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VORTEX CURVATURE EQUATIONS ON VORTEX SURFACES

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Abstract. The aim of this work is to derive a partial differential equation that explains the movement of vortex lines on a vortex trajectory surface in a three dimensional incompressible inviscid flow.

1. Main interests

This paper deals with the three dimensional incompressible inviscid flows governed by the Euler-vorticity equations;

$$\frac{\partial \omega}{\partial t} = (\omega, \nabla)u - (u, \nabla)\omega = -[u, \omega] = -\mathcal{L}_u \omega, \quad \text{div } u = 0, \quad \omega := \text{curl } u, \tag{1.1}$$

where $(u, \nabla)\omega_k := \sum_{i=1}^{3} u_i \frac{\partial \omega_k}{\partial x_i}$ $(k = 1, 2, 3)$, and the symbol [,] represents the Lie bracket and $\mathcal{L}_u \omega$ is the Lie derivative of the *vorticity* field $\omega = (\omega_1, \omega_2, \omega_3)$ with respect to the *velocity* field $u = (u_1, u_2, u_3)$ of a given fluid flow. Associated with the Euler equations, we have a system of ordinary differential equations

$$\begin{cases} \frac{\partial}{\partial t} X(a, t) = u(X(a, t), t), \\ X(a, 0) = a, \end{cases} \tag{1.2}$$

which defines *particle trajectory flows* $X(a, t)$ along the velocity u, starting from the initial position a.

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By virtue of the Biot-Savart's law, the velocity $u(x,t)$ can be retrieved by the vorticity $\omega(x,t)$ as follows:

$$u(x,t) = \frac{1}{4\pi}\int_{\mathbb{R}^3}\left[\left(\nabla_y \frac{1}{|y|}\right)\times \omega(x-y,t)\right]dy, \tag{1.3}$$

where $\times$ denotes the cross product between 3-D vectors. So the vorticity flow is one of the essential factors for the understanding of the fluid flow. The equation (1.1) illustrates that at time t, the vorticity $\omega(x,t)$ at the position x moves in the opposite direction to the Lie derivative $\mathcal{L}_{u(\cdot,t)}\omega(\cdot,t)$:

$$\mathcal{L}_{u(a,t)}\omega(a,t) := \lim_{h\to 0}\frac{1}{h}\left[(Y_{-h})_*\omega(Y_h(a,t),t) - \omega(a,t)\right],$$

where $\{Y_s\}$ is a one-parameter local group of local diffeomorphisms with respect to the velocity field $u(\cdot,t)$ at time t:

$$\begin{cases} \dfrac{\partial}{\partial s}Y_s(a,t) = u(Y_s(a,t),t), \\ Y_0(a,t) = a, \qquad a\in\mathbb{R}^3 \end{cases} \tag{1.4}$$

and the symbol $(Y_{-h})_*\omega$ represents the push-forward along the flow Y_{-h}: $(Y_{-h})_*\omega(Y_h(a,t),t) = D_{Y_h(a,t)}Y_{-h}(\omega)$. Also, we introduce another one-parameter local group $\{Z_s\}$ of local diffeomorphisms with respect to the vorticity field $\omega(\cdot,t)$ at time t:

$$\begin{cases} \dfrac{\partial}{\partial s}Z_s(a,t) = \omega(Z_s(a,t),t), \\ Z_0(a,t) = a, \qquad a\in\mathbb{R}^3. \end{cases} \tag{1.5}$$

Since, for vector fields, the Lie derivative and the Lie bracket coincide as indicated in (1.1), the temporal movement of the vorticity $\omega(x,\cdot)$ remains constant when the flow $\{Y_s\}$ of the velocity field $u(\cdot,t)$ and the flow $\{Z_s\}$ of the vorticity field $\omega(\cdot,t)$ *commute*.

On the other hand, we can also represent the vorticity equations (1.1) via the material derivative as

$$\frac{D}{Dt}\omega := \frac{\partial\omega}{\partial t} + (u,\nabla)\omega = (\omega,\nabla)u. \tag{1.6}$$

This formulation illustrates that the vortex lines and vortex sheets[1] induced from the vorticity ω move with the Euler flow.

[1]Our definition of vortex sheet is a surface that is tangent to the vorticity vector at each of its points.

We choose $a \in \mathbb{R}^3$ with $\omega(Z_s(a,0),0) \neq 0$ for all $-1 < s < 1$, and select a vortex line

$$C = \{Z_s(a,0) \in \mathbb{R}^3 : -1 < s < 1\}.^2$$

Then we employ a parametric surface S as

$$S := \{X(Z_s(a,0),t) : -1 < s < 1,\ 0 \leq t < T^*\},$$

where $T^* \in (0,\infty]$ is the first blow-up time for the Euler flow. The surface S permits self-intersections and non-orientablity[3].

Let $\alpha(s,t) := X(Z_s(a,0),t)$. Then from the equation (1.2), we get

$$\frac{\partial}{\partial t}\alpha(s,t) = u(\alpha(s,t),t) \tag{1.7}$$

and we also notice that

$$\frac{\partial}{\partial s}\,\alpha(s,t) = \omega(\alpha(s,t),t). \tag{1.8}$$

In fact, from the well-known vorticity transport formula

$$\omega(X(a,t),t) = \nabla_a X(a,t)\omega(a,0),$$

we have

$$\begin{aligned}\frac{\partial}{\partial s}\,\alpha(s,t) &= \nabla_a X(Z_s(a,0),t)\frac{\partial}{\partial s}Z_s(a,0)\\ &= \nabla_a X(Z_s(a,0),t)\,\omega(Z_s(a,0),0)\\ &= \omega(X(Z_s(a,0),t),t) = \omega(\alpha(s,t),t).\end{aligned}$$

The notations $' \equiv \frac{\partial}{\partial s}$ and $\dot{} \equiv \frac{\partial}{\partial t}$ will be used throughout the paper. Also, all flows are assumed to possess enough temporal and spacial regularities.

2. An evolution equation for geodesic curvature

For each t, the regular curve $\alpha(t,\cdot)$ can be reparametrized in a way that it has unit speed, and s^t represents its arc-length parameter with respect to the variable s at time t. Let $\mathbf{T}(s,t)$ be the unit tangent vector to the curve $\alpha(\cdot,t)$ and define the unit vector $\mathbf{U}(s,t)$ as

$$\mathbf{U}(s,t) := \frac{\omega(s,t) \times u(s,t)}{|\omega(s,t) \times u(s,t)|}.$$

[2]The choice of the interval $(-1,1)$ simply means it contains the zero point for convenience. The interval can be chosen to be the whole real line $\mathbb{R}$.

[3]The surface S is called a *vortex (trajectory) surface.*

The vector $\mathbf{U}(s,t)$ is orthogonal to the tangent plane T_pS at $p=\alpha(s,t)$. Finally, we take $\mathbf{N} := \mathbf{U}\times\mathbf{T}$. Then $\{\mathbf{T},\mathbf{N},\mathbf{U}\}$ constitutes a frame field on $\mathbb{R}^3$ and for a fixed t, we have

$$\frac{D}{ds^t}\mathbf{T} = \kappa_g\mathbf{N} = \frac{1}{|\alpha'|}\frac{D}{ds}\mathbf{T}, \tag{2.1}$$

$$\frac{D}{ds^t}\mathbf{N} = -\kappa_g\mathbf{T} = \frac{1}{|\alpha'|}\frac{D}{ds}\mathbf{N}. \tag{2.2}$$

On the other hand, since each vector $u(\alpha,t)$ is on the tangent space T_pS $(p=\alpha(s,t))$, the Euler equation (1.7) can be written as

$$\frac{\partial}{\partial t}\alpha = u(\alpha,t) := \eta\mathbf{N}+\zeta\mathbf{T} \tag{2.3}$$

for some scalar functions η and ζ. We note that

$$\zeta(s,t) = \frac{1}{|\alpha'|} < \omega(\alpha(s,t),t), u(\alpha(s,t),t) >_p .$$

We now state our main theorem:

Theorem 2.1. *The vorticity on a vortex trajectory surface S for the Euler flow (1.1) is represented by an indefinite integral of the form:*

$$|\omega(\alpha(s,t),t)| = \int^s e^{-\int_r^s \kappa_g\eta\, d\bar{r}}\,\zeta'(r,t)\,dr. \tag{2.4}$$

The geodesic curvature κ_g and the coordinate functions ζ, η in (2.4) satisfy a nonlinear evolution equation with respect to κ_g:

$$\dot{\kappa_g} - \triangle_{s^t}\eta - \zeta\,\partial_{s^t}\kappa_g + (K-\kappa_g^2)\,\eta = 0,$$

where we set $\frac{\partial^2}{\partial(s^t)^2} := \triangle_{s^t}$, $\frac{1}{|\alpha'|}\frac{\partial}{\partial s} := \partial_{s^t}$ and K represents the Gaussian curvature on the surface S.

Proof. From the fact that

$$\begin{aligned}\frac{\partial}{\partial s}|\alpha'| = \frac{1}{|\alpha'|} < \alpha', \frac{D}{ds}\alpha' >_p &=< \mathbf{T}, \frac{D}{ds}(\zeta\mathbf{T}+\eta\mathbf{N}) >_p \\ &=< \mathbf{T}, (\zeta' - |\alpha'|\kappa_g\eta)\mathbf{T} >_p \\ &= \zeta' - |\alpha'|\kappa_g\eta,\end{aligned}$$

we get the first order linear ODE with respect to $|\alpha'(\cdot,t)|$ for any t, and its solutions can be represented by

$$|\alpha'(s,t)| = |\omega(\alpha(\cdot,t),t)| = \int^s e^{-\int_r^s \kappa_g\eta\, d\bar{r}}\,\zeta'(r,t)\,dr.$$

Next, we have

$$\frac{D}{ds}\left(\frac{\partial \alpha}{\partial t}\right) = \frac{D}{ds}(\eta \mathbf{N} + \zeta \mathbf{T}) = (\eta' + \kappa_g \zeta |\alpha'|)\mathbf{N} + (\zeta' - \kappa_g\, \eta\, |\alpha'|)\mathbf{T}.$$

From the symmetric condition $\frac{D}{ds}\frac{\partial\alpha}{\partial t} = \frac{D}{dt}\frac{\partial \alpha}{\partial s}$ together with the fact that

$$\frac{D}{dt}\left(\frac{\partial \alpha}{\partial s}\right) = \frac{D}{dt}(|\alpha'|\mathbf{T}) = |\dot{\alpha'}|\mathbf{T} + |\alpha'|\frac{D}{dt}\mathbf{T},$$

we have

$$\frac{D}{dt}\mathbf{T} = \frac{1}{|\alpha'|}\left[(\zeta' - \kappa_g \eta |\alpha'| - |\dot{\alpha'}|)\mathbf{T} + (\eta' + \kappa_g \zeta |\alpha'|)\,\mathbf{N}\right]. \tag{2.5}$$

We now consider $\frac{D}{dt}\mathbf{N}$. For it, we can display it as

$$\frac{D}{dt}\mathbf{N} =< \frac{D}{dt}\mathbf{N}, \mathbf{T} >_p \mathbf{T} + < \frac{D}{dt}\mathbf{N}, \mathbf{N} >_p \mathbf{N},$$

where $< \cdot,\ \cdot >_p$ is the inner product on the tangent plane T_pS. It is obvious to have $< \frac{D}{dt}\mathbf{N}, \mathbf{N} >_p = 0$, and we can also see that

$$\begin{aligned} &< \frac{D}{dt}\mathbf{N}, \mathbf{T} >_p \\ &\quad = - < \mathbf{N}, \frac{D}{dt}\mathbf{T} >_p \\ &\quad = - < \mathbf{N}, \frac{1}{|\alpha'|}\left[(\zeta' - \kappa_g \eta |\alpha'| - |\dot{\alpha'}|)\mathbf{T} + (\eta' + \kappa_g \zeta |\alpha'|)\,\mathbf{N}\right] >_p \\ &\quad = -\kappa_g \zeta - \frac{\eta'}{|\alpha'|}. \end{aligned}$$

From this we obtain

$$\frac{D}{dt}\mathbf{N} = -\left(\kappa_g \zeta + \frac{\eta'}{|\alpha'|}\right)\mathbf{T}. \tag{2.6}$$

Now we will make a use of the following identity:

$$\frac{D}{dt}\frac{D}{ds}\mathbf{N} - \frac{D}{ds}\frac{D}{dt}\mathbf{N} = K\left(\frac{\partial \alpha}{\partial s} \times \frac{\partial \alpha}{\partial t}\right) \times \mathbf{N}, \tag{2.7}$$

where K is the Gaussian curvature on the surface S. We will start with the computations of $\frac{D}{dt}\frac{D}{ds}\mathbf{N}$ and $\frac{D}{ds}\frac{D}{dt}\mathbf{N}$. From the equation (2.1) together with (2.5), we have

$$\begin{aligned} \frac{D}{dt}\frac{D}{ds}\mathbf{N} &= -\left(\dot{\kappa_g}|\alpha'| + \kappa_g |\dot{\alpha'}|\right)\mathbf{T} - \kappa_g |\alpha'| \frac{D}{dt}\mathbf{T} \\ &= -\left(\dot{\kappa_g}|\alpha'| - \kappa_g^2 \eta |\alpha'| + k_g \zeta'\right)\mathbf{T} - \kappa_g(\eta' + \kappa_g \zeta |\alpha'|)\mathbf{N}. \end{aligned} \tag{2.8}$$

The equation (2.6) together with (2.2) yields

$$\frac{D}{ds}\left(\frac{D}{dt}\mathbf{N}\right) = -\left(\kappa_g\zeta + \frac{\eta'}{|\alpha'|}\right)'\mathbf{T} - \left(\kappa_g\zeta + \frac{\eta'}{|\alpha'|}\right)\frac{D}{ds}\mathbf{T}$$

$$= -\left(\kappa_g\zeta + \frac{\eta'}{|\alpha'|}\right)'\mathbf{T} - \kappa_g(\eta' + \kappa_g\zeta|\alpha'|)\mathbf{N}. \tag{2.9}$$

The facts that $\frac{\partial\alpha}{\partial s} = \frac{ds^t}{ds}\frac{d\alpha}{ds^t} = |\alpha'|\mathbf{T}$ and $\frac{\partial\alpha}{\partial t} = \eta\mathbf{N} + \zeta\mathbf{T}$ lead to

$$\left(\frac{\partial\alpha}{\partial s} \times \frac{\partial\alpha}{\partial t}\right) \times \mathbf{N} = -|\alpha'|\eta\mathbf{T}. \tag{2.10}$$

Taking the equations (2.7) through (2.10) together, we derive the nonlinear evolution equation

$$\dot{\kappa_g} - \frac{1}{|\alpha'|}\left(\frac{\eta'}{|\alpha'|}\right)' - \frac{\kappa_g'}{|\alpha'|}\zeta - \kappa_g^2\eta + K\eta = 0. \tag{2.11}$$

With the help of the notation $\frac{\partial^2}{\partial(s^t)^2} \equiv \triangle_{s^t}$ and $\frac{1}{|\alpha'|}\frac{\partial}{\partial s} = \partial_{s^t}$, (2.11) can be rewritten as

$$\dot{\kappa_g} - \triangle_{s^t}\eta - (\partial_{s^t}\kappa_g)\zeta - \kappa_g^2\eta + K\eta = 0.$$

The proof is now completed. □

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