Commun. Korean Math. Soc. **33** (2018), No. 4, pp. 1303–1308
https://doi.org/10.4134/CKMS.c170466
pISSN: 1225-1763 / eISSN: 2234-3024

ON THE MAXIMUM AND MINIMUM MODULUS OF POLYNOMIALS ON CIRCLES

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Abstract. In this paper, we consider both maximum modulus and minimum modulus on a circle of some polynomials. These give rise to interesting examples that are about moduli of Chebyshev polynomials and certain sums of polynomials on a circle. Moreover, we obtain some root locations of difference quotients of Chebyshev polynomials.

1. Introduction

A classic question regarding polynomials is where the maximum modulus of such a polynomial on a circle occurs. Many papers ([2], [5], [7]) and books (e.g. Chapter 12 of [6]) have been written about this question and more. For example, given a complex polynomial $p(z)$, the maximum of $|p(z)|$ on the unit disc $D = \{z : |z| \leq 1\}$, denoted by $\|p\|_\infty$, is a quantity which arises in many interesting results in mathematics. By the maximum modulus principle, $\|p\|_\infty$ is attained at a boundary point of D. In this paper, we start to consider both maximum modulus and minimum modulus of some polynomials on a circle. These will give rise to interesting examples that are about moduli of Chebyshev polynomials and certain sums of polynomials on a circle. In [4], Kim and Lee obtained a result about root locations of difference quotients of Chebyshev polynomials of the first kind. From our example in this paper, we will obtain root locations of difference quotients of Chebyshev polynomials of the second kind as well as of the first kind.

2. Moduli of polynomials

We first prove the following.

Theorem 1. *For a positive integer n, let*

$$p(z) = \prod_{k=1}^{n} (z - \alpha_k)$$

Received December 4, 2017; Revised January 24, 2018; Accepted August 27, 2018.
2010 *Mathematics Subject Classification.* Primary 30C10; Secondary 11B83.
Key words and phrases. modulus, polynomials, difference quotients, Chebyshev polynomials.

be a polynomial of degree n *with real coefficients with*

$$\alpha_k = -\alpha_{n+1-k} > 0 \qquad (1 \le k \le \lfloor n/2 \rfloor)$$

and $\alpha_{\frac{n+1}{2}} = 0$ *when* n *is odd. Then for* $a > 0$*,*

$$\min_{|z|=a} |p(z)| = |p(a)| (= |p(-a)|)$$

and

$$\max_{|z|=a} |p(z)| = |p(i\,a)| (= |p(-i\,a)|).$$

Proof. For $0 \le \theta < 2\pi$, we define a function $f(\theta)$ by

$$\begin{aligned} f(\theta) = |p(ae^{i\theta})|^2 &= \prod_{k=1}^{n} |(a\cos\theta + ia\sin\theta) - \alpha_k|^2 \\ &= \prod_{k=1}^{n} \left[(a\cos\theta - \alpha_k)^2 + a^2 \sin^2\theta \right] = \prod_{k=1}^{n} \left[a^2 - 2a\alpha_k \cos\theta + \alpha_k^2 \right]. \end{aligned}$$

Then

$$f'(\theta) = \sum_{k=1}^{n} \left[2a\alpha_k \sin\theta \frac{f(\theta)}{a^2 - 2a\alpha_k\cos\theta + \alpha_k^2} \right]. \tag{1}$$

Since $\alpha_k = -\alpha_{n+1-k}$ for $1 \le k \le \lfloor n/2 \rfloor$, we may rewrite $f'(\theta)$ as

$$f'(\theta) = \sum_{k=1}^{\lfloor n/2 \rfloor} \left[2a\alpha_k \sin\theta \cdot \frac{f(\theta)}{a^2 - 2a\alpha_k\cos\theta + \alpha_k^2} - 2a\alpha_k \sin\theta \cdot \frac{f(\theta)}{a^2 + 2a\alpha_k\cos\theta + \alpha_k^2} \right],$$

where, for n odd, the mid summand is equal to zero since $\alpha_{\frac{n+1}{2}} = 0$. We now assume that n is even. Then

$$\begin{aligned} f'(\theta) &= \sum_{k=1}^{n/2} 2a\alpha_k \sin\theta\, f(\theta) \left[\frac{1}{a^2 - 2a\alpha_k\cos\theta + \alpha_k^2} - \frac{1}{a^2 + 2a\alpha_k\cos\theta + \alpha_k^2} \right] \\ &= \sum_{k=1}^{n/2} 2a\alpha_k \sin\theta\, f(\theta) \frac{4a\alpha_k\cos\theta}{a^4 - 2a^2\alpha_k^2 \cos 2\theta + \alpha_k^4} \\ &= \sum_{k=1}^{n/2} 4a^2\alpha_k \sin 2\theta\, f(\theta) \frac{\alpha_k}{a^4 - 2a^2\alpha_k^2\cos 2\theta + \alpha_k^4}. \end{aligned}$$

But

$$a^4 - 2a^2\alpha_k^2 \cos 2\theta + \alpha_k^4 > a^4 - 2a^2\alpha_k^2 + \alpha_k^4 = \left(a^2 - \alpha_k^2\right)^2 > 0.$$

Since $\alpha_k > 0$ for $1 \le k \le n/2$ and $f(\theta) > 0$, all summands of the sum of $f'(\theta)$ have the same sign. This implies that the sign of $f'(\theta)$ depends only on the sign of $\sin 2\theta$. Hence

$$f'(\theta)\begin{cases} >0 & \text{if } \theta\in\left(0,\frac{\pi}{2}\right)\cup\left(\pi,\frac{3\pi}{2}\right),\\ <0 & \text{if } \theta\in\left(\frac{\pi}{2},\pi\right)\cup\left(\frac{3\pi}{2},2\pi\right),\end{cases}$$

and $f'(\theta)=0$ only when $\theta=0,\ \pi/2,\ \pi,\ 3\pi/2$. Letting

$$H_k(\theta)=\frac{f(\theta)}{a^2-2a\alpha_k\cos\theta+\alpha_k^2}$$

gives

$$f''(\theta)=\sum_{k=1}^{n}\left(2a\alpha_k\cos\theta\right)H_k(\theta)+\sum_{k=1}^{n}\left(2a\alpha_k\sin\theta\right)H_k'(\theta)$$

from (1). We now apply the second derivative test for each critical point to obtain the results. First,

$$\begin{aligned} f''(0)&=\sum_{k=1}^{n}2a\alpha_k\prod_{j\neq k}\left(a^2-2a\alpha_j+\alpha_j^2\right)=\sum_{k=1}^{n}2a\alpha_k\prod_{j\neq k}\left(a-\alpha_j\right)^2\\ &=2a\sum_{k=1}^{n}\alpha_k\frac{p(a)^2}{\left(a-\alpha_k\right)^2}\\ &=2ap(a)^2\sum_{k=1}^{n/2}\left[\frac{\alpha_k}{\left(a-\alpha_k\right)^2}+\frac{\alpha_{n+1-k}}{\left(a-\alpha_{n+1-k}\right)^2}\right]\\ &=2ap(a)^2\sum_{k=1}^{n/2}\alpha_k\left[\frac{1}{\left(a-\alpha_k\right)^2}-\frac{1}{\left(a+\alpha_k\right)^2}\right]\\ &=2ap(a)^2\sum_{k=1}^{n/2}\alpha_k\frac{4a\alpha_k}{\left(a^2-\alpha_k^2\right)^2}>0, \end{aligned}$$

which implies that $f(\theta)$ has a local minimum at $\theta=0$. It follows from $\cos\pi=-1$ and $p(x)=p(-x)$ that

$$\begin{aligned} f''(\pi)&=-\sum_{k=1}^{n}2a\alpha_k\prod_{j\neq k}\left(a^2+2a\alpha_j+\alpha_j^2\right)=-\sum_{k=1}^{n}2a\alpha_k\prod_{j\neq k}\left(a+\alpha_j\right)^2\\ &=-2a\sum_{k=1}^{n}\alpha_k\frac{p(a)^2}{\left(a+\alpha_k\right)^2}=-2ap(a)^2\sum_{k=1}^{n}\frac{\alpha_k}{\left(a+\alpha_k\right)^2}\\ &=-2ap(a)^2\sum_{k=1}^{n/2}\left[\frac{\alpha_k}{\left(a+\alpha_k\right)^2}+\frac{\alpha_{n+1-k}}{\left(a+\alpha_{n+1-k}\right)^2}\right]\\ &=-2ap(a)^2\sum_{k=1}^{n/2}\alpha_k\left[\frac{1}{\left(a+\alpha_k\right)^2}-\frac{1}{\left(a-\alpha_k\right)^2}\right]>0. \end{aligned}$$

So $f(\theta)$ has a local minimum at $\theta = \pi$. On the other hand,

$$f''\left(\frac{\pi}{2}\right) = \sum_{k=1}^{n} 2a\alpha_k H_k'\left(\frac{\pi}{2}\right) \text{ and } f''\left(\frac{3\pi}{2}\right) = -\sum_{k=1}^{n} 2a\alpha_k H_k'\left(\frac{3\pi}{2}\right).$$

Since

$$H_k'\left(\frac{\pi}{2}\right) = -\frac{f\left(\frac{\pi}{2}\right)2a\alpha_k}{\left(a^2+\alpha_k^2\right)^2} \text{ and } H_k'\left(\frac{3\pi}{2}\right) = \frac{f\left(\frac{3\pi}{2}\right)2a\alpha_k}{\left(a^2+\alpha_k^2\right)^2},$$

we have

$$f''\left(\frac{\pi}{2}\right) = -\sum_{k=1}^{n} \frac{4a^2\alpha_k^2}{\left(a^2+\alpha_k^2\right)^2} f\left(\frac{\pi}{2}\right) < 0 \text{ and}$$
$$f''\left(\frac{3\pi}{2}\right) = -\sum_{k=1}^{n} \frac{4a^2\alpha_k^2}{\left(a^2+\alpha_k^2\right)^2} f\left(\frac{3\pi}{2}\right) < 0,$$

respectively, which imply that $f(\theta)$ has a local maximum at $\theta = \pi/2,\ 3\pi/2$. In all $f(\theta)$ has the minimum at $\theta = 0,\ \pi$, and the maximum at $\theta = \pi/2,\ 3\pi/2$ since $f(0) = f(\pi)$ and $f(\pi/2) = f(3\pi/2)$. □

From Theorem 1, we obtain the following.

Theorem 2. *With the notations used in Theorem 1, if* $a > \max_{1\le k\le \lfloor n/2\rfloor} \alpha_k$, *then the roots of* $p(z) = p(a)$ *lie inside* $|z| \le a$.

Proof. If $\max_{1\le k\le \lfloor n/2\rfloor} \alpha_k < a < b$, then for each k,

$$0 < a - \alpha_k < b - \alpha_k$$

and

$$|p(a)| = \prod_{k=1}^{n} |a-\alpha_k| < \prod_{k=1}^{n} |b-\alpha_k| = |p(b)|. \tag{2}$$

If z is a complex number with $|z| = b$, then by (2) and Theorem 1, we have

$$|p(a)| < |p(b)| \le |p(z)|.$$

This follows the result. □

3. Examples

In this section, we provide some interesting examples of Theorems 1 and 2.

Example 3. Chebyshev polynomials are of great importance in many areas of mathematics, particularly approximation theory. Let $T_n(z)$ and $U_n(z)$ be the Chebyshev polynomials of first kind and of the second kind, respectively. These polynomials satisfy the recurrence relations

$$T_0(z) = 1, \quad T_1(z) = z, \quad T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z) \quad (n \ge 1),$$
$$U_0(z) = 1, \quad U_1(z) = 2z, \quad U_{n+1}(z) = 2zU_n(z) - U_{n-1}(z) \quad (n \ge 1).$$

The roots of $T_n(z)$ and $U_n(z)$ are

$$\cos\frac{(2k-1)\pi}{2n} \quad \text{and} \quad \cos\frac{k\pi}{n+1},$$

respectively, where $1 \le k \le n$. Letting

$$\alpha_k = \cos\frac{(2k-1)\pi}{2n} \quad \text{or} \quad \cos\frac{k\pi}{n+1}$$

gives

$$0 < \alpha_k = -\alpha_{n+1-k} < 1 \qquad (1 \le k \le \lfloor n/2 \rfloor)$$

and $\alpha_{\frac{n+1}{2}} = 0$ when n is odd. Thus by Theorem 1, for $a > 0$

$$\min_{|z|=a} |T_n(z)| = |T_n(a)|(= |T_n(-a)|), \quad \max_{|z|=a} |T_n(z)| = |T_n(i\,a)|(= |T_n(-i\,a)|),$$

and

$$\min_{|z|=a} |U_n(z)| = |U_n(a)|(= |U_n(-a)|), \quad \max_{|z|=a} |U_n(z)| = |U_n(i\,a)|(= |U_n(-i\,a)|).$$

Moreover, by Theorem 2, if $a > 1$, then all roots of

$$QT_n(a,z) := \frac{T_n(z) - T_n(a)}{z - a} \quad \text{and} \quad QU_n(a,z) := \frac{U_n(z) - U_n(a)}{z - a},$$

where $z \ne a$, lie inside $|z| \le a$. In fact, it was shown in [4] that the roots of $QT_n(a,z)$ lie on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - 1} = 1 \qquad (z = x + iy)$$

that is inside $|z| \le a$. But there have not been known a specific location of roots of $QU_n(a,z)$. The sufficient condition $a > \max_{1 \le k \le \lfloor n/2 \rfloor} \alpha_k$ in Theorem 2 seems to be required from many examples. Two of them are $QT_{10}(0.9, z)$ and $QU_{10}(0.9, z)$ having roots $\pm 0.984\ldots$ and $\pm 0.932\ldots$, respectively, whose absolute values are greater than 0.9.

Perhaps the most immediate question of sums of polynomials, $A + B = C$, is “given bounds for the roots of A and B, what bounds can be given for the roots of C?” By Fell [1], if all roots of A and B lie in $[-1, 1]$ with A, B monic and $\deg A = \deg B = n$, then no root of C can have modulus exceeding $\cot(\pi/2n)$, the largest root of $(z+1)^n + (z-1)^n$. This suggests to study polynomials having a form something like $A(z) + B(z)$ where all roots of $A(z)$ are negative and all roots of $B(z)$ are positive.

Example 4. We consider the polynomial

$$q_n(z) := \prod_{j=1}^{n} (z - r_j) + \prod_{j=1}^{n} (z + r_j), \tag{3}$$

where $0 < r_1 \le r_2 \le \cdots \le r_n$. It is known by Kim [3] that all (conjugate) roots of the polynomial $q_n(z)$ with real coefficients lie on the imaginary axis, and no two of the roots of $q_n(z)$ can be equal and the gaps between the roots of $q_n(z)$ in the upper half-plane strictly increase as one proceeds upward. The polynomial $q_n(z)$ has purely imaginary roots symmetric with the other about the real axis, i.e., if $i\alpha_k$, where $\alpha_k > 0$, is a root, so is $-i\alpha_k$, and letting $i\alpha_{n+1-k} = -i\alpha_k$ deduces $\alpha_k = -\alpha_{n+1-k}$ which is a sufficient condition of Theorem 1. So we may apply Theorem 1 to the polynomial $q_n(z)$ so that for $a > 0$,

$$\max_{|z|=a} |q(z)| = |q(a)|(= |q(-a)|)$$

and

$$\min_{|z|=a} |q(z)| = |q(i\,a)|(= |q(-i\,a)|).$$

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