Commun. Korean Math. Soc. **33** (2018), No. 4, pp. 1285–1301
https://doi.org/10.4134/CKMS.c170461
pISSN: 1225-1763 / eISSN: 2234-3024

TURÁN-TYPE INEQUALITIES FOR GAUSS AND CONFLUENT HYPERGEOMETRIC FUNCTIONS VIA CAUCHY-BUNYAKOVSKY-SCHWARZ INEQUALITY

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Abstract. This paper is devoted to the study of Turán-type inequalities for some well-known special functions such as Gauss hypergeometric functions, generalized complete elliptic integrals and confluent hypergeometric functions which are derived by using a new form of the Cauchy-Bunyakovsky-Schwarz inequality. We also apply these inequalities for some sample of interest such as incomplete beta function, incomplete gamma function, elliptic integrals and modified Bessel functions to obtain their corresponding Turán-type inequalities.

1. Introduction

Special functions constitute a very old branch of mathematics, the origins of their unified and rather complete theory date to the nineteenth century. Several special functions such as Gauss hypergeometric functions, generalized complete elliptic integrals, confluent hypergeometric functions, play a very important and interesting role in various branches of mathematics and mathematical physics. The importance of the Gauss hypergeometric functions is that many elementary and special functions of mathematical physics can directly be represented in terms of these functions. Similar to Gauss hypergeometric functions, the first and second kind of confluent hypergeometric functions are so important that many well-known special functions can directly be expressed in terms of these functions.

Also, the inequalities of the type

$$f_n(x)\,f_{n+2}(x) - [f_{n+1}(x)]^2 \le 0, \tag{1}$$

where $n = 0, 1, 2, \ldots$, have great importance in different fields of mathematics.

These inequalities are named by Karlin and Szegö [9] as Turán-type inequalities because the first of this type of inequality was introduced by Hungarian

Received November 21, 2017; Revised January 27, 2018; Accepted February 13, 2018.
2010 *Mathematics Subject Classification.* Primary 26D07; Secondary 33C15.
Key words and phrases. Cauchy-Bunyakovsky-Schwarz inequality, Turán-type inequalities, Gauss hypergeometric functions, generalized complete elliptic integrals, confluent hypergeometric functions.

mathematician P. Turán [18] in 1950, while studying the zeros of Legendre polynomials.

This classical result has been extended in several directions. Many Turán-type inequalities have been investigated in the literature. For example, Joshi and Bissu [8] presented some two-sided inequalities for the ratio of modified Bessel functions of first kind. They also deduced some Turán-type and Wronskian type inequalities for Bessel and modified Bessel functions. Mehrez [13] presented Turán-type inequalities for the Wright functions. Baricz [3] presented Turán-type inequalities for regular Coulomb wave functions. Ali, Mondal and Nisar [1] established some monotonicity properties of the generalized Struve functions. Nisar, Mondal and Choi [16] presented certain inequalities involving the k-Struve function.

Motivated by the immense research on Turán-type inequalities, in the present paper, we investigate Turán-type inequalities for some well-known special functions such as Gauss hypergeometric functions, generalized complete elliptic integrals and confluent hypergeometric functions and their examples, by using a new form of the Cauchy-Bunyakovsky-Schwarz inequality.

It is well-known that the discrete version of Cauchy-Schwarz inequality (see for instant; [14])

$$\left(\sum_{i=1}^{n}\left[a_i b_i\right]\right)^2 \leq \sum_{i=1}^{n}\left[a_i\right]^2 \sum_{i=1}^{n}\left[b_i\right]^2,\ \left(a_i, b_i \in \mathbb{R}\right), \tag{2}$$

and its integral representation in the space of continuous real-valued functions $C\left(\left[a,b\right], \mathbb{R}\right)$, i.e., the Cauchy-Bunyakovsky-Schwarz (CBS) inequality (see for instant; [14])

$$\left(\int_a^b \left[u\left(t\right)\right]^{\frac{1}{2}} \left[v\left(t\right)\right]^{\frac{1}{2}} dt\right)^2 \leq \left(\int_a^b u\left(t\right) dt\right)\left(\int_a^b v\left(t\right) dt\right), \tag{3}$$

play an important role in various branches of modern mathematics. To date, a large number of generalizations and refinements of the inequalities (2) and (3) have been investigated in the literature, e.g., [2, 4, 5, 7, 10–12, 17].

Recently, in [6], we have proved Turán-type inequalities for Struve functions, modified Struve functions, Anger Weber functions and Hurwitz zeta function, by using the following new form of the Cauchy-Bunyakovsky-Schwarz inequality:

$$\begin{aligned} &\left(\int_a^b \left[g\left(t\right)\right]^{\alpha} \left[h\left(t\right)\right]^{\beta} dt\right)^2 \\ &\leq \left(\int_a^b \left[g\left(t\right)\right]^{\alpha-\lambda} \left[h\left(t\right)\right]^{\beta-\gamma} dt\right)\left(\int_a^b \left[g\left(t\right)\right]^{\alpha+\lambda} \left[h\left(t\right)\right]^{\beta+\gamma} dt\right), \end{aligned} \tag{4}$$

where g and h are two non-negative functions of a real variable and α, β, λ and γ belong to a set S of real numbers, such that the involved integrals in (4) exist.

Motivated by this remark, we have the idea to replace $u(t)$ and $v(t)$ in (3) by $[g(t)]^{\xi-l}[h(t)]^{\xi-m}[f(t)]^{\xi-n}$ and $[g(t)]^{\xi+l}[h(t)]^{\xi+m}[f(t)]^{\xi+n}$ respectively, to introduce the following new generalized CBS inequality:

$$\begin{aligned}(5)\qquad &\left(\int_a^b [g(t)]^{\xi}[h(t)]^{\xi}[f(t)]^{\xi}\,dt\right)^2 \\ &\le \left(\int_a^b [g(t)]^{\xi-l}[h(t)]^{\xi-m}[f(t)]^{\xi-n}\,dt\right)\left(\int_a^b [g(t)]^{\xi+l}[h(t)]^{\xi+m}[f(t)]^{\xi+n}\,dt\right),\end{aligned}$$

in which $\xi, l, m, n \in \mathbb{R}$ and g, h and f are real integrable functions such that the involved integrals in (5) exist.

The aim of this paper is to apply the above new form of CBS inequality (5) to obtain Turán-type inequalities for some well-known special functions.

2. The results

In this section, we apply the inequality (5) to establish Turán-type inequalities for Gauss hypergeometric functions, generalized complete elliptic integrals and confluent hypergeometric functions. We also apply this inequality to some other functions such as incomplete beta function, incomplete gamma function, elliptic integrals, modified Bessel functions and obtain their corresponding Turán-type inequalities.

2.1. Turán-type inequalities for Gauss hypergeometric functions

Let us start our discussion with the well-known Gauss differential equation [15]:

$$(6)\qquad x(1-x)\frac{d^2y}{dx^2}+(\gamma-(\alpha+\beta+1)x)\frac{dy}{dx}-\alpha\beta y(x)=0,$$

where α, β and γ are constant parameters.

The indicial equation of (6) has respectively two roots $r_1=0$ and $r_2=1-\gamma$, and the series solution (for $r_1=0$) of the Gauss differential equation is given by

$$(7)\qquad y_1(x)={}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix}\middle|x\right)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!},$$

in which $\gamma \neq 0, -1, -2, \ldots,$ and $(\alpha)_n$ is the Pochhammer (or Appell) symbol defined by

$$\begin{aligned}(\alpha)_n &\equiv \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}=\alpha(\alpha+1)\cdots(\alpha+n-1),\ n\in\mathbb{N},\\ (\alpha)_0 &= 1,\ (\alpha+1)_{-1}=\frac{1}{\alpha},\ \alpha\neq 0\end{aligned}$$

with the gamma function $\Gamma(x)\equiv\int_0^{\infty}e^{-t}t^{x-1}dt$ for $x>0$ and series converges for $-1\le x\le 1$.

The function (7) is known as Gauss hypergeometric function and has a specific integral representation as

$$
{}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix}\Big| x\right)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-xt)^{-\beta}\,dt, \tag{8}
$$
$$
(\gamma>\alpha>0;|x|\le 1).
$$

Now if $g(t)=t^{\alpha-1}$, $h(t)=(1-t)^{\gamma-\alpha-1}$ and $f(t)=(1-xt)^{-\beta}$ are substituted in inequality (5) for $[a,b]=[0,1]$, the following inequality is derived for the class of Gauss hypergeometric functions:

$$
\begin{aligned}
&\left(\int_0^1 t^{\xi(\alpha-1)}(1-t)^{\xi(\gamma-\alpha-1)}(1-xt)^{-\xi\beta}\,dt\right)^2\\
&\le\left(\int_0^1 t^{(\xi-l)(\alpha-1)}(1-t)^{(\xi-m)(\gamma-\alpha-1)}(1-xt)^{-(\xi-n)\beta}\,dt\right)\\
&\quad\times\left(\int_0^1 t^{(\xi+l)(\alpha-1)}(1-t)^{(\xi+m)(\gamma-\alpha-1)}(1-xt)^{-(\xi+n)\beta}\,dt\right).
\end{aligned}
$$

By applying the identity (8), the following result will eventually be obtained

$$
\begin{aligned}
&\frac{[\Gamma\{\xi(\alpha-1)+1\}]^2[\Gamma\{\xi(\gamma-\alpha-1)+1\}]^2}{[\Gamma\{\xi(\gamma-2)+2\}]^2}\left[{}_2F_1\left(\begin{matrix}\xi(\alpha-1)+1,\ \xi\beta\\ \xi(\gamma-2)+2\end{matrix}\Big| x\right)\right]^2\\
&\le\frac{\Gamma[(\xi-l)(\alpha-1)+1]\,\Gamma[(\xi-m)(\gamma-\alpha-1)+1]}{\Gamma[(\xi-l)(\alpha-1)+(\xi-m)(\gamma-\alpha-1)+2]}\\
&\quad\times{}_2F_1\left(\begin{matrix}(\xi-l)(\alpha-1)+1,\ (\xi-n)\beta\\ (\xi-l)(\alpha-1)+(\xi-m)(\gamma-\alpha-1)+2\end{matrix}\Big| x\right)\\
&\quad\times\frac{\Gamma[(\xi+l)(\alpha-1)+1]\,\Gamma[(\xi+m)(\gamma-\alpha-1)+1]}{\Gamma[(\xi+l)(\alpha-1)+(\xi+m)(\gamma-\alpha-1)+2]}\\
&\quad\times{}_2F_1\left(\begin{matrix}(\xi+l)(\alpha-1)+1,\ (\xi+n)\beta\\ (\xi+l)(\alpha-1)+(\xi+m)(\gamma-\alpha-1)+2\end{matrix}\Big| x\right)
\end{aligned}
$$

which can also be written as

$$
\begin{aligned}
&\left[{}_2F_1\left(\begin{matrix}\xi(\alpha-1)+1,\ \xi\beta\\ \xi(\gamma-2)+2\end{matrix}\Big| x\right)\right]^2\\
&\le\frac{[\Gamma\{\xi(\gamma-2)+2\}]^2}{[\Gamma\{\xi(\alpha-1)+1\}]^2[\Gamma\{\xi(\gamma-\alpha-1)+1\}]^2}\\
&\quad\times\frac{\Gamma[(\xi-l)(\alpha-1)+1]\,\Gamma[(\xi-m)(\gamma-\alpha-1)+1]}{\Gamma[(\xi-l)(\alpha-1)+(\xi-m)(\gamma-\alpha-1)+2]}\\
&\quad\times\frac{\Gamma[(\xi+l)(\alpha-1)+1]\,\Gamma[(\xi+m)(\gamma-\alpha-1)+1]}{\Gamma[(\xi+l)(\alpha-1)+(\xi+m)(\gamma-\alpha-1)+2]}\\
&\quad\times{}_2F_1\left(\begin{matrix}(\xi-l)(\alpha-1)+1,\ (\xi-n)\beta\\ (\xi-l)(\alpha-1)+(\xi-m)(\gamma-\alpha-1)+2\end{matrix}\Big| x\right)
\end{aligned}
$$

$$\times {}_2F_1\left(\begin{matrix}(\xi+l)(\alpha-1)+1,\ (\xi+n)\beta\\ (\xi+l)(\alpha-1)+(\xi+m)(\gamma-\alpha-1)+2\end{matrix}\middle|x\right). \tag{9}$$

If we replace $\nu_1 = \xi(\alpha-1)+1,\ \nu_2 = \xi(\gamma-\alpha-1)+1,\ \nu_3 = \xi\beta$ and $\mu_1 = l(\alpha-1),\ \mu_2 = m(\gamma-\alpha-1),\ \mu_3 = n\beta$ in inequality (9), we obtain the Turán-type inequality for the class of Gauss hypergeometric functions as follows:

$$\begin{aligned}\left[{}_2F_1\left(\begin{matrix}\nu_1,\ \nu_3\\ \nu_1+\nu_2\end{matrix}\middle|x\right)\right]^2 \leq{}& \frac{[\Gamma(\nu_1+\nu_2)]^2}{[\Gamma(\nu_1)]^2[\Gamma(\nu_2)]^2}\frac{\Gamma(\nu_1-\mu_1)\Gamma(\nu_2-\mu_2)}{\Gamma[(\nu_1-\mu_1)+(\nu_2-\mu_2)]}\\ &\times\frac{\Gamma(\nu_1+\mu_1)\Gamma(\nu_2+\mu_2)}{\Gamma[(\nu_1+\mu_1)+(\nu_2+\mu_2)]}\\ &\times {}_2F_1\left(\begin{matrix}\nu_1-\mu_1,\ \nu_3-\mu_3\\ (\nu_1+\nu_2)-(\mu_1+\mu_2)\end{matrix}\middle|x\right)\\ &\times {}_2F_1\left(\begin{matrix}\nu_1+\mu_1,\ \nu_3+\mu_3\\ (\nu_1+\nu_2)+(\mu_1+\mu_2)\end{matrix}\middle|x\right).\end{aligned} \tag{10}$$

With the help of the well-known following relation between beta and gamma functions,

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p,q>0. \tag{11}$$

Turán-type inequality (10) can also be put in the following form:

$$\begin{aligned}\left[{}_2F_1\left(\begin{matrix}\nu_1,\ \nu_3\\ \nu_1+\nu_2\end{matrix}\middle|x\right)\right]^2 \leq{}& \frac{B(\nu_1-\mu_1,\nu_2-\mu_2)\,B(\nu_1+\mu_1,\nu_2+\mu_2)}{[B(\nu_1,\nu_2)]^2}\\ &\times {}_2F_1\left(\begin{matrix}\nu_1-\mu_1,\ \nu_3-\mu_3\\ (\nu_1+\nu_2)-(\mu_1+\mu_2)\end{matrix}\middle|x\right)\\ &\times {}_2F_1\left(\begin{matrix}\nu_1+\mu_1,\ \nu_3+\mu_3\\ (\nu_1+\nu_2)+(\mu_1+\mu_2)\end{matrix}\middle|x\right),\end{aligned} \tag{12}$$

provided that $\nu_1 > |\mu_1|$, $\nu_2 > |\mu_2|$ and $|x| \leq 1$.

In the particular case when $\xi = 1$, the inequality (9) reduces to following inequality

$$\begin{aligned}&\left[{}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix}\middle|x\right)\right]^2\\ &\leq \frac{[\Gamma(\gamma)]^2}{[\Gamma(\alpha)]^2[\Gamma(\gamma-\alpha)]^2}\\ &\quad\times\frac{\Gamma[\alpha-l(\alpha-1)]\,\Gamma[(\gamma-\alpha)-m(\gamma-\alpha-1)]\,\Gamma[\alpha+l(\alpha-1)]\,\Gamma[(\gamma-\alpha)+m(\gamma-\alpha-1)]}{\Gamma[\gamma-l(\alpha-1)-m(\gamma-\alpha-1)]\,\Gamma[\gamma+l(\alpha-1)+m(\gamma-\alpha-1)]}\\ &\quad\times {}_2F_1\left(\begin{matrix}\alpha-l(\alpha-1),\ (1-n)\beta\\ \gamma-l(\alpha-1)-m(\gamma-\alpha-1)\end{matrix}\middle|x\right)\\ &\quad\times {}_2F_1\left(\begin{matrix}\alpha+l(\alpha-1),\ (1+n)\beta\\ \gamma+l(\alpha-1)+m(\gamma-\alpha-1)\end{matrix}\middle|x\right).\end{aligned} \tag{13}$$

For $l\left(\alpha-1\right)=\lambda,\ m\left(\gamma-\alpha-1\right)=\mu$ and $n\beta=w$, the inequality (13) transforms to the much nicer version of Turán-type inequality for Gauss hypergeometric functions as follows:

$$\begin{aligned}
&\left[{}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix}\Big|x\right)\right]^2\\
&\le\frac{\left[\Gamma\left(\gamma\right)\right]^2\Gamma\left(\alpha-\lambda\right)\Gamma\left[\left(\gamma-\alpha\right)-\mu\right]\Gamma\left(\alpha+\lambda\right)\Gamma\left[\left(\gamma-\alpha\right)+\mu\right]}{\left[\Gamma\left(\alpha\right)\right]^2\left[\Gamma\left(\gamma-\alpha\right)\right]^2\Gamma\left[\gamma-\left(\lambda+\mu\right)\right]\Gamma\left[\gamma+\left(\lambda+\mu\right)\right]}\\
&\quad\times{}_2F_1\left(\begin{matrix}\alpha-\lambda,\ \beta-w\\ \gamma-\left(\lambda+\mu\right)\end{matrix}\Big|x\right){}_2F_1\left(\begin{matrix}\alpha+\lambda,\ \beta+w\\ \gamma+\left(\lambda+\mu\right)\end{matrix}\Big|x\right).
\end{aligned}\tag{14}$$

Or equivalently

$$\begin{aligned}
\left[{}_2F_1\left(\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix}\Big|x\right)\right]^2&\le\frac{B\left[\alpha-\lambda,\left(\gamma-\alpha\right)-\mu\right]B\left[\alpha+\lambda,\left(\gamma-\alpha\right)+\mu\right]}{\left[B\left(\alpha,\ \gamma-\alpha\right)\right]^2}\\
&\quad\times{}_2F_1\left(\begin{matrix}\alpha-\lambda,\ \beta-w\\ \gamma-\left(\lambda+\mu\right)\end{matrix}\Big|x\right){}_2F_1\left(\begin{matrix}\alpha+\lambda,\ \beta+w\\ \gamma+\left(\lambda+\mu\right)\end{matrix}\Big|x\right),
\end{aligned}\tag{15}$$

provided that $\alpha>|\lambda|,\ \gamma-\alpha>|\mu|$ and $|x|\le 1$.

Example 1 (Turán-type inequalities for the incomplete beta function)**.** If one considers the integral representation of the incomplete beta function

$$\begin{aligned}
B_x\left(u;\upsilon\right)=&\int_0^x t^{u-1}\left(1-t\right)^{\upsilon-1}dt=u^{-1}x^u{}_2F_1\left(\begin{matrix}u,\ 1-\upsilon\\ u+1\end{matrix}\Big|x\right),\\
&\left(0\le x\le 1;u,\upsilon>0\right).
\end{aligned}\tag{16}$$

and replaces $\alpha=u,\ \beta=1-\upsilon$ and $\gamma=u+1$ with $\lambda=l\left(u-1\right),\ \mu=0$ and $w=n\left(1-\upsilon\right)$ in inequality (14), then one obtains

$$\begin{aligned}
\left[{}_2F_1\left(\begin{matrix}u,\ 1-\upsilon\\ u+1\end{matrix}\Big|x\right)\right]^2&\le\frac{\left[\Gamma\left(u+1\right)\right]^2\Gamma\left(u-\lambda\right)\Gamma\left(u+\lambda\right)}{\left[\Gamma\left(u\right)\right]^2\Gamma\left(u-\lambda+1\right)\Gamma\left(u+\lambda+1\right)}\\
&\quad\times{}_2F_1\left(\begin{matrix}u-\lambda,\ \left(1-\upsilon\right)-w\\ u+1-\lambda\end{matrix}\Big|x\right)\\
&\quad\times{}_2F_1\left(\begin{matrix}u+\lambda,\ \left(1-\upsilon\right)+w\\ u+1+\lambda\end{matrix}\Big|x\right).
\end{aligned}$$

Corresponding to the definition (16), above inequality is equivalent to

$$\begin{aligned}
\frac{u^2}{x^{2u}}\left[B_x\left(u;\upsilon\right)\right]^2\le&\frac{\left[\Gamma\left(u+1\right)\right]^2\Gamma\left(u-\lambda\right)\Gamma\left(u+\lambda\right)}{\left[\Gamma\left(u\right)\right]^2\Gamma\left(u-\lambda+1\right)\Gamma\left(u+\lambda+1\right)}\\
&\times\frac{\left(u-\lambda\right)}{x^{u-\lambda}}B_x\left(u-\lambda;\upsilon+w\right)\times\frac{\left(u+\lambda\right)}{x^{u+\lambda}}B_x\left(u+\lambda;\upsilon-w\right).
\end{aligned}$$

By applying the identity of gamma function

$$\Gamma\left(p+1\right)=p\Gamma\left(p\right),\ p>0.\tag{17}$$

We have the Turán-type inequality for incomplete beta function as follows:

$$[B_x(u;\upsilon)]^2 \le B_x(u-\lambda;\upsilon+w)\, B_x(u+\lambda;\upsilon-w)\,, \tag{18}$$

provided that $0 \le x \le 1; u > |\lambda|$ and $\upsilon > |w|$.

2.2. Turán-type inequalities for generalized complete elliptic integrals

The first and second kind of generalized complete elliptic integrals for $x \in (0,1)\,, x^{'} = \sqrt{1-x^2}$ and $\varphi \in (0,1)$ are defined by

$$\begin{cases} \kappa_\varphi(x) = \frac{\pi}{2} {}_2F_1\left(\begin{smallmatrix} 1-\varphi, & \varphi \\ & 1 \end{smallmatrix} \Big| x^2\right) \\ \kappa'_\varphi(x) = \kappa_\varphi(x') \\ \kappa_\varphi(0) = \frac{\pi}{2}, \kappa_\varphi(1) = \infty \end{cases}$$

and

$$\begin{cases} E_\varphi(x) = \frac{\pi}{2} {}_2F_1\left(\begin{smallmatrix} 1-\varphi, & \varphi-1 \\ & 1 \end{smallmatrix} \Big| x^2\right) \\ E'_\varphi(x) = E_\varphi(x') \\ E_\varphi(0) = \frac{\pi}{2}, E_\varphi(1) = \frac{\sin\pi\varphi}{2(1-\varphi)}. \end{cases}$$

and can respectively be represented by the following integrals

$$\kappa_\varphi(x) = \frac{\pi}{2}\frac{1}{\Gamma(\varphi)\Gamma(1-\varphi)}\int_0^1 t^{-\varphi}(1-t)^{\varphi-1}\left(1-x^2t\right)^{-\varphi} dt, \tag{19}$$
$$(0<\varphi<1; 0<x<1)$$

and

$$E_\varphi(x) = \frac{\pi}{2}\frac{1}{\Gamma(\varphi)\Gamma(1-\varphi)}\int_0^1 t^{-\varphi}(1-t)^{\varphi-1}\left(1-x^2t\right)^{-(\varphi-1)} dt, \tag{20}$$
$$(0<\varphi<1; 0<x<1)\,.$$

Now, applying inequality (5) for $g(t) = t^{-\varphi}$, $h(t) = (1-t)^{\varphi-1}$, $f(t) = \left(1-x^2t\right)^{-\varphi}$ and $[a,b] = [0,1]$, results in the following inequality for first kind of generalized complete elliptic integrals:

$$\begin{aligned} &\left(\int_0^1 t^{-\xi\varphi}(1-t)^{\xi(\varphi-1)}\left(1-x^2t\right)^{-\xi\varphi} dt\right)^2 \\ &\le \left(\int_0^1 t^{-(\xi-l)\varphi}(1-t)^{(\xi-m)(\varphi-1)}\left(1-x^2t\right)^{-(\xi-n)\varphi} dt\right) \\ &\quad\times\left(\int_0^1 t^{-(\xi+l)\varphi}(1-t)^{(\xi+m)(\varphi-1)}\left(1-x^2t\right)^{-(\xi+n)\varphi} dt\right). \end{aligned}$$

Corresponding to the identity (8), this is transformed to

$$\begin{aligned} &\frac{[\Gamma(1-\xi\varphi)]^2[\Gamma\{\xi(\varphi-1)+1\}]^2}{[\Gamma(2-\xi)]^2}\left[{}_2F_1\left(\begin{matrix} 1-\xi\varphi, \ \xi\varphi \\ 2-\xi \end{matrix}\Big| x^2\right)\right]^2 \\ &\le \frac{\Gamma[1-(\xi-l)\varphi]\,\Gamma[(\xi-m)(\varphi-1)+1]}{\Gamma[2-(\xi-l)\varphi+(\xi-m)(\varphi-1)]} \end{aligned}$$

$$
\begin{aligned}
& {}_2F_1\left(\begin{matrix} 1-(\xi-l)\,\varphi,\ (\xi-n)\,\varphi \\ 2-(\xi-l)\,\varphi+(\xi-m)\,(\varphi-1) \end{matrix}\middle| x^2\right) \\
& \times \frac{\Gamma\left[1-(\xi+l)\,\varphi\right]\Gamma\left[(\xi+m)\,(\varphi-1)+1\right]}{\Gamma\left[2-(\xi+l)\,\varphi+(\xi+m)\,(\varphi-1)\right]} \\
& {}_2F_1\left(\begin{matrix} 1-(\xi+l)\,\varphi,\ (\xi+n)\,\varphi \\ 2-(\xi+l)\,\varphi+(\xi+m)\,(\varphi-1) \end{matrix}\middle| x^2\right) \\
\Rightarrow & \left[{}_2F_1\left(\begin{matrix} 1-\xi\varphi,\ \xi\varphi \\ 2-\xi \end{matrix}\middle| x^2\right)\right]^2 \\
\leq & \frac{\left[\Gamma\,(2-\xi)\right]^2}{\left[\Gamma\,(1-\xi\varphi)\right]^2\left[\Gamma\left\{\xi\,(\varphi-1)+1\right\}\right]^2} \\
& \times \frac{\Gamma\left[1-(\xi-l)\,\varphi\right]\Gamma\left[(\xi-m)\,(\varphi-1)+1\right]\Gamma\left[1-(\xi+l)\,\varphi\right]\Gamma\left[(\xi+m)\,(\varphi-1)+1\right]}{\Gamma\left[2-(\xi-l)\,\varphi+(\xi-m)\,(\varphi-1)\right]\Gamma\left[2-(\xi+l)\,\varphi+(\xi+m)\,(\varphi-1)\right]} \\
& {}_2F_1\left(\begin{matrix} 1-(\xi-l)\,\varphi,\ (\xi-n)\,\varphi \\ 2-(\xi-l)\,\varphi+(\xi-m)\,(\varphi-1) \end{matrix}\middle| x^2\right) \\
& \times {}_2F_1\left(\begin{matrix} 1-(\xi+l)\,\varphi,\ (\xi+n)\,\varphi \\ 2-(\xi+l)\,\varphi+(\xi+m)\,(\varphi-1) \end{matrix}\middle| x^2\right).
\end{aligned} \tag{21}
$$

If we replace $\nu_1 = 1-\xi\varphi$, $\nu_2 = \xi\,(\varphi-1)+1$, $\nu_3 = \xi\varphi$ and $\mu_1 = -l\varphi,\ \mu_2 = m\,(\varphi-1)\,,\ \mu_3 = n\varphi$ in inequality (21), we obtain the Turán-type inequality for the class of generalized complete elliptic integrals of first kind as follows:

$$
\begin{aligned}
\left[{}_2F_1\left(\begin{matrix} \nu_1,\ \nu_3 \\ \nu_1+\nu_2 \end{matrix}\middle| x^2\right)\right]^2 \leq & \frac{\left[\Gamma\,(\nu_1+\nu_2)\right]^2}{\left[\Gamma\,(\nu_1)\right]^2\left[\Gamma\,(\nu_2)\right]^2}\frac{\Gamma\,(\nu_1-\mu_1)\,\Gamma\,(\nu_2-\mu_2)}{\Gamma\left[(\nu_1-\mu_1)+(\nu_2-\mu_2)\right]} \\
& \times \frac{\Gamma\,(\nu_1+\mu_1)\,\Gamma\,(\nu_2+\mu_2)}{\Gamma\left[(\nu_1+\mu_1)+(\nu_2+\mu_2)\right]} \\
& \times {}_2F_1\left(\begin{matrix} \nu_1-\mu_1,\ \nu_3-\mu_3 \\ (\nu_1+\nu_2)-(\mu_1+\mu_2) \end{matrix}\middle| x^2\right) \\
& \times {}_2F_1\left(\begin{matrix} \nu_1+\mu_1,\ \nu_3+\mu_3 \\ (\nu_1+\nu_2)+(\mu_1+\mu_2) \end{matrix}\middle| x^2\right).
\end{aligned} \tag{22}
$$

On the other hand, by applying the well-known relationship between beta and gamma functions (11) we can obtain the final result as:

$$
\begin{aligned}
\left[{}_2F_1\left(\begin{matrix} \nu_1,\ \nu_3 \\ \nu_1+\nu_2 \end{matrix}\middle| x^2\right)\right]^2 \leq & \frac{B\,(\nu_1-\mu_1,\nu_2-\mu_2)\,B\,(\nu_1+\mu_1,\nu_2+\mu_2)}{\left[B\,(\nu_1,\nu_2)\right]^2} \\
& \times {}_2F_1\left(\begin{matrix} \nu_1-\mu_1,\ \nu_3-\mu_3 \\ (\nu_1+\nu_2)-(\mu_1+\mu_2) \end{matrix}\middle| x^2\right) \\
& \times {}_2F_1\left(\begin{matrix} \nu_1+\mu_1,\ \nu_3+\mu_3 \\ (\nu_1+\nu_2)+(\mu_1+\mu_2) \end{matrix}\middle| x^2\right),
\end{aligned} \tag{23}
$$

provided that $x \in (0,1)\,;\ \nu_1 > |\mu_1|$ and $\nu_2 > |\mu_2|$.

When $\xi = 1$, inequality (21) reduces to the following inequality for the function $\kappa_\varphi(x)$

$$\begin{aligned}&\left[{}_2F_1\left(\begin{matrix}1-\varphi,\ \varphi\\ 1\end{matrix}\middle| x^2\right)\right]^2\\&\le \frac{[\Gamma(1)]^2\,\Gamma[1-\varphi+l\varphi]\,\Gamma[\varphi-m(\varphi-1)]\,\Gamma[1-\varphi-l\varphi]\,\Gamma[\varphi+m(\varphi-1)]}{[\Gamma(1-\varphi)]^2\,[\Gamma(\varphi)]^2\,\Gamma[1+l\varphi-m(\varphi-1)]\,\Gamma[1-l\varphi+m(\varphi-1)]}\\&\quad\times {}_2F_1\left(\begin{matrix}1-\varphi+l\varphi,\ (1-n)\varphi\\ 1+l\varphi-m(\varphi-1)\end{matrix}\middle| x^2\right){}_2F_1\left(\begin{matrix}1-\varphi-l\varphi,\ (1+n)\varphi\\ 1-l\varphi+m(\varphi-1)\end{matrix}\middle| x^2\right).\end{aligned}$$

If we put $(-l\varphi)=\lambda,\ m(\varphi-1)=\mu$ and $n\varphi=w$ in the above inequality, we obtain the Turán-type inequality for generalized complete elliptic integrals of the first kind as follows:

$$\begin{aligned}&\left[{}_2F_1\left(\begin{matrix}1-\varphi,\ \varphi\\ 1\end{matrix}\middle| x^2\right)\right]^2\\&\le \frac{[\Gamma(1)]^2\,\Gamma[1-\varphi-\lambda]\,\Gamma[\varphi-\mu]\,\Gamma[1-\varphi+\lambda]\,\Gamma[\varphi+\mu]}{[\Gamma(1-\varphi)]^2\,[\Gamma(\varphi)]^2\,\Gamma[1-(\lambda+\mu)]\,\Gamma[1+(\lambda+\mu)]}\\&\quad\times {}_2F_1\left(\begin{matrix}1-\varphi-\lambda,\ \varphi-w\\ 1-(\lambda+\mu)\end{matrix}\middle| x^2\right){}_2F_1\left(\begin{matrix}1-\varphi+\lambda,\ \varphi+w\\ 1+(\lambda+\mu)\end{matrix}\middle| x^2\right).\end{aligned}\tag{24}$$

From the definition of generalized complete elliptic integrals of first kind $\kappa_\varphi(x)$ and relation (11), we get

$$\begin{aligned}[\kappa_\varphi(x)]^2 &\le \frac{\pi^2}{4}\times\frac{B(1-\varphi-\lambda,\varphi-\mu)\ B(1-\varphi+\lambda,\varphi+\mu)}{[B(1-\varphi,\ \varphi)]^2}\\&\quad\times {}_2F_1\left(\begin{matrix}1-\varphi-\lambda,\ \varphi-w\\ 1-(\lambda+\mu)\end{matrix}\middle| x^2\right){}_2F_1\left(\begin{matrix}1-\varphi+\lambda,\ \varphi+w\\ 1+(\lambda+\mu)\end{matrix}\middle| x^2\right),\end{aligned}\tag{25}$$

provided that $x\in(0,1)\,;1-\varphi>|\lambda|$ and $\varphi>|\mu|$.

By a similar approach, substituting $g(t)=t^{-\varphi},\ h(t)=(1-t)^{\varphi-1}$ and $f(t)=\left(1-x^2t\right)^{-(\varphi-1)}$ in inequality (5) for $[a,b]=[0,1]$, we obtain

$$\begin{aligned}&\left(\int_0^1 t^{-\xi\varphi}(1-t)^{\xi(\varphi-1)}\left(1-x^2t\right)^{-\xi(\varphi-1)}dt\right)^2\\&\le\left(\int_0^1 t^{-(\xi-l)\varphi}(1-t)^{(\xi-m)(\varphi-1)}\left(1-x^2t\right)^{-(\xi-n)(\varphi-1)}dt\right)\\&\quad\times\left(\int_0^1 t^{-(\xi+l)\varphi}(1-t)^{(\xi+m)(\varphi-1)}\left(1-x^2t\right)^{-(\xi+n)(\varphi-1)}dt\right).\end{aligned}$$

Corresponding to relation (8), after simplification eventually yields the following inequality

$$\left[{}_2F_1\left(\begin{matrix}1-\xi\varphi,\ \xi(\varphi-1)\\ 2-\xi\end{matrix}\middle| x^2\right)\right]^2$$

$$
\begin{aligned}
&\le \frac{[\Gamma(2-\xi)]^2}{[\Gamma(1-\xi\varphi)]^2[\Gamma\{\xi(\varphi-1)+1\}]^2} \\
&\quad\times \frac{\Gamma[1-(\xi-l)\varphi]\Gamma[(\xi-m)(\varphi-1)+1]\Gamma[1-(\xi+l)\varphi]\Gamma[(\xi+m)(\varphi-1)+1]}{\Gamma[2-(\xi-l)\varphi+(\xi-m)(\varphi-1)]\Gamma[2-(\xi+l)\varphi+(\xi+m)(\varphi-1)]} \\
&\quad\times {}_2F_1\left(\begin{matrix}1-(\xi-l)\varphi,\ (\xi-n)(\varphi-1)\\ 2-(\xi-l)\varphi+(\xi-m)(\varphi-1)\end{matrix}\middle| x^2\right) \\
&\quad\times {}_2F_1\left(\begin{matrix}1-(\xi+l)\varphi,\ (\xi+n)(\varphi-1)\\ 2-(\xi+l)\varphi+(\xi+m)(\varphi-1)\end{matrix}\middle| x^2\right). \qquad (26)
\end{aligned}
$$

Hence, if we take $\nu_1 = 1-\xi\varphi,\ \nu_2 = \xi(\varphi-1)+1,\ \nu_3 = \xi(\varphi-1)$ and $\mu_1 = -l\varphi,\ \mu_2 = m(\varphi-1),\ \mu_3 = n(\varphi-1)$ in inequality (26), then we get Turán-type inequality for the class of generalized complete elliptic integral of second kind as follows:

$$
\begin{aligned}
\left[{}_2F_1\left(\begin{matrix}\nu_1,\ \nu_3\\ \nu_1+\nu_2\end{matrix}\middle| x^2\right)\right]^2 &\le \frac{[\Gamma(\nu_1+\nu_2)]^2}{[\Gamma(\nu_1)]^2[\Gamma(\nu_2)]^2}\frac{\Gamma(\nu_1-\mu_1)\Gamma(\nu_2-\mu_2)}{\Gamma[(\nu_1-\mu_1)+(\nu_2-\mu_2)]} \\
&\quad\times \frac{\Gamma(\nu_1+\mu_1)\Gamma(\nu_2+\mu_2)}{\Gamma[(\nu_1+\mu_1)+(\nu_2+\mu_2)]} \\
&\quad\times {}_2F_1\left(\begin{matrix}\nu_1-\mu_1,\ \nu_3-\mu_3\\ (\nu_1+\nu_2)-(\mu_1+\mu_2)\end{matrix}\middle| x^2\right) \\
&\quad\times {}_2F_1\left(\begin{matrix}\nu_1+\mu_1,\ \nu_3+\mu_3\\ (\nu_1+\nu_2)+(\mu_1+\mu_2)\end{matrix}\middle| x^2\right). \qquad (27)
\end{aligned}
$$

Or equivalently

$$
\begin{aligned}
\left[{}_2F_1\left(\begin{matrix}\nu_1,\ \nu_3\\ \nu_1+\nu_2\end{matrix}\middle| x^2\right)\right]^2 &\le \frac{B(\nu_1-\mu_1,\nu_2-\mu_2)B(\nu_1+\mu_1,\nu_2+\mu_2)}{[B(\nu_1,\nu_2)]^2} \\
&\quad\times {}_2F_1\left(\begin{matrix}\nu_1-\mu_1,\ \nu_3-\mu_3\\ (\nu_1+\nu_2)-(\mu_1+\mu_2)\end{matrix}\middle| x^2\right) \\
&\quad\times {}_2F_1\left(\begin{matrix}\nu_1+\mu_1,\ \nu_3+\mu_3\\ (\nu_1+\nu_2)+(\mu_1+\mu_2)\end{matrix}\middle| x^2\right), \qquad (28)
\end{aligned}
$$

provided that $x\in(0,1)\,;\nu_1>|\mu_1|$ and $\nu_2>|\mu_2|$.

Also when $\xi = 1$, we get the following inequality for the function $E_\varphi(x)$ from the inequality (26)

$$
\begin{aligned}
&\left[{}_2F_1\left(\begin{matrix}1-\varphi,\ \varphi-1\\ 1\end{matrix}\middle| x^2\right)\right]^2 \\
&\le \frac{[\Gamma(1)]^2\Gamma[1-\varphi+l\varphi]\Gamma[\varphi-m(\varphi-1)]\Gamma[1-\varphi-l\varphi]\Gamma[\varphi+m(\varphi-1)]}{[\Gamma(1-\varphi)]^2[\Gamma(\varphi)]^2\Gamma[1+l\varphi-m(\varphi-1)]\Gamma[1-l\varphi+m(\varphi-1)]} \\
&\quad\times {}_2F_1\left(\begin{matrix}1-\varphi+l\varphi,\ (1-n)(\varphi-1)\\ 1+l\varphi-m(\varphi-1)\end{matrix}\middle| x^2\right) {}_2F_1\left(\begin{matrix}1-\varphi-l\varphi,\ (1+n)(\varphi-1)\\ 1-l\varphi+m(\varphi-1)\end{matrix}\middle| x^2\right).
\end{aligned}
$$

For $(-l\varphi) = \lambda,\ m(\varphi-1) = \mu$ and $n(\varphi-1) = w$, the above inequality reduces to the Turán-type inequality for generalized complete elliptic integrals of second kind as follows:

$$\begin{aligned} &\left[{}_2F_1\left(\begin{matrix}1-\varphi,\ \varphi-1\\ 1\end{matrix}\middle| x^2\right)\right]^2 \\ &\leq \frac{[\Gamma(1)]^2\,\Gamma[1-\varphi-\lambda]\,\Gamma[\varphi-\mu]\,\Gamma[1-\varphi+\lambda]\,\Gamma[\varphi+\mu]}{[\Gamma(1-\varphi)]^2\,[\Gamma(\varphi)]^2\,\Gamma[1-(\lambda+\mu)]\,\Gamma[1+(\lambda+\mu)]} \\ &\quad\times {}_2F_1\left(\begin{matrix}1-\varphi-\lambda,\ (\varphi-1)-w\\ 1-(\lambda+\mu)\end{matrix}\middle| x^2\right){}_2F_1\left(\begin{matrix}1-\varphi+\lambda,\ (\varphi-1)+w\\ 1+(\lambda+\mu)\end{matrix}\middle| x^2\right). \end{aligned} \tag{29}$$

From the definition of generalized complete elliptic integrals of second kind $E_\varphi(x)$ and relation (11), we obtain

$$\begin{aligned} [E_\varphi(x)]^2 &\leq \frac{\pi^2}{4} \times \frac{B(1-\varphi-\lambda,\varphi-\mu)\,B(1-\varphi+\lambda,\varphi+\mu)}{[B(1-\varphi,\ \varphi)]^2} \\ &\quad\times {}_2F_1\left(\begin{matrix}1-\varphi-\lambda,\ (\varphi-1)-w\\ 1-(\lambda+\mu)\end{matrix}\middle| x^2\right) \\ &\quad\times {}_2F_1\left(\begin{matrix}1-\varphi+\lambda,\ (\varphi-1)+w\\ 1+(\lambda+\mu)\end{matrix}\middle| x^2\right), \end{aligned} \tag{30}$$

provided that $x \in (0,1)\,; 1-\varphi > |\lambda|$ and $\varphi > |\mu|$.

In the particular case when $\varphi = 1/2$, the functions $\kappa_\varphi(x)$ and $E_\varphi(x)$ reduces to the elliptic integrals of the first and second kind $\kappa_{1/2}(x)$ and $E_{1/2}(x)$ respectively.

On replacing $\varphi = 1/2$ with $\lambda = -l/2,\ \mu = -m/2$ and $w = n/2$ in inequality (24), then we get

$$\begin{aligned} \left[{}_2F_1\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ 1\end{matrix}\middle| x^2\right)\right]^2 &\leq \frac{\Gamma\left(\frac{1+l}{2}\right)\Gamma\left(\frac{1+m}{2}\right)\Gamma\left(\frac{1-l}{2}\right)\Gamma\left(\frac{1-m}{2}\right)}{\left[\Gamma\left(\frac{1}{2}\right)\right]^4\Gamma\left[1+\frac{(l+m)}{2}\right]\Gamma\left[1-\frac{(l+m)}{2}\right]} \\ &\quad\times {}_2F_1\left(\begin{matrix}\frac{1+l}{2},\ \frac{1-n}{2}\\ 1+\frac{(l+m)}{2}\end{matrix}\middle| x^2\right){}_2F_1\left(\begin{matrix}\frac{1-l}{2},\ \frac{1+n}{2}\\ 1-\frac{(l+m)}{2}\end{matrix}\middle| x^2\right). \end{aligned}$$

Applying the well-known identities (17) and

$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}\ ,\ 0 < p < 1, \tag{31}$$

with $\Gamma(1/2) = \sqrt{\pi}$, we have

$$\begin{aligned} \left[\kappa_{1/2}(x)\right]^2 &\leq \frac{\pi}{2} \times \frac{\sin\left[(l+m)\,\frac{\pi}{2}\right]}{(l+m)\sin\left[(1-l)\,\frac{\pi}{2}\right]\sin\left[(1-m)\,\frac{\pi}{2}\right]} \\ &\quad\times {}_2F_1\left(\begin{matrix}\frac{1+l}{2},\ \frac{1-n}{2}\\ 1+\frac{(l+m)}{2}\end{matrix}\middle| x^2\right){}_2F_1\left(\begin{matrix}\frac{1-l}{2},\ \frac{1+n}{2}\\ 1-\frac{(l+m)}{2}\end{matrix}\middle| x^2\right), \end{aligned} \tag{32}$$

where $x \in (0,1)\,; l, m \in (-1,1)$ and $n \in \mathbb{R}$.

By a similar approach, on substituting $\varphi = 1/2$, with $\lambda = -l/2$, $\mu = -m/2$ and $w = -n/2$ in inequality (29), then one finally obtains

$$\begin{aligned}\left[E_{1/2}(x)\right]^2 \le \frac{\pi}{2} \times & \frac{\sin\left[(l+m)\frac{\pi}{2}\right]}{(l+m)\sin\left[(1-l)\frac{\pi}{2}\right]\sin\left[(1-m)\frac{\pi}{2}\right]} \\ & \times {}_2F_1\left(\begin{matrix}\frac{1+l}{2},\ -\left(\frac{1-n}{2}\right) \\ 1+\frac{(l+m)}{2}\end{matrix}\middle| x^2\right) {}_2F_1\left(\begin{matrix}\frac{1-l}{2},\ -\left(\frac{1+n}{2}\right) \\ 1-\frac{(l+m)}{2}\end{matrix}\middle| x^2\right), \end{aligned} \tag{33}$$

where $x \in (0,1)\,; l, m \in (-1,1)$ and $n \in \mathbb{R}$.

2.3. Turán-type inequalities for confluent hypergeometric functions

Due to the importance of these functions, let us consider the confluent differential equation:

$$x\frac{d^2y}{dx^2} + (\gamma - x)\frac{dy}{dx} - \alpha y(x) = 0, \tag{34}$$

where α and γ are constant parameters. One of the basic solutions of the confluent differential equation is the hypergeometric function of order (1, 1), i.e.,

$$y_1(x) = {}_1F_1\left(\begin{matrix}\alpha \\ \gamma\end{matrix}\middle| x\right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\gamma)_k}\frac{x^k}{k!}, \tag{35}$$

in which $\gamma \ne 0, -1, -2, \ldots, (\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$ and the series converges for $-1 \le x \le 1$.

This Taylor series at $x = 0$ is known as the confluent hypergeometric function of first kind [15] and has a specific integral representation as

$$\begin{aligned}{}_1F_1\left(\begin{matrix}\alpha \\ \gamma\end{matrix}\middle| x\right) = & \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 e^{xt}t^{\alpha-1}(1-t)^{\gamma-\alpha-1}\,dt, \\ & (\gamma > \alpha > 0; |x| \le 1), \end{aligned} \tag{36}$$

which we apply in inequality (5) to obtain a new inequality for the class of confluent hypergeometric functions of first kind. For this purpose, let us first replace $g(t) = t^{\alpha-1}$, $h(t) = (1-t)^{\gamma-\alpha-1}$ and $f(t) = e^{xt}$ in inequality (5) for $[a,b] = [0,1]$ to reach

$$\begin{aligned} & \left(\int_0^1 t^{\xi(\alpha-1)}(1-t)^{\xi(\gamma-\alpha-1)} e^{\xi x t}dt\right)^2 \\ \le & \left(\int_0^1 t^{(\xi-l)(\alpha-1)}(1-t)^{(\xi-m)(\gamma-\alpha-1)} e^{(\xi-n)xt}dt\right) \\ & \times \left(\int_0^1 t^{(\xi+l)(\alpha-1)}(1-t)^{(\xi+m)(\gamma-\alpha-1)} e^{(\xi+n)xt}dt\right). \end{aligned}$$

Therefore, according to (36), we obtain

$$\begin{aligned}
&\left[{}_1F_1\left(\begin{matrix}\xi\,(\alpha-1)+1\\ \xi\,(\gamma-2)+2\end{matrix}\middle|\xi x\right)\right]^2\\
&\le \frac{\left[\Gamma\left\{\xi\,(\gamma-2)+2\right\}\right]^2}{\left[\Gamma\left\{\xi\,(\alpha-1)+1\right\}\right]^2\left[\Gamma\left\{\xi\,(\gamma-\alpha-1)+1\right\}\right]^2}\\
&\quad\times\frac{\Gamma\left[(\xi-l)\,(\alpha-1)+1\right]\Gamma\left[(\xi-m)\,(\gamma-\alpha-1)+1\right]}{\Gamma\left[(\xi-l)\,(\alpha-1)+(\xi-m)\,(\gamma-\alpha-1)+2\right]}\\
&\quad\times\frac{\Gamma\left[(\xi+l)\,(\alpha-1)+1\right]\Gamma\left[(\xi+m)\,(\gamma-\alpha-1)+1\right]}{\Gamma\left[(\xi+l)\,(\alpha-1)+(\xi+m)\,(\gamma-\alpha-1)+2\right]}\\
&\quad\times{}_1F_1\left(\begin{matrix}(\xi-l)\,(\alpha-1)+1\\ (\xi-l)\,(\alpha-1)+(\xi-m)\,(\gamma-\alpha-1)+2\end{matrix}\middle|\,(\xi-n)\,x\right)\\
&\quad\times{}_1F_1\left(\begin{matrix}(\xi+l)\,(\alpha-1)+1\\ (\xi+l)\,(\alpha-1)+(\xi+m)\,(\gamma-\alpha-1)+2\end{matrix}\middle|\,(\xi+n)\,x\right).
\end{aligned}\tag{37}$$

For $\xi=1$, inequality (37) is transformed to the following inequality for confluent hypergeometric function of first kind:

$$\begin{aligned}
&\left[{}_1F_1\left(\begin{matrix}\alpha\\ \gamma\end{matrix}\middle|x\right)\right]^2\\
&\le\frac{\left[\Gamma\left(\gamma\right)\right]^2}{\left[\Gamma\left(\alpha\right)\right]^2\left[\Gamma\left(\gamma-\alpha\right)\right]^2}\\
&\quad\times\frac{\Gamma\left[\alpha-l\,(\alpha-1)\right]\Gamma\left[(\gamma-\alpha)-m\,(\gamma-\alpha-1)\right]\Gamma\left[\alpha+l\,(\alpha-1)\right]\Gamma\left[(\gamma-\alpha)+m\,(\gamma-\alpha-1)\right]}{\Gamma\left[\gamma-l\,(\alpha-1)-m\,(\gamma-\alpha-1)\right]\Gamma\left[\gamma+l\,(\alpha-1)+m\,(\gamma-\alpha-1)\right]}\\
&\quad\times{}_1F_1\left(\begin{matrix}\alpha-l\,(\alpha-1)\\ \gamma-l\,(\alpha-1)-m\,(\gamma-\alpha-1)\end{matrix}\middle|\,(1-n)\,x\right)\\
&\quad\times{}_1F_1\left(\begin{matrix}\alpha+l\,(\alpha-1)\\ \gamma+l\,(\alpha-1)+m\,(\gamma-\alpha-1)\end{matrix}\middle|\,(1+n)\,x\right).
\end{aligned}\tag{38}$$

If we put $l\,(\alpha-1)=\lambda$, $m\,(\gamma-\alpha-1)=\mu$ and $n=0$ in inequality (38), we obtain the Turán-type inequality for confluent hypergeometric functions of first kind as follows:

$$\begin{aligned}
\left[{}_1F_1\left(\begin{matrix}\alpha\\ \gamma\end{matrix}\middle|x\right)\right]^2 &\le\frac{\left[\Gamma\left(\gamma\right)\right]^2\Gamma\left[\alpha-\lambda\right]\Gamma\left[(\gamma-\alpha)-\mu\right]\Gamma\left[\alpha+\lambda\right]\Gamma\left[(\gamma-\alpha)+\mu\right]}{\left[\Gamma\left(\alpha\right)\right]^2\left[\Gamma\left(\gamma-\alpha\right)\right]^2\Gamma\left[\gamma-(\lambda+\mu)\right]\Gamma\left[\gamma+(\lambda+\mu)\right]}\\
&\quad\times{}_1F_1\left(\begin{matrix}\alpha-\lambda\\ \gamma-(\lambda+\mu)\end{matrix}\middle|x\right){}_1F_1\left(\begin{matrix}\alpha+\lambda\\ \gamma+(\lambda+\mu)\end{matrix}\middle|x\right).
\end{aligned}\tag{39}$$

Or equivalently

$$\begin{aligned}
\left[{}_1F_1\left(\begin{matrix}\alpha\\ \gamma\end{matrix}\middle|x\right)\right]^2 &\le\frac{B\left[\alpha-\lambda,(\gamma-\alpha)-\mu\right]B\left[\alpha+\lambda,(\gamma-\alpha)+\mu\right]}{\left[B\left(\alpha,\ \gamma-\alpha\right)\right]^2}\\
&\quad\times{}_1F_1\left(\begin{matrix}\alpha-\lambda\\ \gamma-(\lambda+\mu)\end{matrix}\middle|x\right){}_1F_1\left(\begin{matrix}\alpha+\lambda\\ \gamma+(\lambda+\mu)\end{matrix}\middle|x\right),
\end{aligned}\tag{40}$$

provided that $\alpha > |\lambda|$, $\gamma - \alpha > |\mu|$ and $|x| \leq 1$.

The second basic solution of equation (34) is known as the confluent hypergeometric function of second kind [15] and usually given by

$$y_2(x) = \phi(\alpha;\gamma;x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-xt} t^{\alpha-1} (1+t)^{\gamma-\alpha-1}\, dt,\ (\alpha > 0; x > 0). \tag{41}$$

As well, now again replace $g(t) = t^{\alpha-1}$, $h(t) = (1+t)^{\gamma-\alpha-1}$, $f(t) = e^{-xt}$ and $[a,b] \to [0,\infty)$ in inequality (5) to reach

$$\begin{aligned} &\left(\int_0^\infty t^{\xi(\alpha-1)} (1+t)^{\xi(\gamma-\alpha-1)} e^{-\xi xt} dt\right)^2 \\ \leq &\left(\int_0^\infty t^{(\xi-l)(\alpha-1)} (1+t)^{(\xi-m)(\gamma-\alpha-1)} e^{-(\xi-n)xt} dt\right) \\ \times &\left(\int_0^\infty t^{(\xi+l)(\alpha-1)} (1+t)^{(\xi+m)(\gamma-\alpha-1)} e^{-(\xi+n)xt} dt\right). \end{aligned}$$

Corresponding to definition (41), the following result will eventually be obtained

$$\begin{aligned} &[\phi(\xi(\alpha-1)+1; \xi(\gamma-2)+2; \xi x)]^2 \\ \leq &\frac{\Gamma[(\xi-l)(\alpha-1)+1]\,\Gamma[(\xi+l)(\alpha-1)+1]}{[\Gamma\{\xi(\alpha-1)+1\}]^2} \\ &\times \phi[(\xi-l)(\alpha-1)+1; (\xi-l)(\alpha-1)+(\xi-m)(\gamma-\alpha-1)+2; (\xi-n)x] \\ &\times \phi[(\xi+l)(\alpha-1)+1; (\xi+l)(\alpha-1)+(\xi+m)(\gamma-\alpha-1)+2; (\xi+n)x]. \end{aligned} \tag{42}$$

In the particular case when $\xi = 1$, the inequality (42) reduces to the following inequality for the confluent hypergeometric function of second kind

$$\begin{aligned} [\phi(\alpha;\gamma;x)]^2 \leq &\frac{\Gamma[\alpha - l(\alpha-1)]\,\Gamma[\alpha + l(\alpha-1)]}{[\Gamma(\alpha)]^2} \\ &\times \phi[\alpha - l(\alpha-1); \gamma - l(\alpha-1) - m(\gamma-\alpha-1); (1-n)x] \\ &\times \phi[\alpha + l(\alpha-1); \gamma + l(\alpha-1) + m(\gamma-\alpha-1); (1+n)x]. \end{aligned} \tag{43}$$

If we put $l(\alpha-1) = \lambda,\ m(\gamma-\alpha-1) = \mu$ and $n = 0$ in inequality (43), we obtain the Turán-type inequality for confluent hypergeometric functions of second kind as follows:

$$\begin{aligned} [\phi(\alpha;\gamma;x)]^2 \leq &\frac{\Gamma(\alpha-\lambda)\,\Gamma(\alpha+\lambda)}{[\Gamma(\alpha)]^2} \\ &\times \phi[\alpha-\lambda; \gamma-(\lambda+\mu); x] \times \phi[\alpha+\lambda; \gamma+(\lambda+\mu); x], \end{aligned} \tag{44}$$

provided that $\alpha > |\lambda|$ and $x > 0$.

Example 2 (Turán-type inequalities for the lower incomplete gamma function)**.** The lower incomplete Euler's gamma function is defined for $u, x > 0$

as

$$\Gamma(u;x)=\int_0^x e^{-t}t^{u-1}dt,\ \ (u,x>0)\,. \tag{45}$$

This function can be written in terms of the first kind of confluent hypergeometric function as

$$\Gamma(u;x)=\int_0^x e^{-t}t^{u-1}dt=u^{-1}x^u{}_1F_1\left(\begin{matrix}u\\u+1\end{matrix}\Big|-x\right),\ \ (u,x>0)\,. \tag{46}$$

If $\alpha=u,\ \gamma=u+1$ with $l(u-1)=\lambda$ and $\mu=0$ are substituted in inequality (39), then one finally gets

$$\begin{aligned}\left[{}_1F_1\left(\begin{matrix}u\\u+1\end{matrix}\Big|-x\right)\right]^2 &\le \frac{[\Gamma(u+1)]^2\,\Gamma(u-\lambda)\,\Gamma(u+\lambda)}{[\Gamma(u)]^2\,\Gamma(u-\lambda+1)\,\Gamma(u+\lambda+1)}\\ &\quad\times{}_1F_1\left(\begin{matrix}u-\lambda\\u+1-\lambda\end{matrix}\Big|-x\right){}_1F_1\left(\begin{matrix}u+\lambda\\u+1+\lambda\end{matrix}\Big|-x\right).\end{aligned} \tag{47}$$

This means that according to (46), we have

$$\begin{aligned}\frac{u^2}{x^{2u}}\left[\Gamma(u;x)\right]^2 &\le \frac{[\Gamma(u+1)]^2\,\Gamma(u-\lambda)\,\Gamma(u+\lambda)}{[\Gamma(u)]^2\,\Gamma(u-\lambda+1)\,\Gamma(u+\lambda+1)}\\ &\quad\times\frac{(u-\lambda)}{x^{u-\lambda}}\Gamma(u-\lambda;x)\times\frac{(u+\lambda)}{x^{u+\lambda}}\Gamma(u+\lambda;x)\,.\end{aligned}$$

Now applying the identity (17), we have the Turán-type inequality for lower incomplete gamma function as follows:

$$\left[\Gamma(u;x)\right]^2\le\Gamma(u-\lambda;x)\,\Gamma(u+\lambda;x)\,,\ \ u>\lambda>0,\ \ x>0. \tag{48}$$

Example 3 (Turán-type inequalities for various kinds of modified Bessel functions)**.** It is known that the modified Bessel functions $I_\upsilon(x)$ and $K_\upsilon(x)$ can respectively be represented in the terms of the first and second kind of confluent hypergeometric functions by the following relations for $x>0$ and $\upsilon>-1/2$ (see for instant; [15]):

$$I_\upsilon(x)=I(\upsilon;x)=\frac{e^{-x}\left(\frac{x}{2}\right)^\upsilon}{\Gamma(\upsilon+1)}{}_1F_1\left(\begin{matrix}\upsilon+\frac{1}{2}\\2\upsilon+1\end{matrix}\Big|2x\right), \tag{49}$$

$$K_\upsilon(x)=K(\upsilon;x)=\sqrt{\pi}e^{-x}(2x)^\upsilon\,\phi\left(\upsilon+\frac{1}{2};2\upsilon+1;2x\right). \tag{50}$$

If we put $\alpha=\ \upsilon+1/2$, $\gamma=2\upsilon+1$ with $l(\upsilon-1/2)=\lambda$ and $m(\upsilon-1/2)=\mu$ in inequality (39), we obtain the Turán-type inequality for modified Bessel functions of first kind $I_\upsilon(x)$ as follows:

$$\left[{}_1F_1\left(\begin{matrix}\upsilon+\frac{1}{2}\\2\upsilon+1\end{matrix}\Big|2x\right)\right]^2$$

$$\leq \frac{[\Gamma(2\upsilon+1)]^2\,\Gamma\left(\upsilon+\frac{1}{2}-\lambda\right)\Gamma\left(\upsilon+\frac{1}{2}-\mu\right)\Gamma\left(\upsilon+\frac{1}{2}+\lambda\right)\Gamma\left(\upsilon+\frac{1}{2}+\mu\right)}{\left[\Gamma\left(\upsilon+\frac{1}{2}\right)\right]^2\left[\Gamma\left(\upsilon+\frac{1}{2}\right)\right]^2\Gamma\left[2\upsilon+1-(\lambda+\mu)\right]\Gamma\left[2\upsilon+1+(\lambda+\mu)\right]}$$

$$\times\,{}_1F_1\left(\begin{matrix}\upsilon+\frac{1}{2}-\lambda\\ 2\upsilon+1-(\lambda+\mu)\end{matrix}\middle|2x\right){}_1F_1\left(\begin{matrix}\upsilon+\frac{1}{2}+\lambda\\ 2\upsilon+1+(\lambda+\mu)\end{matrix}\middle|2x\right). \tag{51}$$

The above inequality (51) can also be expressed as

$$\left[{}_1F_1\left(\begin{matrix}\upsilon+\frac{1}{2}\\ 2\upsilon+1\end{matrix}\middle|2x\right)\right]^2 \leq \frac{B\left(\upsilon+\frac{1}{2}-\lambda,\upsilon+\frac{1}{2}-\mu\right)B\left(\upsilon+\frac{1}{2}+\lambda,\upsilon+\frac{1}{2}+\mu\right)}{\left[B\left(\upsilon+\frac{1}{2},\upsilon+\frac{1}{2}\right)\right]^2}$$

$$\times\,{}_1F_1\left(\begin{matrix}\upsilon+\frac{1}{2}-\lambda\\ 2\upsilon+1-(\lambda+\mu)\end{matrix}\middle|2x\right)$$

$$\times\,{}_1F_1\left(\begin{matrix}\upsilon+\frac{1}{2}+\lambda\\ 2\upsilon+1+(\lambda+\mu)\end{matrix}\middle|2x\right). \tag{52}$$

For $\lambda = \mu$ and corresponding to definition (49), the following result will eventually be obtained

$$[I_\upsilon(x)]^2 \leq \frac{B\left(\upsilon+\frac{1}{2}-\lambda,\upsilon+\frac{1}{2}-\lambda\right)B\left(\upsilon+\frac{1}{2}+\lambda,\upsilon+\frac{1}{2}+\lambda\right)}{\left[B\left(\upsilon+\frac{1}{2},\upsilon+\frac{1}{2}\right)\right]^2}$$

$$\times\left[1-\left(\frac{\lambda}{\upsilon}\right)^2\right]\times\frac{\Gamma(\upsilon-\lambda)\Gamma(\upsilon+\lambda)}{[\Gamma(\upsilon)]^2}\times I_{\upsilon-\lambda}(x)\;\; I_{\upsilon+\lambda}(x), \tag{53}$$

provided that $\upsilon > |\lambda|$ and $x > 0$.

By a similar approach, substituting $\alpha = \upsilon + 1/2$ and $\gamma = 2\upsilon + 1$ with $l(\upsilon - 1/2) = \lambda$ and $m(\upsilon - 1/2) = \mu$ in inequality (44), we obtain the Turán-type inequality for modified Bessel functions of second kind $K_\upsilon(x)$ as follows:

$$\left[\phi\left(\upsilon+\frac{1}{2};\; 2\upsilon+1; 2x\right)\right]^2 \leq \frac{\Gamma\left(\upsilon+\frac{1}{2}-\lambda\right)\Gamma\left(\upsilon+\frac{1}{2}+\lambda\right)}{\left[\Gamma\left(\upsilon+\frac{1}{2}\right)\right]^2}$$

$$\times\,\phi\left[\upsilon+\frac{1}{2}-\lambda; 2\upsilon+1-(\lambda+\mu); 2x\right]$$

$$\times\,\phi\left[\upsilon+\frac{1}{2}+\lambda; 2\upsilon+1+(\lambda+\mu); 2x\right].$$

This means that according to (50) and for $\lambda = \mu$, we have

$$[K_\upsilon(x)]^2 \leq \frac{\Gamma\left(\upsilon+\frac{1}{2}-\lambda\right)\Gamma\left(\upsilon+\frac{1}{2}+\lambda\right)}{\left[\Gamma\left(\upsilon+\frac{1}{2}\right)\right]^2}\times K_{\upsilon-\lambda}(x)\,K_{\upsilon+\lambda}(x), \tag{54}$$

provided that $\upsilon + \frac{1}{2} > |\lambda|$ and $x > 0$.

Acknowledgements. The authors appreciate and would like to thank the anonymous referee for his valuable comments and suggestions, which have led to a better presentation of the paper in its present form.

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