $\theta_\tau)$ is continuous.*

Corollary 2.5. *If (X,ζ) is a regular topological space, then the topology field ζ:$(X,\tau) \to (\tau(X),\, \theta_\tau)$ is continuous.*

Additionally, if f is open and closed and $\omega \in \theta(f^{-1}(G))$, then $f^{-1}(G)$ is θ-open in (X,ω). By Theorem 1.5 ([3]), G is θ-open in $(X, f_*(\omega))$, i.e. $f_*(\omega) \in \theta(G)$. That is, $\omega \in f_*^{-1}(\theta(G))$. Consequently we have;

Theorem 2.6. *If $f : (X,\tau) \to (Y,\eta)$ is a continuous and open and closed surjective map, then for any open G in Y*

$$f_*^{-1}(\theta(G)) = \theta(f^{-1}(G)).$$

Let (X,τ) and (Y,ζ) be topological spaces. We may assume that $\tau(X)$ and $\tau(Y)$ are given the topologies θ_τ and θ_ζ respectively and assume that $\tau(X \times Y)$ is given topology $\theta_{\tau\times\zeta}$. Then we have the next theorem by [3].

Theorem 2.7. *The multiplication $\times : \tau(X) \times \tau(Y) \to \tau(X \times Y)$ is continuous.*

Theorem 2.8. *Let (X,τ) and (Y,ζ) be topological spaces. Then*

$$\tau_\theta \times \zeta_\theta = (\tau \times \zeta)_\theta.$$

Consequently we have the following commutative diagram:

$$\begin{array}{ccc} \tau(X)\times\tau(Y) & \xrightarrow{\times} & \tau(X\times Y) \\ \downarrow \theta\times\theta & & \downarrow \theta \\ \tau(X)\times\tau(Y) & \xrightarrow{\times} & \tau(X\times Y). \end{array}$$

Again we consider Θ as a map from $\tau(X)$ to $\tau(\tau(X))$ defined by $\Theta(\eta) = \theta_\eta$, then we have next result.

Theorem 2.9. *Let (X,τ) be a topological space. Then the induced map*

$$\Theta : (\tau(X), \Upsilon) \to (\tau(\tau(X)), \Upsilon)$$

is continuous.

Corollary 2.10. *Let (X,τ) be a topological space. Then the induced map*

$$\Theta : (\tau(X), \theta_\tau) \to (\tau(\tau(X)), \theta_{\theta_\tau})$$

is continuous.

3. Topology on the family $\tau(X)$ related to the θ topologies on X.

Definition 3.1. Let (X, τ) be a topological space, and $G \in \tau$. Let $w\theta(G) = \{\zeta \in \tau(X) \mid \text{there exist } \theta\text{-open O in } \zeta \text{ such that } O \subset G\}$. Denote $\beta = \{w\theta(G) \mid G \in \tau\}$, a family of subset of $\tau(X)$. Then there is exactly one topology $w\theta_\tau$ on $\tau(X)$ with β as a subbasis. We will call this topology $w\theta_\tau$ *the weak θ topology* induced by the topology τ.

It is natural $\theta_\tau \leq w\theta_\tau$.

Theorem 3.2. *If $\tau \leq \zeta \leq 1$, then $w\theta_\tau \leq w\theta_\zeta \leq w\theta_1 \leq \Upsilon$.*

Proof. For any $G \in \tau \leq \zeta$, by definition of $w\theta(G)$ we can naturally have $w\theta_\tau \leq w\theta_\zeta$. Now we will prove that every $w\theta(G)$ is upper set in $\tau(X)$. Let $\delta \in w\theta(G)$. Then there exists a θ-open O in (X, δ) such that $O \subset G$. Hence $O \in \delta_\theta$. If $\delta \leq \gamma$, we have by Theorem 2.1, $O \in \gamma_\theta$. This means O is θ-open in (X, γ) such that $O \subset G$. This implies $\gamma \in w\theta(G)$. Hence $w\theta(G)$ is upper set in $\tau(X)$. This completes the proof. □

If we consider θ as a map from $\tau(X)$ to $\tau(X)$ defined by $\theta(\eta) = \eta_\theta$, then we have;

Theorem 3.3. *Let (X, τ) be a topological space. Then the induced map*

$$\theta : (\tau(X), w\theta_\tau) \to (\tau(X), w\theta_\tau)$$

is continuous.

Proof. Let $\zeta \in \tau(X)$ and $w\theta(K)$ is a neighborhood of $\theta(\zeta) = \zeta_\theta$ where $K \in \tau$. Then since $\zeta_\theta = \{U \in \zeta | U : \theta - \text{open in } (\text{X}, \zeta)\}$, there exists a θ-open set H in (X, ζ_θ) such that $H \subset K$. Hence H is also θ-open in (X, ζ) such that $H \subset K$. Consequently $\zeta \in w\theta(K)$, i.e. $w\theta(K)$ is a neighborhood of ζ which satisfied that $\theta(\theta(K)) \subset \theta(K)$. This completes the Theorem. □

Corollary 3.4. *If (X, ζ) is a θ topological space, then the topology field $\zeta : (X, \tau \to (\tau(X), w\theta_\tau)$ is continuous.*

Proof. Let $p \in X$ and $w\theta(G)$ be a subbasic open neighborhood of $\zeta(p){=}\zeta_p$. Then there is a θ-open O in (X, ζ_p) such that $O \subset G$. This implies O is θ-open in (X, ζ) because O is open set in (X, ζ) which contains the point p. Moreover since $G{\in}\tau$, G is a neighborhood of p. Hence if $q{\in}G$, $\zeta(q){=}\zeta_q{\in}w\theta(G)$, so that $\zeta(G) \subset w\theta(G)$. This shows that topology field ζ is continuous. □

Corollary 3.5. *If (X, ζ) is a regular topological space, then the topology field ζ:$(X,\tau) \to (\tau(X), w\theta_\tau)$ is continuous.*

Let f:$(X, \tau) \to (Y, \eta)$ be a continuous surjective map. If we define a map f_*:$(\tau(X), w\theta_\tau) \to (\tau(Y), w\theta_\eta)$ by $f_*(w)$=$\{U \subset Y | f^{-1}(U) \in w\}$, then $f_*(0) = 0$ and $f_*(1) = 1$. Let $\omega \in \tau(X)$. For any subbasic open neighborhood $w\theta(G)$ of $f_*(\omega)$ in $(\tau(Y), w\theta_\eta)$, where G is open in (Y, η), there is a θ-open O in $(Y, f_*(\omega))$ such that $O \subset G$. By Theorem 1.1, $f^{-1}(O)$ is θ-open in (X, ω) such that $f^{-1}(O) \subset f^{-1}(G)$. Thus $\omega \in w\theta(f^{-1}(G))$. Hence $w\theta(f^{-1}(G))$ is an open neighborhood of ω in $(\tau(X), w\theta_\tau)$. Consequently we have the next result.

Theorem 3.6. *Let f:$(X, \tau) \to (Y, \eta)$ be a continuous surjective map. If we define a map f_*:$(\tau(X), w\theta_\tau) \to (\tau(Y), w\theta_\eta)$ by $f_*(w)$=$\{U \subset Y | f^{-1}(U) \in w\}$, then the map f_* is continuous. If $\gamma \leq \delta$, then $f_*(\gamma) \leq f_*(\delta)$ and $f_*(\tau) \geq \eta$. And for any θ topology field ζ, the diagram*

$$\begin{array}{ccc} (X, \tau) & \xrightarrow{f} & (Y, \eta) \\ \downarrow \zeta & & \downarrow f_*(\zeta) \\ (\tau(X), w\theta_\tau) & \xrightarrow{f_*} & (\tau(Y), w\theta_\eta) \end{array}$$

commutes. Furthermore if (Z, λ) is a topological space and g: $(Y, \eta) \to (Z, \lambda)$ is a map, then

$$(g \circ f)_* = g_* \circ f_*.$$

Finally, if f:$(X, \tau) \to (X, \tau)$ is the identity homeomorphism, then so is f_*.

Proof. The continuity of the map f_*:$(\tau(X), w\theta_\tau) \to (\tau(Y), w\theta_\eta)$ was proved already. And the commutativity of the diagram follows from the next fact.

$$\begin{aligned} f_*(\zeta_p) &= \{U | f^{-1}(U) \in \zeta_p\} = \{U | p \in f^{-1}(U) \in \zeta\} \\ &= \{U | f(p) \in U, f^{-1}(U) \in \zeta\} = \{U | U \in f_*(\zeta), f(p) \in U\} \\ &= f_*(\zeta)_{f(p)}. \end{aligned}$$

All other statements follow directly from the definitions. □

Additionally, if f is open and closed and $\omega \in w\theta(f^{-1}(G))$, then there is a θ-open O in (X, ω) such that $O \subset f^{-1}(G)$. By Theorem 1.5 ([3]), $f(O)$ is θ-open in $(X, f_*(\omega))$ such that $f(O) \subset G$, i.e., $f_*(\omega) \in w\theta(G)$. That is, $\omega \in f_*^{-1}(w\theta(G))$. Consequently we have the following theorem.

THEOREM 3.7. *If $f : (X, \tau) \to (Y, \eta)$ is a continuous and open and closed surjective map, then for any open G in Y*

$$f_*^{-1}(w\theta(G)) = w\theta(f^{-1}(G)).$$

Let (X, τ) and (Y, ζ) be topological spaces. We may assume that $\tau(X)$ and $\tau(Y)$ are given the topologies $w\theta_\tau$ and $w\theta_\zeta$ respectively and assume that $\tau(X \times Y)$ is given topology $w\theta_{\tau\times\zeta}$. Then we get the following result.

THEOREM 3.8. *The multiplication $\times : \tau(X) \times \tau(Y) \to \tau(X \times Y)$ is continuous.*

Proof. Let $(\alpha, \beta) \in \tau(X) \times \tau(Y)$. Then $\alpha \times \beta \in \tau(X \times Y)$. If $w\theta(W)$ is a neighborhood of $\times(\alpha, \beta) = \alpha \times \beta$, where W is open in $(X \times Y, \tau \times \zeta)$. Then there exists an θ-open set O in $\alpha \times \beta$ such that $O \subset W$. We may assume that $O = O_X \times O_Y$ is a basic open set in $(\tau(X \times Y), \alpha \times \beta)$. Since O is θ-open in $(\tau(X \times Y), \alpha \times \beta)$. Hence projection O_X and O_Y are θ-opens in (X, α) and (Y, β) respectively. Hence $(\alpha, \beta) \in \theta(O_X) \times \theta(O_Y)$. Moreover $\times(\theta(O_X) \times \theta(O_Y)) \subset \theta(O)$. In fact, if $\delta \in \theta(O_X)$ and $\gamma \in \theta(O_Y)$, then O_X is θ-open in (X, δ) and O_Y is θ-open in (Y, γ). Since the product of θ-opens is θ-open [5], $O = O_X \times O_Y$ is θ-open in $(X \times Y, \delta \times \gamma)$ such that $O \subset W$. Hence $\delta \times \gamma \in w\theta(W)$. This completes the proof. $\square$

4. Topology on the family $\tau(X)$ related to the semi-regular topology on X.

DEFINITION 4.1. Let (X, τ) be a topological space, and $G \in \tau$. Let $s(G) = \{\zeta \in \tau(X) | G \text{isregular} - \text{openin}\zeta\}$. Denote $\beta' = \{s(G) | G \in \tau\}$, a family of subset of $\tau(X)$. Then there is exactly one topology s_τ on $\tau(X)$ with β' as a subbasis. We will call this topology s_τ *the semireg topology* induced by the topology τ.

THEOREM 4.2. *If $\tau \le \zeta \le 1$, then $s_\tau \le s_\zeta \le s_1$.*

Proof. Let $s(G) \in s_\tau$. Then for any $\delta \in s(G)$, G is regular-open in (X, δ). Since $\tau \le \zeta \le 1$, $G \in \tau$ implies $G \in \zeta$. Consequently $\delta \in s(G) \in s_\zeta$. $\square$

Similarly as in the θ case if we consider s as a map from $\tau(X)$ to $\tau(X)$ defined by $s(\eta) = \eta_s$, then we have next result:

THEOREM 4.3. *Let(X, τ) be a topological space. Then the induced map*

$$s : (\tau(X), s_\tau) \to (\tau(X), s_\tau)$$

is continuous.

Proof. Let $\zeta \in \tau(X)$ and $s(K)$ is a neighborhood of $s(\zeta) = \zeta_s$ where $K \in \tau$. Then since $\zeta_s = \{U \in \zeta | U : \text{regur} - \text{open in } (\mathrm{X}, \zeta)\}$, K is a regular-open set in (X, ζ_s). Hence it is also regular-open in (X, ζ). Consequently $\zeta \in s(K)$, i.e. $s(K)$ is a neighborhood of ζ which satisfied that $s(s(K)) \subset s(K)$. This completes the proof of theorem. □

Such map $s : (\tau(X), s_\tau) \to (\tau(X), s_\tau)$ will be called semi-regularization operator. Moreover this map satisfies that the following corollary.

COROLLARY 4.4. *$s(\zeta \wedge \eta) \leq s(\zeta) \wedge s(\eta)$ and $s(\zeta) \vee s(\eta) \leq s(\zeta \vee \eta)$.*

Proof. This corollary follows from the above definition and Theorem 3.1. □

Now we want to know the relations between s_τ and In_{τ_s}. Let $\eta \in s(G) \in s_\tau$, then $G \in \eta$, i.e. $\eta \in i(G)$. Hence it is natural that $s(G) \subset i(G)$. Let $i(G)$ be a sub basic open in In_{τ_s}. Then $G \in \tau_s$. Hence G is regular-open in τ, that is $G = int^\tau \bar{G}^\tau$. Hence if $\zeta \in i(G)$ and (X, η) is regular, then by above Theorem 1.2, G is also regular-open in (X, η). Hence $\eta \in s(G)$. Thus we have;

THEOREM 4.5. *Let (X, τ) is a regular space. If we denote $\tau_{reg}(X)$ by the subset of all regular topologies in $\tau(X)$. Then the subspace $\tau_{reg}(X)$ of the space $(\tau(X), s_\tau)$ and the subspace $\tau_{reg}(X)$ of the space $(\tau(X), In_{\tau_s})$ are identical.*

THEOREM 4.6. *If (X, ζ) is semi-regular space, then the topology field ζ:$(X,\tau) \to (\tau(X), s_\tau)$ is continuous.*

Proof. Let $p \in X$ and $s(G)$ be a subbasic open neighborhood of $\zeta(p)=\zeta_p$. Then G is regular-open in (X, ζ_p). This implies G is regular-open in (X, ζ) because G is open set in (X, ζ) which contains the point p. Moreover since $G \in \tau$, G is a neighborhood of p. Hence if $q \in G$, $\zeta(q)=\zeta_q \in s(G)$, so that $\zeta(G) \subset s(G)$. This shows that topology field ζ is continuous. □

COROLLARY 4.7. *If (X, ζ) is regular space, then the topology field ζ:$(X,\tau) \to (\tau(X), s_\tau)$ is continuous.*

Let f:$(X, \tau) \to (Y, \eta)$ be a continuous surjective map. If we define a map f_*:$(\tau(X), s_\tau) \to (\tau(Y), s_\eta)$ by $f_*(w)=\{U \subset Y | f^{-1}(U) \in w\}$, then $f_*(0) = 0$ and $f_*(1) = 1$. Let $\omega \in \tau(X)$. For any subbasic open neighborhood $s(G)$ of $f_*(\omega)$ in $(\tau(Y), s_\eta)$, where G is open in (Y, η), G is regular-open in $(Y, f_*(\omega))$. Since f is continuous, it is naturally almost-continuous. Hence $f^{-1}(G)$ is regular-open in (X, ω). Thus $\omega \in s(f^{-1}(G))$. Hence $s(f^{-1}(G))$ is an open neighborhood of ω in $(\tau(X), s_\tau)$.

Now we will prove that $f_*(s(f^{-1}(G))) \subset s(G)$. Let $\zeta \in s(f^{-1}(G))$. Then $f^{-1}(G)$ is regular-open in (X, ζ). Since naturally the map $f : (X, \zeta) \to (Y, f_*(\zeta))$ is continuous, G is regular-open in $(Y, f_*(\zeta))$. This implies that $f_*(\zeta) \in s(G)$. Thus we have next theorem.

THEOREM 4.8. *Let f:$(X, \tau) \to (Y, \eta)$ be a continuous surjective map. If we define a map f_*:$(\tau(X), s_\tau) \to (\tau(Y), s_\eta)$ by $f_*(w)$=$\{U \subset Y | f^{-1}(U) \in w\}$, then the map f_* is continuous. If $\gamma \leq \delta$, then $f_*(\gamma) \leq f_*(\delta)$ and $f_*(\tau) \geq \eta$. And for any semi-regular topology field ζ, the diagram*

$$\begin{array}{ccc} (X, \tau) & \xrightarrow{f} & (Y, \eta) \\ \downarrow \zeta & & \downarrow f_*(\zeta) \\ (\tau(X), s_\tau) & \xrightarrow{f_*} & (\tau(Y), s_\eta) \end{array}$$

commutes. If, furthermore, (Z, λ) is a topological space and g: $(Y, \eta) \to (Z, \lambda)$ is a map, then

$$(g \circ f)_* = g_* \circ f_*.$$

Finally, if f:$(X, \tau) \to (X, \tau)$ is the identity homeomorphism, then so is f_*.

Proof. The proof of this theorem is very closed to the case of θ topological case. □

Additionally, if f is open and closed and $\omega \in s(f^{-1}(G))$, then $f^{-1}(G)$ is regular-open in (X, ω). By above Theorem 1.5, G is regular-open in $(X, f_*(\omega))$, i.e. $f_*(\omega) \in s(G)$. That is, $\omega \in f_*^{-1}(s(G))$. Consequently we have the next result.

LEMMA 4.9. *If $f : (X, \tau) \to (Y, \eta)$ is continuous and open and closed surjective map, then for any open G in Y*

$$f_*^{-1}(s(G)) = s(f^{-1}(G)).$$

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