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# BIHARMONIC SPACELIKE CURVES IN LORENTZIAN HEISENBERG SPACE

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Abstract. In this paper, we show that proper biharmonic spacelike curve $\gamma$ in Lorentzian Heisenberg space $(\mathbb{H}_3, g)$ is pseudo-helix with $\kappa^2 - \tau^2 = -1 + 4\eta(B)^2$. Moreover, $\gamma$ has the spacelike normal vector field and is a slant curve. Finally, we find the parametric equations of them.

## 1. Introduction

J. Eells and J. H. Sampson ([6]) defined harmonic and biharmonic map between Riemannian manifolds. G. Y. Jiang ([9] and [10]) derived the first variation formula of the bienergy from the Euler-Lagrange equation. Harmonic maps are clearly biharmonic. Non-harmonic biharmonic maps are called *proper* biharmonic maps. B. Y. Chen and S. Ishikawa [3] showed nonexistence of proper biharmonic curves in Euclidean 3-space $\mathbb{E}^3$. Moreover they classified all proper biharmonic curves in Minkowski 3-space $\mathbb{E}^3_1$ (See [8]). Recently, T. Sasahara ([12]) introduced biharmonic maps between pseudo-Riemannian manifolds and studied proper biharmonic submanifolds in Lorentzian 3-space forms.

A *contact manifold* $(M, \eta)$ is a smooth manifold $M^{2n+1}$ together with a global differential one-form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on $M$. In [2] G. Calvaruso found relationship between Riemannian and Lorentzian metrics associated to the same contact structure. Given contact structure $(M^{2n+1}, \eta)$, there is a one-to-one correspondence between the two associated structure by the relation

$$g = \widetilde{g} - 2\eta \otimes \eta,$$

where $g$ and $\widetilde{g}$ are the Lorentzian and Riemannian metric. $(M^{2n+1}, \eta, \xi, \varphi, g)$ is a contact Lorentzian manifold with $\xi$ timelike, and the structure is Sasakian if and only if the corresponding Riemannian structure is Sasakian.

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As a generalization of Legendre curve, the notion of slant curves was introduced in [4]. A curve in a contact 3-manifold is said to be *slant* if its tangent vector field has constant angle with the Reeb vector field. In [5], we found that biharmonic curves in 3-dimensional Sasakian space forms are slant helices.

In this paper, we study biharmonic curves in 3-dimensional Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. In Section 3 we show that proper biharmonic spacelike curve $\gamma$ in Lorentzian Heisenberg space $(\mathbb{H}_3, g)$ is pseudo-helix with $\kappa^2 - \tau^2 = -1 + 4\eta(B)^2$. Moreover, $\gamma$ has the spacelike normal vector field and is a slant curve. Finally, we find the parametric equations of them.

## 2. Preliminaries

### 2.1. Contact Lorentzian manifold

Let $M$ be a $(2n+1)$-dimensional differentiable manifold. $M$ has an almost contact structure $(\varphi, \xi, \eta)$ if it admits a $(1,1)$-tensor field $\varphi$, a vector field $\xi$ and a 1-form $\eta$ satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1. \tag{1}$$

Suppose $M$ has an almost contact structure $(\varphi, \xi, \eta)$. Then $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. Moreover, the endomorphism $\varphi$ has rank $2n$.

If a $(2n+1)$-dimensional smooth manifold $M$ with almost contact structure $(\varphi, \xi, \eta)$ admits a compatible Lorentzian metric such that

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y),$$

then we say $M$ has an almost contact Lorentzian structure $(\eta, \xi, \varphi, g)$. Setting $Y = \xi$ we have

$$\eta(X) = -g(X, \xi).$$

Next, if the compatible Lorentzian metric $g$ satisfies

$$d\eta(X, Y) = g(X, \varphi Y),$$

then $\eta$ is a contact form on $M$, $\xi$ the associated Reeb vector field, $g$ an associated metric and $(M, \varphi, \xi, \eta, g)$ is called a *contact Lorentzian manifold.*

For a contact Lorentzian manifold $M$, one may define naturally an almost complex structure $J$ on $M \times \mathbb{R}$ by

$$J(X, f\frac{\mathrm{d}}{\mathrm{d}t}) = (\varphi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}),$$

where $X$ is a vector field tangent to $M$, $t$ the coordinate of $\mathbb{R}$ and $f$ a function on $M \times \mathbb{R}$. If the almost complex structure $J$ is integrable, then the contact Lorentzian manifold $M$ is said to be *normal* or *Sasakian.* It is known that a contact Lorentzian manifold $M$ is normal if and only if $M$ satisfies

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0,$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of $\varphi$.

**Proposition 2.1** ([2]). *An almost contact Lorentzian manifold* $(M^{2n+1}, \eta, \xi, \varphi, g)$ *is Sasakian if and only if*

$$(\nabla_X \varphi)Y = g(X,Y)\xi + \eta(Y)X.$$

Using the similar arguments and computations in [1] we obtain:

**Proposition 2.2** ([2]). *Let* $(M^{2n+1}, \eta, \xi, \varphi, g)$ *be a contact Lorentzian manifold. Then*

$$\nabla_X \xi = \varphi X - \varphi h X, \quad h = \frac{1}{2} L_\xi \varphi.$$

If $\xi$ is a killing vector field with respect to the Lorentzian metric $g$, then we have

$$\nabla_X \xi = \varphi X.$$

### 2.2. Frenet-Serret equations

Let $\gamma : I \to M^3$ be a unit speed curve in Lorentzian 3-manifolds $M^3$ such that $\gamma'$ satisfies $g(\gamma', \gamma') = \varepsilon_1 = \pm 1$. The constant $\varepsilon_1$ is called the *causal character* of $\gamma$. A unit speed curve $\gamma$ is said to be a spacelike or timelike if its causal character is 1 or $-1$, respectively. A unit speed curve $\gamma$ is said to be a *Frenet curve* if $g(\gamma'', \gamma'') \neq 0$. A Frenet curve $\gamma$ admits a orthonormal frame field $\{T = \gamma', N, B\}$ along $\gamma$. Then the *Frenet-Serret* equations are following ([7], [8]):

$$\begin{cases} \nabla_{\gamma'} T = \qquad \varepsilon_2 \kappa N, \\ \nabla_{\gamma'} N = -\varepsilon_1 \kappa T \quad + \varepsilon_3 \tau B, \\ \nabla_{\gamma'} B = \qquad -\varepsilon_2 \tau N, \end{cases} \tag{2}$$

where $\kappa = |\nabla_{\gamma'} \gamma'|$ is the *geodesic curvature* of $\gamma$ and $\tau$ its *geodesic torsion*. The vector fields $T$, $N$ and $B$ are called tangent vector field, principal normal vector field, and binormal vector field of $\gamma$, respectively.

The constant $\varepsilon_2$ and $\varepsilon_3$ defined by $g(N,N) = \varepsilon_2$ and $g(B,B) = \varepsilon_3$, and called *second causal character* and *third causal character* of $\gamma$, respectively. Thus it satisfied $\varepsilon_1 \varepsilon_2 = -\varepsilon_3$.

A Frenet curve $\gamma$ is a *geodesic* if and only if $\kappa = 0$. A Frenet curve $\gamma$ with constant geodesic curvature and zero geodesic torsion is called a *pseudo-circle*. A *pseudo-helix* is a Frenet curve $\gamma$ whose geodesic curvature and torsion are constants.

**Proposition 2.3.** *Let* $\{T, N, B\}$ *are orthonomal Frame field in a Lorentzian 3-manifold. Then*

$$T \wedge_L N = \varepsilon_3 B, \quad N \wedge_L B = \varepsilon_1 T, \quad B \wedge_L T = \varepsilon_2 N.$$

### 2.3. Biharmonic curve

The harmonic maps $\phi : (M^m, g) \to (N^n, h)$ between two pseudo-Riemannian manifolds as critical points of the energy $E(\phi) = \int_M |d\phi|^2 dv$. The *tension field* $\tau_\phi$ is defined by

$$\tau_\phi = trace\nabla^\phi d\phi = \Sigma_{i=1}^m \varepsilon_i (\nabla^\phi_{e_i} d\phi(e_i) - d\phi(\nabla_{e_i} e_i)),$$

where $\nabla^\phi$ and $\{e_i\}$ denote the induced connection by $\phi$ on the bundle $\phi^* TN^n$. A smooth map $\phi$ is called a *harmonic map* if its tension field vanishes.

Next, the bienergy $E_2(\phi)$ of a map $\phi$ is defined by $E_2(\phi) = \int_M |\tau_\phi|^2 dv$, and say that $\phi$ is biharmonic if it is a critical point of the bienergy. Harmonic maps are clearly biharmonic. Non-harmonic biharmonic maps are called *proper* biharmonic maps. We define the *bitension field* $\tau_2(\phi)$ by

$$\tau_2(\phi) := \Sigma_{i=1}^m \varepsilon_i ((\nabla^\phi_{e_i} \nabla^\phi_{e_i} - \nabla^\phi_{\nabla_{e_i} e_i}) \tau_\phi - R^N(\tau_\phi, d\phi(e_i)) d\phi(e_i)),$$

where $R^N$ is the curvature tensor of $N^n$ and defined by $R^N(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ (see [12]).

We now restrict our attention to isometric immersions $\gamma : I \to (M, g)$ from an interval $I$ to a pseudo-Riemannian manifold. The image $C = \gamma(I)$ is the trace of a curve in $M$ and $\gamma$ is a parametrization of $C$ by arc length. In this case the tension field becomes $\tau_\gamma = \varepsilon_1 \nabla_{\gamma'} \gamma'$ and the biharmonic equation reduces to

$$\tau_2(\gamma) = \varepsilon_1 (\nabla^2_{\gamma'} \tau_\gamma - R(\tau_\gamma, \gamma')\gamma') = 0. \quad (3)$$

Note that $C = \gamma(I)$ is part of a geodesic of $M$ if and only if $\gamma$ is harmonic. Moreover, from the biharmonic equation if $\gamma$ is harmonic, thus geodesics are a subclass of biharmonic curves.

For a $n$-dimensional Lorentzian space forms of constant curvature $k$ by $M_1^n(k)$. The curvature tensor $R$ of $M_1^n(k)$ is given by

$$R(X,Y)Z = k(g(Z,X)Y - g(Z,Y)X),$$

where $g$ is the Lorentzian metric tensor of $M_1^n(k)$ (see [11, p. 80]). Hence we have:

**Proposition 2.4** ([12]). *Let $\gamma : I \to M_1^3(k)$ be a Frenet curve. Then $\gamma$ is proper biharmonic if and only if $\gamma$ is a helix with*

$$k = -\varepsilon_3(\varepsilon_1 \kappa^2 + \varepsilon_3 \tau^2), \quad \kappa \neq 0.$$

## 3. Biharmonic spacelike curves in $(\mathbb{H}_3, g)$

The Heisenberg group $\mathbb{H}_3$ is a Lie group which is diffeomorphic to $\mathbb{R}^3$ and the group operation is defined by

$$(x, y, z) * (\overline{x}, \overline{y}, \overline{z}) = (x + \overline{x}, y + \overline{y}, z + \overline{z} + \frac{x\overline{y}}{2} - \frac{\overline{x}y}{2}).$$

The mapping

$$\mathbb{H}_3 \to \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\} : (x, y, z) \mapsto \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

is an isomorphism between $\mathbb{H}_3$ and a subgroup of $GL(3, \mathbb{R})$.

Now, we take the contact form

$$\eta = dz + (ydx - xdy).$$

Then the characteristic vector field of $\eta$ is $\xi = \frac{\partial}{\partial z}$.

Now, we equip the Lorentzian metric as following:

$$g = dx^2 + dy^2 - (dz + (ydx - xdy))^2 .$$

We take a left-invariant Lorentzian orthonormal frame field $(e_1, e_2, e_3)$ on $(\mathbb{H}_3, g)$:

$$e_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ e_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial z},$$

and the commutative relations are derived as follows:

$$[e_1, e_2] = 2e_3,\ [e_2, e_3] = [e_3, e_1] = 0.$$

Then the endomorphism field $\varphi$ is defined by

$$\varphi e_1 = e_2,\ \varphi e_2 = -e_1,\ \varphi e_3 = 0.$$

The Levi-Civita connection $\nabla$ of $(\mathbb{H}_3, g)$ is described as

$$\begin{aligned} &\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0, && \nabla_{e_1} e_2 = e_3 = -\nabla_{e_2} e_1, \\ &\nabla_{e_2} e_3 = -e_1 = \nabla_{e_3} e_2, && \nabla_{e_3} e_1 = e_2 = \nabla_{e_1} e_3. \end{aligned} \tag{4}$$

The contact form $\eta$ satisfies $d\eta(X, Y) = g(X, \varphi Y)$. Moreover the structure $(\eta, \xi, \varphi, g)$ is Sasakian. The Riemannian curvature tensor $R$ of $(\mathbb{H}_3, g)$ is given by

$$\begin{aligned} &R(e_1, e_2)e_1 = 3e_2, && R(e_1, e_2)e_2 = -3e_1, \\ &R(e_2, e_3)e_2 = -e_3, && R(e_2, e_3)e_3 = -e_2, \\ &R(e_3, e_1)e_3 = e_1, && R(e_3, e_1)e_1 = e_3, \end{aligned} \tag{5}$$

the others are zero.

The sectional curvature is given by ([2])

$$K(\xi, e_i) = -R(\xi, e_i, \xi, e_i) = -1 \quad \text{for } i = 1, 2,$$

and

$$K(e_1, e_2) = R(e_1, e_2, e_1, e_2) = 3.$$

Hence Lorentzian Heisenberg space $(\mathbb{H}_3, g)$ is the Lorentzian Sasakian space forms with constant holomorphic sectional curvature $\mu = 3$.

The tension field $\tau_\gamma = \varepsilon_1 \nabla_{\gamma'} \gamma'$ and from the Frenet-Serret equation (2), $\nabla_{\gamma'} \gamma' = 0$ if and only if $\kappa = 0$, hence we have:

**Proposition 3.1.** *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a Frenet curve in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. Then $\gamma$ is harmonic if and only if $\gamma$ is a geodesic.*

Next, using (2) we get

$$\nabla_T^3 T = 3\varepsilon_3\kappa\kappa' T + \varepsilon_2(\kappa'' - \varepsilon_2\kappa(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2))N - \varepsilon_1(2\kappa'\tau + \kappa\tau')B.$$

Let $\gamma : I \to (\mathbb{H}_3, g)$ be a curve parametrized by arc-length with the Frenet frame field $(T, N, B)$. Expand $T, N, B$ as $T = T_1e_1 + T_2e_2 + T_3e_3$, $N = N_1e_1 + N_2e_2 + N_3e_3$, $B = B_1e_1 + B_2e_2 + B_3e_3$ with respect to the pseudo-orthonormal basis $\{e_1, e_2, e_3 = \xi\}$ with timelike $\xi$. From Proposition 2.3 we see that $\varepsilon_3 B = T \wedge_L N$, that is,

$$\varepsilon_3B_1 = T_2N_3 - T_3N_2, \varepsilon_3B_2 = T_3N_1 - T_1N_3, \varepsilon_3B_3 = T_2N_1 - T_1N_2. \tag{6}$$

Moreover, using the Riemannian curvature tensor (5) we have

$$\begin{aligned} &R(\kappa N, T)T \\ &= \kappa R(N_1e_1 + N_2e_2 + N_3e_3, T_1e_1 + T_2e_2 + T_3e_3)(T_1e_1 + T_2e_2 + T_3e_3) \\ &= -\varepsilon_2\kappa\big[(\varepsilon_3 + 4B_3^2)N - (4N_3B_3)B\big]. \end{aligned}$$

From the biharmonic equation (3) we have

$$\begin{aligned} \tau_2(\gamma) &= \nabla^3{}_T T - \varepsilon_2 R(\kappa N, T)T \\ &= 3\varepsilon_3\kappa\kappa' T + \big[\varepsilon_2(\kappa'' - \varepsilon_2\kappa(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2)) + \kappa(\varepsilon_3 + 4B_3^2)\big]N \\ &\quad + \big[-\varepsilon_1(2\kappa'\tau + \kappa\tau)' - 4\kappa N_3B_3\big]B \\ &= 0. \end{aligned}$$

Hence we have:

**Proposition 3.2.** *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a Frenet curve parametrized by arc-length in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. Then $\gamma$ is a proper biharmonic curve if and only if*

$$\begin{aligned} \kappa &= \textit{constant} \neq 0, \\ \varepsilon_1\kappa^2 + \varepsilon_3\tau^2 &= \varepsilon_3 + 4\eta(B)^2, \\ \tau' &= -4\varepsilon_1\eta(N)\eta(B). \end{aligned} \tag{7}$$

### 3.1. Biharmonic spacelike curves

In this subsection, we study a spacelike curve such that biharmonic equation (3) in Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. We fix $\varepsilon_1 = 1$ then we have:

**Corollary 3.3.** *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a spacelike curve in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. Then $\gamma$ is proper biharmonic if and only if $\gamma$ is a pseudo-helix with*

$$\kappa^2 + \varepsilon_3\tau^2 = \varepsilon_3 + 4\eta(B)^2, \quad \eta(N)\eta(B) = 0, \quad \kappa \neq 0. \tag{8}$$

*Proof.* By using (4) we get first

$$\nabla_T T = (T_1' - 2T_2T_3)e_1 + (T_2' + 2T_1T_3)e_2 + (T_3')e_3.$$

By using the 1st Frenet-Serret equation, it follows that

$$\varepsilon_2 \kappa N = (T_1' - 2T_2T_3)e_1 + (T_2' + 2T_1T_3)e_2 + (T_3')e_3. \tag{9}$$

From this, we obtain $T_3' = \varepsilon_2\kappa N_3$. Here we may put $T_3(s) = \kappa F(s)$ and $f(s) = F^{'}(s)$. Then we get $N_3(s) = \varepsilon_2 f(s)$. We may also write

$$T = \sqrt{1+\kappa^2F^2}\cos\beta(s)e_1 + \sqrt{1+\kappa^2F^2}\sin\beta(s)e_2 + \kappa F(s)e_3.$$

Then (9) is rewritten as

$$\begin{aligned}\varepsilon_2\kappa N = &\Big\{-\big(2\kappa F(s)+\beta'(s)\big)\sqrt{1+\kappa^2F^2}\sin\beta(s) + \frac{\kappa^2 F f}{\sqrt{1+\kappa^2F^2}}\cos\beta(s)\Big\}e_1\\ &+\Big\{\big(2\kappa F(s)+\beta'(s)\big)\sqrt{1+\kappa^2F^2}\cos\beta(s) + \frac{\kappa^2 F f}{\sqrt{1+\kappa^2F^2}}\sin\beta(s)\Big\}e_2\\ &+\kappa f(s)e_3.\end{aligned} \tag{10}$$

Since $g(\varepsilon_2\kappa N, \varepsilon_2\kappa N) = \varepsilon_2\kappa^2$, we have

$$2\kappa F + \beta' = \pm\kappa\frac{\sqrt{\varepsilon_2 + f^2 + \varepsilon_2\kappa^2F^2}}{1+\kappa^2F^2}.$$

If we replace $2\kappa F + \beta'$ in (10), then we obtain

$$\begin{aligned}\varepsilon_2 N = &\Big(\mp\frac{\sqrt{\varepsilon_2 + f^2 + \varepsilon_2\kappa^2F^2}}{\sqrt{1+\kappa^2F^2}}\sin\beta(s) + \frac{\kappa F f}{\sqrt{1+\kappa^2F^2}}\cos\beta(s)\Big)e_1\\ &+\Big(\pm\frac{\sqrt{\varepsilon_2 + f^2 + \varepsilon_2\kappa^2F^2}}{\sqrt{1+\kappa^2F^2}}\cos\beta(s) + \frac{\kappa F f}{\sqrt{1+\kappa^2F^2}}\sin\beta(s)\Big)e_2 + f(s)e_3.\end{aligned}$$

As $\varepsilon_3 B = T\wedge_L N$, we have $\varepsilon_3 B_3 = -T_1N_2 + N_1T_2 = \mp\varepsilon_2\sqrt{\varepsilon_2 + f^2 + \varepsilon_2\kappa^2F^2}$. The second Frenet-Serret equation gives

$$g(\nabla_T N, e_3) = g(-\kappa T + \varepsilon_3\tau B, e_3) = \kappa T_3 - \varepsilon_3\tau B_3. \tag{11}$$

On the other hand, we have

$$\begin{aligned}g(\nabla_T N, e_3) &= g\Big(\nabla_T(N_1e_1 + N_2e_2 + N_3e_3), e_3\Big)\\ &= g\Big((N_1^{'} - T_3N_2 - T_2N_3)e_1 + (N_2^{'} + T_3N_1 + T_1N_3)e_2\\ &\qquad + (N_3^{'} - T_2N_1 + T_1N_2)e_3, e_3\Big)\\ &= -N_3^{'} + \varepsilon_3 B_3.\end{aligned} \tag{12}$$

Comparing (11) with (12), we obtain

$$N_3^{'} - \varepsilon_3 B_3 = -\kappa T_3 + \varepsilon_3\tau B_3. \tag{13}$$

Next, we replace $N_3 = \varepsilon_2 f$, $B_3 = \pm\sqrt{\varepsilon_2 + f^2 + \varepsilon_2\kappa^2 F^2}$ and $T_3 = \kappa F$ in (13), then we get

$$\tau = \pm\varepsilon_3\varepsilon_2 \frac{f^{'} + \varepsilon_2\kappa^2 F}{\sqrt{\varepsilon_2 + f^2 + \varepsilon_2\kappa^2 F^2}} - 1 = \varepsilon_3 \frac{B_3^{'}}{N_3} - 1. \tag{14}$$

We now assume that $\gamma$ is biharmonic and suppose that $\tau^{'} = -4B_3N_3 \neq 0$ and by using (14),

$$\tau\tau' = -4\varepsilon_3 B_3 B_3' + 4N_3B_3 = -4\varepsilon_3 B_3 B_3' - \tau'.$$

Hence we obtain

$$(\tau + 1)^2 = -4\varepsilon_3 B_3^2 + a, \tag{15}$$

where $a$ is a constant. From the second equation in (7)

$$\varepsilon_3 + 4B_3^2 = \kappa^2 + \varepsilon_3\tau^2. \tag{16}$$

Using (16), since $\kappa$ is a constant, the equation (15) becomes

$$\tau^2 + \tau = b,$$

where $b$ is a constant, and hence $\tau$ is also constant, which yields a contradiction. □

Therefore we have:

**Theorem 3.4.** *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a spacelike curve in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. Then $\gamma$ is proper biharmonic if and only if $\gamma$ is a pseudo-helix with*

$$\kappa^2 - \tau^2 = -1 + 4\eta(B)^2, \quad \eta(N) = 0, \quad \eta(B) = \textit{constant}, \quad \kappa \neq 0. \tag{17}$$

*Proof.* Let $\gamma$ be a spacelike curve in the Lorentzian Heisenberg space $(\mathbb{H}_3, g)$. Then we write

$$T = \cos\beta\cosh\alpha e_1 + \sin\beta\cosh\alpha e_2 + \sinh\alpha e_3,$$

where $\alpha = \alpha(s)$, $\beta = \beta(s)$. Using the Frenet-Serret equation (2) and (4), we get

$$\begin{aligned}\varepsilon_2\kappa N = {} & (\alpha'\cos\beta\sinh\alpha - \sin\beta\cosh\alpha(\beta' + 2\sinh\alpha))e_1 \\ & + (\alpha'\sin\beta\sinh\alpha + \cos\beta\cosh\alpha(\beta' + 2\sinh\alpha))e_2 \\ & + (\alpha'\cosh\alpha)e_3.\end{aligned}$$

Next, we compute

$$\varepsilon_3 B_3 = -T_1N_2 + T_2N_1 = -\frac{\varepsilon_2}{\kappa}(\beta' + 2\sinh\alpha)\cosh^2\alpha.$$

We suppose that $B_3 = 0$ then since $\cosh^2\alpha$ is non-zero, we have to $\beta' + 2\sinh\alpha = 0$. Without loss of generality we assume that $\kappa =\mid \nabla_T T \mid_L = \alpha' > 0$ then we have

$$N = -\cos\beta\sinh\alpha e_1 - \sin\beta\sinh\alpha e_2 - \cosh\alpha e_3.$$

This normal vector field is timelike. Moreover, the binormal vector field is spacelike as

$$B = -\sin\beta e_1 + \cos\beta e_2.$$

Differentiating of $N$ along $\gamma$, we get

$$\begin{aligned}\nabla_T N = &-(\alpha'\cos\beta\cosh\alpha - \sin\beta)e_1 \\ &-(\alpha'\sin\beta\cosh\alpha + \cos\beta)e_2 - (\alpha'\sinh\alpha)e_3.\end{aligned}$$

Using the second Frenet-Serret equation, since $\varepsilon_3 = g(B, B) = 1$, we have

$$\tau = \varepsilon_3\tau = g(\nabla_T N, B) = -1.$$

Hence from (8), $\kappa = 0$ and $\gamma$ is not proper biharmonic. □

### 3.2. Slant curves

A one-dimensional integral submanifold of $D$ in 3-dimensional contact manifold is called a *Legendre curve*, especially to avoid confusion with an integral curve of the vector field $\xi$. As a generalization of Legendre curve, the notion of slant curves was introduced in [4] for a contact Riemannian 3-manifold, that is, a curve in a contact 3-manifold is said to be *slant* if its tangent vector field has constant angle with the Reeb vector field. The *contact angle* $\theta(s)$ is a function defined by $\cos\theta(s) = g(\gamma'(s), \xi)$.

Similarly as in the contact Riemnnian 3-manifolds, a curve in a contact Lorentzian 3-manifold is said to be *slant* if its tangent vector field has constant angle with the Reeb vector field (i.e., $g(\gamma', \xi)$ is a constant). In particular, if $g(\gamma', \xi) = 0$, then $\gamma$ is a Legendre curve.

Let $\gamma$ be a non-geodesic spacelike curve in a Sasakian Lorentzian 3-manifold $M^3$. Differentiating $g(\gamma', \xi) = -\sinh\alpha$, then

$$-\varepsilon_2\kappa\eta(N) = g(\nabla_{\gamma'}\gamma', \xi) + g(\gamma', \nabla_{\gamma'}\xi) = -\alpha'\cosh\alpha.$$

This equation implies:

**Proposition 3.5.** *A non-geodesic spacelike curve $\gamma$ in a Sasakian Lorentzian 3-manifold $M^3$ is a slant curve if and only if $\eta(N) = 0$.*

From Theorem 3.4 and Proposition 3.5, if $\gamma$ is a spacelike proper biharmonic curve in Lorentzian Heisenberg space $(\mathbb{H}_3, g)$, then $\gamma$ has a spacelike normal vector field and is a slant pseudo-helix.

Let $\gamma$ be a spacelike slant curve in Lorentzian Heisenberg group $(\mathbb{H}_3, g)$. Then the tangent vector field has the form

$$T = \gamma' = \cos\beta\cosh\alpha_0 e_1 + \sin\beta\cosh\alpha_0 e_2 + \sinh\alpha_0 e_3, \quad \beta = \beta(s). \tag{18}$$

Using (4) we get

$$\nabla_{\gamma'}T = \cosh\alpha_0(\beta' + 2\sinh\alpha_0)(-\sin\beta e_1 + \cos\beta e_2).$$

Using the Frenet-Serret equation (2), we have the curvature

$$\kappa = \cosh\alpha_0(\beta' + 2\sinh\alpha_0).$$

Since $\gamma$ is a non-geodesic, we may assume that $\kappa = \cosh\alpha_0(\beta' + 2\sinh\alpha_0) > 0$ without loss of generality. Then the normal vector field

$$N = -\sin\beta e_1 + \cos\beta e_2.$$

Using Proposition 2.3, the binormal vector field

$$\begin{aligned} B &= -T \wedge_L N \\ &= -(\cos\beta\cosh\alpha_0 e_1 + \sin\beta\cosh\alpha_0 e_2 + \sinh\alpha_0 e_3) \wedge_L (-\sin\beta e_1 + \cos\beta e_2) \\ &= \cos\beta\sinh\alpha_0 e_1 + \sin\beta\sinh\alpha_0 e_2 + \cosh\alpha_0 e_3. \end{aligned}$$

Differentiation the normal vector field $N$

$$\begin{aligned} \nabla_{\gamma'} N &= \nabla_{\gamma'}(-\sin\beta e_1 + \cos\beta e_2) \\ &= -(\beta' + \sinh\alpha_0)(\cos\beta e_1 + \sin\beta e_2) + \cosh\alpha_0 e_3, \end{aligned}$$

and using the Frenet-Serret equation (2), we have

$$\tau = -1 - \sinh\alpha_0(\beta' + 2\sinh\alpha_0).$$

Therefore we get:

**Lemma 3.6.** *Let $\gamma$ be a spacelike slant curve in Lorentzian Heisenberg group $(\mathbb{H}_3, g)$ parametrized by arc-length. Then $\gamma$ admits a pseudo-orthonormal frame field $\{T, N, B\}$ with timelike $B$ along $\gamma$ and*

$$\begin{aligned} \kappa &= \cosh\alpha_0(\beta' + 2\sinh\alpha_0), \\ \tau &= -1 - \sinh\alpha_0(\beta' + 2\sinh\alpha_0). \end{aligned} \tag{19}$$

Thus we have:

**Corollary 3.7.** *Let $\gamma$ be a Legendre curve in Lorentzian Heisenberg group $(\mathbb{H}_3, g)$ parametrized by arc-length. Then $\gamma$ admits a pseudo-orthonormal frame field $\{\gamma', \varphi\gamma', \xi\}$ with timelike $\xi$ along $\gamma$ and $\tau = -1$.*

Using the equation (17) and (19) we have:

**Proposition 3.8.** *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a spacelike curve parametrized by arc-length in the Lorentzian Heisenberg group $(\mathbb{H}_3, g)$. Then $\gamma$ satisfies proper biharmonic if and only if $\gamma$ is a slant pseudo-helix with*

$$\beta'(s) = -\sinh\alpha_0 \pm \sqrt{-1 + 5\cosh^2\alpha_0}. \tag{20}$$

Let $\gamma(s) = (x(s), y(s), z(s))$ be a curve in $(\mathbb{H}_3, g)$. Then the tangent vector field $T$ of $\gamma$ is

$$T = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) = \frac{dx}{ds}\frac{\partial}{\partial x} + \frac{dy}{ds}\frac{\partial}{\partial y} + \frac{dz}{ds}\frac{\partial}{\partial z}.$$

Using the relations:

$$\frac{\partial}{\partial x} = e_1 + y e_3, \ \frac{\partial}{\partial y} = e_2 - x e_3, \ \frac{\partial}{\partial z} = e_3.$$

If $\gamma$ is a spacelike slant curve with spacelike normal vector field in $(\mathbb{H}_3, g)$, then from (18) the system of differential equations for $\gamma$ are given by

$$\frac{dx}{ds}(s) = \cosh\alpha_0 \cos\beta(s), \tag{21}$$

$$\frac{dy}{ds}(s) = \cosh\alpha_0 \sin\beta(s), \tag{22}$$

$$\frac{dz}{ds}(s) = \sinh\alpha_0 + \cosh\alpha_0(x(s)\sin\beta(s) - y(s)\cos\beta(s)).$$

Then (20) is reduced to

$$\beta'(s) = -\sinh\alpha_0 \pm \sqrt{-1 + 5\cosh^2\alpha_0} = \text{constant}.$$

Namely, $\beta'$ is a constant, say $A$, hence $\beta(s) = As + b,\ b \in \mathbb{R}$. Thus, from (21) and (22) we have the following result:

**Theorem 3.9.** *Let $\gamma : I \to (\mathbb{H}_3, g)$ be a spacelike curve parametrized by arc-length $s$ in the Lorentzian Heisenberg group $(\mathbb{H}_3, g)$. If $\gamma$ satisfies proper biharmonic equation, then the parametric equations of $\gamma$ are given by*

$$\begin{cases} x(s) = \frac{1}{A}\cosh\alpha_0 \sin(As + b) + x_0, \\ y(s) = -\frac{1}{A}\cosh\alpha_0 \cos(As + b) + y_0, \\ z(s) = \{\sinh\alpha_0 + \cosh^2\alpha_0/(A)\}s - \frac{\cosh\alpha_0}{A}\{x_0\cos(As+b) + y_0\sin(As+b)\} \\ \qquad\quad +z_0, \end{cases}$$

*where $b, x_0, y_0, z_0$ are constants.*

In particular, using (19) and (20) for a Legendre curve $\gamma$ we get $\kappa = \beta' = 2 = A$.

We assume that Riemannian metric $\widetilde{g}$ is defined by

$$\widetilde{g} = dx^2 + dy^2 + (dz + (ydx - xdy))^2$$

in the Heisenberg group $H_3$, then we get:

*Remark* 3.10 ([5]). Every proper biharmonic helix in Heisenberg spaces $(\mathbb{H}_3, \widetilde{g})$ is represented as

$$\begin{cases} x(s) = \frac{1}{A}\sin\alpha_0 \sin(As + b) + x_0, \\ y(s) = -\frac{1}{A}\sin\alpha_0 \cos(As + b) + y_0, \\ z(s) = \{\cos\alpha_0 + \sin^2\alpha_0/(A)\}s - \frac{\sin\alpha_0}{A}\{x_0\cos(As + b) + y_0\sin(As + b)\} \\ \qquad\quad +z_0, \end{cases}$$

for a constant contact angle $\alpha_0$, where $b, x_0, y_0, z_0$ are constants.

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