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VOLUMES OF GEODESIC BALLS IN HEISENBERG GROUPS

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Abstract. Let $\mathbb{H}^3$ be the 3-dimensional Heisenberg group equipped with a left-invariant metric. In this paper we calculate the volumes of geodesic balls in $\mathbb{H}^3$. Let $B_e(R)$ be the geodesic ball with center e (the identity of $\mathbb{H}^3$) and radius R in $\mathbb{H}^3$. Then, the volume of $B_e(R)$ is given by

$$\begin{aligned} &Vol(B_e(R)) \\ &\quad= \frac{\pi}{6}\Big\{-16R+(R^2+6)\sin R+(R^3+10R)\cos R \\ &\qquad\quad+(R^4+12R^2)\int_0^R \frac{\sin t}{t}dt\Big\}. \end{aligned}$$

1. Introduction

Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with an inner product $<,>$ and N its unique simply connected 2-step nilpotent Lie group with the left invariant metric induced by $<,>$ on $\mathcal{N}$. Let $\mathcal{Z}$ be the center of $\mathcal{N}$. Then $\mathcal{N}$ is represented by the direct sum of $\mathcal{Z}$ and its orthgonal complement $\mathcal{Z}^\perp$.

For each $Z \in \mathcal{Z}$, a skew symmetric linear transformation $j(Z) : \mathcal{Z}^\perp \to \mathcal{Z}^\perp$ is defined by $j(Z)X = (adX)^*Z$ for $X \in \mathcal{Z}^\perp$. Or, equivalently,

$$< j(Z)X, Y > = < [X, Y], Z >$$

for all $X, Y \in \mathcal{Z}^\perp$.

A 2-step nilpotent Lie algebra $\mathcal{N}$ is said to be an algebra of Heisenberg type if

$$j(Z)^2 = -|Z|^2 id$$

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for all $Z \in \mathcal{Z}$. And a Lie group N is said to be a group of Heisenberg type if its Lie algebra $\mathcal{N}$ is of Heisenberg type.

The Heisenberg groups are examples of Heisenberg type. That is, let $n \geq 1$ be any integer and $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$ a basis of $R^{2n} = \mathcal{V}$. Let $\mathcal{Z}$ be an one dimensional vector space spanned by $\{Z\}$. Define

$$[X_i, Y_i] = -[Y_i, X_i] = Z$$

for any $i = 1, 2, \cdots, n$ with all other brackets are zero. Give on $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ the inner product such that the set of vectors $\{X_i, Y_i, Z | i = 1, 2, \cdots, n\}$ forms an orthonormal basis. Let N be the simply connected 2-step nilpotent group of Heisenberg type which is determined by $\mathcal{N}$ and equipped with a left-invariant metric induced by the inner product in $\mathcal{N}$. The group N is called the $(2n+1)$-dimensional Heisenberg group and denoted by $\mathbb{H}^{2n+1}$.

In this paper, we calculate the volumes of the geodesic balls on the Heisenberg group $\mathbb{H}^3$:

Theorem 1.1. *Let $B_e(R)$ be the geodesic ball with center e (the identity of $\mathbb{H}^3$) and radius R in $\mathbb{H}^3$. Then, the following holds.*

$$\begin{aligned} &Vol(B_e(R)) \\ &\quad = \frac{\pi}{6}\Big\{ -16R + (R^2+6)\sin R + (R^3+10R)\cos R \\ &\qquad\qquad + (R^4+12R^2)\int_0^R \frac{\sin t}{t}dt\Big\}. \end{aligned}$$

For a Riemannian manifold M and $p \in M$, the volume growth, $VG_p(M)$ of M at p is defined by

$$VG_p(M) = inf\{x \in R | \lim_{r\to\infty} \frac{Vol(B_p(r))}{r^x} = 0\}.$$

If M is a Lie group with a left invariant metric, then we see that $VG_p(M) = VG_q(M)$ for any $p, q \in M$. In this case, it is denoted by $VG(M)$.

Corollary 1.2. *The volume growth, $VG(\mathbb{H}^3)$ of $\mathbb{H}^3$ is given as follows;*

$$VG(\mathbb{H}^3) = 4.$$

2. Preliminaries

Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with an inner product $<,>$ and N be its unique simply connected 2-step nilpotent Lie group with the left invariant metric induced by $<,>$ on $\mathcal{N}$. The center of $\mathcal{N}$ is denoted by $\mathcal{Z}$. Then $\mathcal{N}$ can be expressed as the direct sum of $\mathcal{Z}$ and its orthogonal complement $\mathcal{Z}^{\perp}$.

Recall that for $Z \in \mathcal{Z}$, a skew symmetric linear transformation $j(Z) : \mathcal{Z}^{\perp} \to \mathcal{Z}^{\perp}$ is defined by $j(Z)X = (\mathrm{ad}X)^{*}Z$ for $X \in \mathcal{Z}^{\perp}$. Or, equivalently,

$$\langle j(Z)X, Y\rangle = \langle [X, Y], Z\rangle$$

for $X, Y \in \mathcal{Z}^{\perp}$. A 2-step nilpotent Lie group N is said to be *of Heisenberg type* if

$$j(Z)^2 = -|Z|^2\,\mathrm{id}$$

for all $Z \in \mathcal{Z}$.

Let $\gamma(t)$ be a curve in N such that $\gamma(0) = e$ (identity element in N) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^{\perp}$ and $Z_0 \in \mathcal{Z}$. Since $\exp : \mathcal{N} \to N$ is a diffeomorphism ([9]), the curve $\gamma(t)$ can be expressed uniquely by $\gamma(t) = \exp\,(X(t) + Z(t)]$ with

$$\begin{array}{lll} X(t) \in \mathcal{Z}^{\perp}, & X'(0) = X_0\,, & X(0) = 0 \\ Z(t) \in \mathcal{Z}\,, & Z'(0) = Z_0\,, & Z(0) = 0\,. \end{array}$$

A. Kaplan ([7, 8]) shows that the curve $\gamma(t)$ is a geodesic in N if and only if

$$\begin{array}{l} X''(t) = j(Z_0)X'(t), \\ Z'(t) + \frac{1}{2}[X'(t), X(t)] \equiv Z_0. \end{array}$$

The following Lemma is useful in the later.

LEMMA 2.1. [2] *Let N be a simply connected 2-step nilpotent Lie group with a left invariant metric, and let $\gamma(t)$ be a geodesic of N with $\gamma(0) = e$ and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^{\perp}$ and $Z_0 \in \mathcal{Z}$. Then, one has*

$$\gamma'(t) = dl_{\gamma(t)}(X'(t) + Z_0), t \in R$$

where $X'(t) = e^{tj(Z_0)}X_0$ and $l_{\gamma(t)}$ is the left translation by $\gamma(t)$.

Throughout this paper, different tangent spaces will be identified with $\mathcal{N}$ via left translation. So, in above lemma, we can consider $\gamma'(t)$ as

$$\gamma'(t) = X'(t) + Z_0 = e^{tj(Z_0)}X_0 + Z_0.$$

Let $\mathbb{H}^3$ be the 3 dimensional Heisenberg group with a left invariant metric and $\mathcal{H}$ its Lie algebra. Let $\gamma(t)$ be an unit speed geodesic on $\mathbb{H}^3$ with $\gamma(0) = e$(the identity element of $\mathbb{H}^3$) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^{\perp}$ and $Z_0 \in \mathcal{Z}$. Assume that $X_0 \neq 0$ and $Z_0 \neq 0$. Then

$$\left\{X_0 + Z_0, \frac{|Z_0|}{|X_0|}X_0 - \frac{|X_0|}{|Z_0|}Z_0, \frac{1}{|Z_0||X_0|}j(Z_0)X_0\right\}$$

is an orthonormal basis of $\mathcal{H}$. Let

$$\begin{aligned} e_1(t) &= \frac{|Z_0|}{|X_0|}X'(t) - \frac{|X_0|}{|Z_0|}Z_0, \\ e_2(t) &= \frac{1}{|Z_0||X_0|}j(Z_0)X'(t). \end{aligned}$$

Then, $\{\gamma'(t), e_1(t), e_2(t)\}$ is an orthonormal frame along $\gamma(t)$ on $\mathbb{H}^3$.

We start the following Proposition.

PROPOSITION 2.2. [5] *For each $k = 1, 2$, let $J_k(t)$ be the Jacobi field with $J_k(0) = 0, J_k'(0) = e_k(0)$. Then, we have that*

$$\begin{bmatrix} J_1(t) \\ J_2(t) \end{bmatrix} = B(t) \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

where

$$B(t) = \frac{1}{|Z_0|^3} \begin{bmatrix} \sin(|Z_0|t) - (1-|Z_0|^2)|Z_0|t & |Z_0|(\cos(|Z_0|t) - 1) \\ |Z_0|(1-\cos(|Z_0|t)) & |Z_0|^2 \sin(|Z_0|t) \end{bmatrix}.$$

COROLLARY 2.3. [1, 6] *Let $\mathbb{H}^3$ be the $(2n+1)$-dimensional Heisenberg group and $\mathcal{N}$ its Lie algebra. Let $\gamma(t)$ be an unit speed geodesic in N with $\gamma(0) = e$(the identity element of N) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^{\perp}$ and $Z_0 \in \mathcal{Z}$. If $Z_0 \neq 0$, then all the conjugate points along γ are at $t \in \frac{2\pi}{|Z_0|}\mathbb{Z}^* \cup \mathbb{A}$ where*

$$\mathbb{Z}^* = \{\pm 1, \pm 2, \ldots\}$$

and

$$\mathbb{A} = \left\{ t \in \mathbb{R} - \{0\} | (1-|Z_0|^2)\frac{|Z_0|t}{2} = \tan\frac{|Z_0|t}{2} \right\}.$$

In particular, $\frac{2\pi}{|Z_0|}$ is the first conjugate point of e along γ.
If $Z_0 = 0$, then there are no conjugate points along γ.

For the conjugate points of another type of Heisenberg groups, Quaternionic Heisenberg groups $\mathbb{H}^{4n+3}$, see [4].

G. Walschap [10] showed that the first conjugate loci and the cut loci are equal in the case of the groups of Heisenberg type or the 2-step nilpotent groups with one-dimensional center. So, we consider the geodesic balls $B_e(r)$ with the radius $r \leq 2\pi$.

In [5], C. Jang, J. Park and K. Park obtained a fomula of the volumes of geodesic balls in the Heisenberg group $\mathbb{H}^3$ as the form of power series.

THEOREM 2.4. [5] *Let $B_e(R)$ be the geodesic ball with center e and radius R in $\mathbb{H}^3$. Then, the following holds.*

$$Vol(B_e(R)) = 4\pi \left(\frac{R^3}{3} + 2\sum_{n=2}^{\infty}(-1)^n \frac{R^{2n+1}}{(2n+1)!(2n-1)(2n-3)} \right).$$

3. Main results

LEMMA 3.1. [5]

$$\begin{aligned}\det(B(t)) &= \frac{1}{|Z_0|^4}\{2(1-\cos(|Z_0|t)) - (1-|Z_0|^2)|Z_0|t\sin(|Z_0|t)\}\\ &\geq 0.\end{aligned}$$

LEMMA 3.2. *Let n be a natural number and $f : [0,x] \to R$ have continuous $n-th$ derivatives. Assume that for $k = 0, 1, \cdots, n-1$, the $\lim_{t\to 0^+} \frac{f^{(k)}(t)}{t^{n-k}}$ exists and $\frac{f^{(n)}(t)}{t}$ is integrable on $[0,x]$. Then, $\frac{f(t)}{t^{n+1}}$ is integrable on $[0,x]$ and the following holds.*

$$\int_0^x \frac{f(t)}{t^{n+1}}dt = -\sum_{k=0}^{n-1} \frac{1}{n(n-1)\cdots(n-k)} \left[\frac{f^{(k)}(t)}{t^{n-k}}\right]_{0^+}^{x^-} + \frac{1}{n!}\int_0^x \frac{f^{(n)}(t)}{t}dt$$

where

$$\left[\frac{f^{(k)}(t)}{t^{n-k}}\right]_{0^+}^{x^-} = \lim_{t\to x^-}\frac{f^{(k)}(t)}{t^{n-k}} - \lim_{t\to 0^+}\frac{f^{(k)}(t)}{t^{n-k}}.$$

Proof. We use the mathematical induction. For $n = 1$, we see that

$$\int_0^x \frac{f(t)}{t^2}dx = \int_0^x f(t)d(-\frac{1}{t}) = -\left[\frac{f(t)}{t}\right]_{0^+}^{x^-} + \int_0^x \frac{f^{(1)}(t)}{t}dt.$$

Suppose that

$$\int_0^x \frac{f(t)}{t^{n+1}}dt = -\sum_{k=0}^{n-1} \frac{1}{n(n-1)\cdots(n-k)} \left[\frac{f^{(k)}(t)}{t^{n-k}}\right]_{0^+}^{x^-} + \frac{1}{n!}\int_0^x \frac{f^{(n)}(t)}{t}dt$$

holds. Then,

$$\begin{aligned}\int_0^x \frac{f(t)}{t^{n+2}}dt &= \int_0^x f(t)d(-\frac{1}{n+1}t^{-(n+1)})\\ &= -\frac{1}{n+1}\left[\frac{f(t)}{t^{n+1}}\right]_{0^+}^{x^-} + \frac{1}{n+1}\int_0^x \frac{f^{(1)}(t)}{t^{n+1}}dt.\end{aligned}$$

Since

$$\int_0^x \frac{f^{(1)}(t)}{t^{n+1}}dx$$
$$= -\sum_{k=0}^{n-1} \frac{1}{n(n-1)\cdots(n-k)} \left[\frac{f^{(k+1)}(t)}{t^{n-k}}\right]_{0^+}^{x^-} + \frac{1}{n!}\int_0^x \frac{f^{(n+1)}(t)}{t}dt,$$

we have that

$$\int_0^x \frac{f(t)}{t^{n+2}}dt = -\sum_{k=0}^{n} \frac{1}{(n+1)(n+1-1)\cdots(n+1-k)} \left[\frac{f^{(k)}(t)}{t^{n+1-k}}\right]_{0^+}^{x^-}$$
$$+ \frac{1}{(n+1)!}\int_0^x \frac{f^{(n+1)}(t)}{t}dt.$$

This completes the proof. □

LEMMA 3.3. *For $R > 0$, the followings are hold.*

(1) $$\int_0^R \frac{2t + t\cos t - 3\sin t}{t^5}dt$$
$$= \frac{R^{-4}}{24}\left(-16R + (R^2+18)\sin R + (R^3 - 2R)\cos R + R^4\int_0^R \frac{\sin t}{t}dt\right).$$

(2) $$\int_0^R \frac{\sin t - t\cos t}{t^3}dt = \frac{R^{-2}}{2}\left(-\sin R + R\cos R + R^2\int_0^R \frac{\sin t}{t}dt\right).$$

Proof. (1) Let $f(t) = 2t + t\cos t - 3\sin t$. Then, direct calculations give that

$$\left[\frac{f(t)}{t^4}\right]_{0^+}^{R} = R^{-4}(2R + R\cos R - 3\sin R),$$
$$\left[\frac{f^{(1)}(t)}{t^3}\right]_{0^+}^{R} = R^{-3}(2 - 2\cos R - R\sin R),$$
$$\left[\frac{f^{(2)}(t)}{t^2}\right]_{0^+}^{R} = R^{-2}(\sin R - R\cos R),$$
$$\left[\frac{f^{(3)}(t)}{t}\right]_{0^+}^{R} = \sin R.$$

Since $f^{(4)}(t) = \sin t + t\cos t$, by Lemma 3.2, we have that

$$\int_0^R \frac{f(t)}{t^5}dt$$
$$= -\sum_{k=0}^{3} \frac{1}{4\times 3\times\cdots\times(4-k)}\left[\frac{f^{(k)}(t)}{t^{4-k}}\right]_{0^+}^{R^-} + \frac{1}{4!}\int_0^R \frac{f^{(4)}(t)}{t}dt$$
$$= \frac{R^{-4}}{24}\left(-16R + (R^2+18)\sin R + (R^3-2R)\cos R + R^4\int_0^R \frac{\sin t}{t}dt\right).$$

Proof of (2) is similar to (1). □

We introduce the volume formula of geodesic balls in Riemannian manifolds, which is well-known. For example, see [3]. Let M be a Riemannian manifold with a metric g and $p \in M$. Take an orthonormal basis $\{u_1, u_2, \cdots, u_n\}$ of T_pM and let $(x_1, x_2, \cdots, x_n)$ be the coordinates determined by $\{u_1, u_2, \cdots, u_n\}$. This local coordinate system is called the normal coordinate system at p. It is easy to show that $\frac{\partial}{\partial x_i}_m = (d\exp_p)_{\sum_{i=1}^n x_i u_i}(u_i)$ where $m = \exp_p(\sum_{i=1}^n x_i u_i)$. Then, the volume form v_g on U_p is given by

$$v_g = \sqrt{\det\left(g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\right)}dx_1\wedge dx_2\wedge\cdots\wedge dx_n$$

where g_{ij} is the metric coefficients of g in U_p. Therefore, the volume of the geodesic ball $B_p(r)$ is given by

$$Vol(B_p(r)) = \int_{\exp_p^{-1}(B_p(r))} \exp_p^* v_g.$$

Let $\gamma(t)$ be the unit speed geodesic in M with $\gamma(0) = p$, $\gamma'(0) = u_1$ and let $J_i(t)$ be the Jacobi field with $J_i(0) = 0$ and $J_i'(0) = u_i$ for each $i = 2, 3, \cdots, n$. Then we know that

$$(d\exp_p)_{tu_1}u_1 = \gamma'(t)$$

and

$$(d\exp_p)_{tu_1}u_i = \frac{1}{t}J_i(t)$$

for each $i = 2, 3, \cdots, n$. So, we see that

$$\sqrt{\det\left(g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\right)} = t^{-(n-1)}\sqrt{\det(g(J_i(t), J_j(t)))}.$$

Hence, we have that

$$\begin{aligned}\exp_p^* v_g &= t^{-(n-1)}\sqrt{\det(g(J_i(t), J_j(t)))}dx_1dx_2\cdots dx_n\\ &= \sqrt{\det(g(J_i(t), J_j(t)))}dtdu\end{aligned}$$

where du denote the canonical measure of the unit sphere S^{n-1}. Therefore, by Fubini's Theorem we get that

$$Vol(B_p(r)) = \int_{S^{n-1}}\int_0^r \sqrt{\det(g(J_i(t), J_j(t)))}dtdu.$$

THEOREM 3.4. *Let $0 \leq R \leq 2\pi$ and $B_e(R)$ be the geodesic ball with center e and radius R in $\mathbb{H}^3$. Then, the following holds.*

$$\begin{aligned}&Vol(B_e(R))\\ &\quad= \frac{\pi}{6}\{-16R + (R^2+6)\sin R + (R^3+10R)\cos R\\ &\qquad + (R^4+12R^2)\int_0^R \frac{\sin t}{t}dt\}.\end{aligned}$$

Proof. Using Proposition 2.2, we have that

$$\begin{aligned}&\det(< J_i(t), J_j(t) >)\\ &= \det(J_i(t)\cdot J_j(t))\\ &= \det\left(\begin{bmatrix}J_1(t)\\ J_2(t)\end{bmatrix}\begin{bmatrix}J_1(t) & J_2(t)\end{bmatrix}\right)\\ &= \det\left(B(t)\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix}\cdot^t\left(B(t)\begin{bmatrix}e_1(t)\\ e_2(t)\end{bmatrix}\right)\right)\\ &= \det\left(B(t)\cdot^t(B(t))\right)\\ &= \left(\frac{1}{|Z_0|^4}\{2(1-\cos(|Z_0|t)) - (1-|Z_0|^2)|Z_0|t\sin(|Z_0|t)\}\right)^2.\end{aligned}$$

Since

$$\det(B(t)) = \frac{1}{|Z_0|^4}\{2(1-\cos(|Z_0|t)) - (1-|Z_0|^2)|Z_0|t\sin(|Z_0|t)\} \geq 0,$$

we see that

$$\begin{aligned}&\sqrt{\det(< J_i(t), J_j(t) >)}\\ &= \frac{1}{|Z_0|^4}\{2(1-\cos(|Z_0|t)) - (1-|Z_0|^2)|Z_0|t\sin(|Z_0|t)\}.\end{aligned}$$

Let $u = (x_1, x_2, x_3) \in S^2$ and

$$f(x_3, t) = \frac{1}{x_3^4}\{2(1 - \cos(x_3 t)) - (1 - x_3^2)x_3 t \sin(x_3 t)\}.$$

Then, we see that

$$Vol(B_e(R)) = \int_{S^2} \int_0^R f(x_3, t) dt du.$$

Since area element du on the sphere S^2 is given by

$$du = \frac{1}{\sqrt{1 - (x_1^2 + x_2^2)}} dx_1 dx_2,$$

we have that

$$\begin{aligned} &Vol(B_e(R)) \\ &\quad = 2 \int_D \int_0^R f(\sqrt{1 - (x_1^2 + x_2^2)}, t) \frac{1}{\sqrt{1 - (x_1^2 + x_2^2)}} dt dx_1 dx_2, \end{aligned}$$

where

$$D = \{(x_1, x_2) | x_1^2 + x_2^2 \leq 1\}.$$

Changing the coordinates on D to polar coordinates, we have

$$Vol(B_e(R)) = 4\pi \int_0^1 \int_0^R f(\sqrt{1 - r^2}, t) \frac{r}{\sqrt{1 - r^2}} dt dr.$$

Replacing $x = \sqrt{1 - r^2}$, we see that

$$Vol(B_e(R)) = 4\pi \int_0^1 \int_0^R f(x, t) dt dx$$

where

$$f(x, t) = \frac{1}{x^4}\{2(1 - \cos(xt)) - (1 - x^2)x_3 t \sin(xt)\}.$$

Since

$$\int_0^R f(x, t) dt = \frac{2Rx + Rx\cos(Rx) - 3\sin(Rx)}{x^5} + \frac{\sin(Rx) - Rx\cos(Rx)}{x^3},$$

we have that

$$
\begin{aligned}
&Vol(B_e(R)) \\
&= 4\pi \int_0^1 \int_0^R f(x,t)dtdx \\
&= 4\pi \int_0^1 \left(\frac{2Rx + Rx\cos(Rx) - 3\sin(Rx)}{x^5} + \frac{\sin(Rx) - Rx\cos(Rx)}{x^3} \right) dx \\
&= 4\pi \left(R^4 \int_0^R \frac{2t + t\cos t - 3\sin t}{t^5} dt + R^2 \int_0^R \frac{\sin t - t\cos t}{t^3} dt \right).
\end{aligned}
$$

By Lemma 3.3, we see that

$$
\begin{aligned}
Vol(B_e(R)) & \\
&= 4\pi \left(R^4 \int_0^R \frac{2t + t\cos t - 3\sin t}{t^5} dt + R^2 \int_0^R \frac{\sin t - t\cos t}{t^3} dt \right) \\
&= \frac{\pi}{6} \Big\{ -16R + (R^2+6)\sin R + (R^3 + 10R)\cos R \\
&\qquad + (R^4 + 12R^2) \int_0^R \frac{\sin t}{t} dt \Big\}.
\end{aligned}
$$

This completes the proof. $\square$

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