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STRONG CONVERGENCE OF MODIFIED ISHIKAWA ITERATES FOR ASYMPTOTICALLY NONEXPANSIVE MAPS WITH NEW CONTROL CONDITIONS

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Abstract. In this paper, we establish strong convergence of the modified Ishikawa iterates of an asymptotically non expansive self-mapping of a nonempty closed bounded and convex subset of a uniformly convex Banach space under a variety of new control conditions.

1. Introduction and preliminaries

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [5] in 1972, they proved that every asymptotically nonexpansive self-mapping of a nonempty closed bounded and convex subset of a uniformly convex Banach space has a fixed point. In the past few decades fixed point iterations of Mann and Ishikawa schemes have been extensively studied by various authors to approximate fixed points of nonexpansive and asymptotically nonexpansive mappings. Mann and Ishikawa process were first studied for nonexpansive operators and later it was modified to study the convergence of fixed points of asymptotically nonexpansive mappings see [2–4, 6–10, 12].

In all these results the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are required to be bounded away from 0 and 1. Our objective is to show that the strong convergence is still true when α_n is allowed to approach 0 or 1 and β_n is allowed to approach 0 and thereby extending the validity of Mann's and Ishikwa iteration scheme. In particular we show that if E is uniformly convex, C a nonempty closed convex and bounded subset of E and $T\colon C \to C$ is a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and the sequence $\{x_n\}$ is defined by the Mann's iteration

$$x_{n+1} = \frac{1}{n}T^n x_n + (1 - \frac{1}{n})x_n$$

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or the Ishikawa iteration

$$\begin{cases} x_{n+1} = (1 - \frac{1}{n})T^n y_n + \frac{1}{n} x_n; \\ y_n = \frac{1}{n} T^n x_n + (1 - \frac{1}{n})x_n, \end{cases}$$

then $\{x_n\}$ strongly converges to a fixed point of T.

Let us see some basic concepts and results related to our work.

Let E be a real Banach space and let C be a nonempty closed convex subset of E. A self mapping $T\colon C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C \tag{1}$$

and asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \ \forall x, y \in C \ \text{ and } \ n \geq 1. \tag{2}$$

Let C be a nonempty convex subset of a normed linear space E, $T\colon C \to C$ a mapping and $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$. The sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$\begin{cases} x_0 \in C; \\ x_{n+1} = \alpha_n T^n y_n + (1 - \alpha_n)x_n; \\ y_n = \beta_n T^n x_n + (1 - \beta_n)x_n, \end{cases} \tag{3}$$

is called the modified Ishikawa iterative process.

The Ishikawa iteration was first introduced by Ishikawa for the class of Lipschitzian pseudo-contractive operators. Under certain assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, the ishikawa iterative process associated with a Lipschitizian pseudocontractive operator converges strongly to a fixed point of T. The result of Ishikawa is stated as follows:

Theorem 1.1 ([7]). *If C is a convex compact subset of a Hilbert space H, T is a lipschitizian pseudo-contractive map from C into itself and x_1 is any point in C, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T, where x_n is defined iteratively for each positive integer n by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T[(1 - \beta_n)x_n + \beta_n T x_n],$$

where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are sequences of positive numbers satisfy the following three conditions:

(i) $0 \leq \alpha_n \leq \beta_n \leq 1, \quad n \geq 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$,

In [9, 10], Schu introduced a modified Mann process to approximate fixed points of asymptotically nonexpansive self-maps defined on nonempty closed convex and bounded subsets of a Hilbert space H.

Theorem 1.2 ([9]). *Let H be a Hilbert space, C a nonempty closed convex and bounded subset of H. Let $T\colon C \to C$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ for all $n \geq 1$, $\lim k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $[0, 1]$ satisfying the condition $\epsilon \leq \alpha_n \leq 1 - \epsilon$ for all $n \geq 1$ and for some $\epsilon > 0$. Then the sequence $\{x_n\}$ generated from arbitrary $x_1 \in C$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1,$$

converges strongly to some fixed point of T.

In [11], Rhoades extended the theorem of Schu to uniformly convex Banach space using modified Ishikawa iteration scheme. The following are the main results of [11].

Theorem 1.3 ([11]). *Let E be uniformly convex, C a nonempty closed convex and bounded subset of E. Let $T\colon C \to C$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$, $r = \max\{2, p\}$; $\epsilon \leq \alpha_n \leq 1 - \epsilon$ for all n and for some $\epsilon > 0$. Choose $x_0 \in C$ by*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 0.$$

Then the sequence $\{x_n\}$ converges strongly to some fixed point of T.

Theorem 1.4. ([11]) *Let E be uniformly convex, C a nonempty closed convex and bounded subset of E. Let $T\colon C \to C$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$, $r = \max\{2, p\}$. Define $\{\alpha_n\}$, $\{\beta_n\}$ to satisfy $\epsilon \leq (1 - \alpha_n)$, $(1 - \beta_n) \leq 1 - \epsilon$ for all n and for some $\epsilon > 0$. Define*

$$\begin{aligned} x_{n+1} &= \alpha_n T^n y_n + (1 - \alpha_n)x_n; \\ y_n &= \beta_n T^n x_n + (1 - \beta_n)x_n. \end{aligned}$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

The following definitions and results will be used in our main results.

Definition 1.5. A Banach space X is said to be

(i) uniformly convex if there exists a strictly increasing function $\delta : (0, 2] \to [0, 1]$ such that for every $x, y, p \in X$, $R > 0$ and $r \in [0, 2R]$, the following implication holds:

$$\begin{cases} \|x - p\| \leq R, \\ \|y - p\| \leq R, \\ \|x - y\| \geq r \end{cases} \Rightarrow \left\| \frac{x + y}{2} - p \right\| \leq (1 - \delta(\frac{r}{R}))R;$$

(ii) strictly convex if for every $x, y, p \in X$ and $R > 0$, the following implication holds:

$$\begin{cases} \|x-p\| \leq R, \\ \|y-p\| \leq R, \\ x \neq y \end{cases} \Rightarrow \left\| \frac{x+y}{2} - p \right\| < R.$$

Lemma 1.6 ([13])**.** *Let $r > 0$ be a fixed real number. Then a Banach space E is uniformly convex if and only if there is a continuous strictly increasing convex map $g\colon [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that for all $x, y \in B_r[0] = \{x \in E : \|x\| \leq r\}$, $\|\lambda x + (1-\lambda)y\|^2 \leq \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$ for all $\lambda \in [0,1]$.*

Lemma 1.7 ([14])**.** *Let $g\colon [0,\infty) \to [0,\infty)$ with $g(0) = 0$ be a strictly increasing map. If a sequence $\{x_n\}$ in $[0,\infty)$ satisfies $\lim_{n\to\infty} g(x_n) = 0$, then $\lim_{n\to\infty} x_n = 0$.*

Lemma 1.8 ([9])**.** *Let C be a nonempty convex subset of a normed space E. Let $T\colon C \to C$ be uniformly L-lipschitzian and $\{\alpha_n\}$ and $\{\beta_n\} \in [0,1]$. Suppose $\{x_n\}$ is defined as in* (3) *and set $c_n = \|T^n(x_n) - x_n\|$ for all $n \in \mathbb{N}$. Then $\|x_n - T(x_n)\| \leqslant c_n + c_{n-1}L(1 + 3L + 2L^2)$ for all $n \in \mathbb{N}$.*

Lemma 1.9 ([3])**.** *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers with $\sum_{n=1}^{\infty} b_n < \infty$. If one of the following condition is satisfied:*

(i) $a_{n+1} \leq a_n + b_n; n \geq 1;$

(ii) $a_{n+1} \leq (1 + b_n)a_n; n \geq 1,$

then $\lim_{n\to\infty} a_n$ exists.

2. Main results

Lemma 2.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T\colon C \to C$ be an asymptotically nonexpansive map with sequence $k_n \subset [1,\infty)$ and $\sum_{n\geq 1}(k_n - 1) < \infty$ with at least one fixed point. Let $\alpha_n \subset (0,1)$. Suppose $\{x_n\}$ is given by* (3)*. Then $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(T)$.*

Proof. Let $p \in F(T)$. Then

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(T^n y_n - p) + (1-\alpha_n)(x_n - p)\| \\
&\leq \alpha_n \|T^n y_n - T^n p\| + (1-\alpha_n)\|x_n - p\| \\
&\leq \alpha_n k_n \|y_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&= \alpha_n k_n \|\beta_n(T^n x_n - p) + (1-\beta_n)(x_n - p)\| + (1-\alpha_n)\|x_n - p\| \\
&\leq \alpha_n \beta_n k_n \|T^n x_n - p\| + \alpha_n k_n (1-\beta_n)\|x_n - p\| \\
&\quad + (1-\alpha_n)\|x_n - p\| \\
&\leq \alpha_n \beta_n k_n^2 \|x_n - p\| + \alpha_n k_n(1-\beta_n)\|x_n - p\| + (1-\alpha_n)\|x_n - p\| \\
&= \|x_n - p\| \left[\alpha_n \beta_n k_n^2 + \alpha_n k_n - \alpha_n \beta_n k_n + 1 - \alpha_n\right]
\end{aligned}$$

$$= \|x_n - p\| [\alpha_n \beta_n k_n (k_n - 1) + \alpha_n (k_n - 1) + 1]$$
$$= \|x_n - p\| [1 + \mu_n],$$

where $\mu_n = \alpha_n \beta_n k_n (k_n - 1) + \alpha_n (k_n - 1)$.

Therefore

$$\|x_{n+1} - p\| \leq \|x_n - p\| [1 + \mu_n]. \tag{4}$$

Also we deduce that

$$\|x_{n+1} - p\| \leq \|x_1 - p\| (1 + \mu_1)(1 + \mu_2) \cdots (1 + \mu_n) \leq \|x_1 - p\| e^{\sum_{n=1}^{\infty}} (\mu_i).$$

Thus $\|x_n - p\|$ is bounded and since $\sum_{n \geq 1} (k_n - 1) < \infty$, applying (4) in Lemma 1.9, we conclude that $\lim_{n \to \infty} \|x_n - p\|$ exists. □

The following lemma is very much useful in proving our results.

Lemma 2.2. *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be positive sequences of real numbers satisfying*

(i) *$\{a_n\}$ is a decreasing sequence,*
(ii) *$\sum a_n = \infty$,*
(iii) *$\sum a_n b_n < \infty$.*

Then there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that the sequence $\{b_{n_k}, b_{n_k+1}\} = b_{n_1}, b_{n_1+1}, b_{n_2}, b_{n_2+1}, \ldots, b_{n_k}, b_{n_k+1}, \ldots$ converges to zero.

Proof. It is enough to show that given $\varepsilon > 0$ and an positive integer k there exists $n_k > k$ such that $b_{n_k} < \varepsilon$ and $b_{n_k+1} < \varepsilon$.

Suppose it is not true then there exist $\varepsilon > 0$ and a positive integer k_0 such that for each $n > k_0$ either $b_n \geq \varepsilon$ or $b_{n+1} \geq \varepsilon$. Let $N_1 = \{n > k_0 : b_n \geq \varepsilon\}$ and $N_2 = \{n > k_0 : b_n < \varepsilon\}$. Arrange the elements of N_1 and N_2 in natural order denoted by n_i and m_i respectively. Since $m_i \in N_2$, $m_i - 1 \in N_1$. From condition (ii) and by our assumption both N_1 and N_2 are infinite.

As $n_i \in N_1$, we have

$$\varepsilon \sum a_{n_k} \leq \sum a_{n_k} b_{n_k} < \infty$$

which implies $\sum a_{n_k} < \infty$.

Also $a_{m_i} \leq a_{m_i - 1}$ and $m_i - 1 \in N_1$, therefore

$$\sum_{i=1}^{\infty} a_{m_i} \leq \sum_{i=1}^{\infty} a_{m_i - 1} \leq \sum_{i=1}^{\infty} a_{n_i} < \infty.$$

And

$$\sum_{i=k+1}^{\infty} a_n = \sum_{i=k+1}^{\infty} a_{n_i} + \sum_{i=k+1}^{\infty} a_{m_i} < \infty$$

a contradiction. Hence the lemma. □

Lemma 2.3. *Let C be a nonempty closed convex subset of a Banach space E. Let $T\colon C \to C$ be continuous and $\{x_n\}$ be defined in C such that $\lim_{n\to\infty} \|x_n - p\|$ exists for each $p \in F(T)$. Suppose there exists a convergent subsequence $\{x_{n_k}\}$ such that $\|x_{n_k} - Tx_{n_k}\| \to 0$ then $\{x_n\}$ converges to a fixed point of T.*

Proof. Let $\{x_{n_k}\}$ be a sequence of $\{x_n\}$ converging to some $x \in C$. As $\|x_{n_k} - Tx_{n_k}\| \to 0$ and since T is continuous we have $Tx = x$. As $\lim_{n\to\infty} \|x_n - x\|$ exists, we conclude that $x_n \to x$. $\square$

Theorem 2.4. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $T\colon C \to C$ be an asymptotically nonexpansive map with at least one fixed point and the sequence $\{k_n\} \geq 1$ be decreasing and satisfy $\sum_{n\geq 1}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,1]$ satisfying one of the following conditions*:

(A) $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$ *and* $\beta_n \downarrow \beta < 1$,
(B) $\sum_{n=1}^{\infty} \beta_n(1-\beta_n) = \infty$ *and* $\alpha_n \downarrow \alpha > 0$ *and* $\beta_n \downarrow \beta < 1$,
(C) $0 \leq \alpha_n \leq b < 1$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\beta_n \to 0$ *as* $n \to \infty$.

Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - Tx_{n_k}\| \to 0$.

Proof. For any $p \in F(T)$, we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n (T^n y_n - p) + (1-\alpha_n)(x_n - p)\|^2 \\
&\leq \alpha_n \|T^n y_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n) g(\|x_n - T^n y_n\|) \\
&\quad \text{(using Lemma 1.6)} \\
&\leq \alpha_n k_n^2 \|y_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n) g(\|x_n - T^n y_n\|) \\
&= \alpha_n k_n^2 \|\beta_n(T^n x_n - p) + (1-\beta_n) x_n - p\|^2 \\
&\quad + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n) g(\|x_n - T^n y_n\|) \\
&\leq \alpha_n k_n^2 \left\{\beta_n \|T^n x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - \beta_n(1-\beta_n) g\|x_n - T^n x_n\|\right\} \\
&\quad + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n) g(\|x_n - T^n y_n)\| \\
&\leq \alpha_n k_n^2 \left\{\beta_n k_n^2 \|x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - \beta_n(1-\beta_n) g(\|x_n - T^n x_n\|)\right\} \\
&\quad + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n) g(\|x_n - T^n y_n\|) \\
&= \alpha_n \beta_n k_n^4 \|x_n - p\|^2 + \alpha_n k_n^2 (1-\beta_n)\|x_n - p\|^2 \\
&\quad - \alpha_n k_n^2 \beta_n (1-\beta_n) g(\|x_n - T^n x_n\|) + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad - \alpha_n(1-\alpha_n) g(\|x_n - T^n y_n\|) \\
&= \alpha_n \beta_n k_n^4 \|x_n - p\|^2 + \alpha_n k_n^2 \|x_n - p\|^2 - \alpha_n \beta_n k_n^2 \|x_n - p\|^2 + \|x_n - p\|^2 \\
&\quad - \alpha_n \|x_n - p\|^2 - \alpha_n k_n^2 \beta_n (1-\beta_n) g(\|x_n - T^n x_n\|)
\end{aligned}$$

$$- \alpha_n(1-\alpha_n)g(\|x_n - T^n y_n\|)$$

and hence

$$\begin{aligned} &\alpha_n(1-\alpha_n)g(\|x_n - T^n y_n\|) + \alpha_n k_n^2 \beta_n(1-\beta_n)g(\|x_n - T^n x_n\|) \\ \le\ & \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \left\{\alpha_n\beta_n k_n^4 - \alpha_n\beta_n k_n^2 + \alpha_n k_n^2 - \alpha_n\right\} \|x_n - p\|^2 . \end{aligned}$$

From which we obtain the following inequalities

$$\begin{aligned} (5)\quad & \alpha_n(1-\alpha_n)g(\|x_n - T^n y_n\|) \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \{\alpha_n\beta_n k_n^2(k_n^2 - 1) + \alpha_n(k_n^2 - 1)\} \|x_n - p\|^2 , \end{aligned}$$

$$\begin{aligned} (6)\quad & \alpha_n\beta_n k_n^2(1-\beta_n)g(\|x_n - T^n x_n\|) \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \{\alpha_n\beta_n k_n^2(k_n^2 - 1) + \alpha_n(k_n^2 - 1)\} \|x_n - p\|^2 . \end{aligned}$$

Case (i). Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy (A). Let $m \ge 1$ then from (5),

$$\begin{aligned} & \sum_{n=1}^{m} \alpha_n(1-\alpha_n)g(\|x_n - T^n y_n\|) \\ \le\ & \sum_{n=1}^{m}(\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \\ & + \sum_{n=1}^{m} \|x_n - p\|^2 \left\{\alpha_n\beta_n k_n^2(k_n^2 - 1) + \alpha_n(k_n^2 - 1)\right\} \\ =\ & \sum_{n=1}^{m} \left(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\right) \\ & + \alpha_n\beta_n k_n^2 \sum_{n=1}^{m} \|x_n - p\|^2 \left(k_n^2 - 1\right) + \sum_{n=1}^{m} \|x_n - p\|^2 \alpha_n \left(k_n^2 - 1\right) \\ =\ & \|x_1 - p\|^2 - \|x_{m+1} - p\|^2 \\ & + \alpha_n\beta_n k_n^2 \sum_{n=1}^{m} \|x_n - p\|^2 \left(k_n^2 - 1\right) + \alpha_n \sum_{n=1}^{m} \|x_n - p\|^2 \left(k_n^2 - 1\right) . \end{aligned}$$

Since $\{\|x_n - p\|\}$ is bounded, allowing $m \to \infty$ we have

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n)g\left(\|x_n - T^n y_n\|\right) < \infty.$$

Let $c_n = \|x_n - T^n x_n\|$ and $d_n = \|x_n - T^n y_n\|$. Put $a_n = \alpha_n(1-\alpha_n)$ and $b_n = g(\|x_n - T^n y_n\|)$, then from Lemma 2.2, $\{g(d_{n_k}), g(d_{n_{k+1}})\}$ converges to zero. Again from Lemma 1.7, $\{d_{n_k}, d_{n_{k+1}}\}$ converges to zero.

Since

$$\begin{aligned} \|x_n - T^n x_n\| &= \|x_n - T^n y_n - T^n x_n + T^n y_n\| \\ &\le \|x_n - T^n y_n\| + \|T^n x_n - T^n y_n\| \end{aligned}$$

$$\begin{aligned}
&\leq \|x_n - T^n y_n\| + k_n \|x_n - y_n\| \\
&= \|x_n - T^n y_n\| + k_n \|x_n - \beta_n T^n x_n - x_n + \beta_n x_n\| \\
&= \|x_n - T^n y_n\| + k_n \beta_n \|x_n - T^n x_n\|, \\
&(1 - k_n \beta_n) c_n \leq d_n.
\end{aligned}$$

Since $\beta_n \downarrow \beta < 1$, and $k_n \downarrow 1$ the sequence c_n converges to zero whenever d_n converges to zero. Hence we conclude that $\{c_{n_k}, c_{n_k+1}\}$ converges to zero.

Case (ii). Let α_n and β_n satisfy (B). Then from (6),

$$\begin{aligned}
&\sum_{n=1}^{m} \alpha_n \beta_n k_n^2 (1 - \beta_n) g(\|x_n - T^n x_n\|) \\
\leq &\sum_{n=1}^{m} (\|x_n - p\|^2 - \|x_{n+1} - p\|^2) \\
&+ \sum_{n=1}^{m} \|x_n - p\|^2 \left\{\alpha_n \beta_n k_n^2 \left(k_n^2 - 1\right) + \alpha_n \left(k_n^2 - 1\right)\right\} \\
= &\|x_1 - p\|^2 - \|x_{m+1} - p\|^2 \\
&+ \sum_{n=1}^{m} \|x_n - p\|^2 \alpha_n \beta_n k_n^2 \left(k_n^2 - 1\right) + \sum_{n=1}^{m} \|x_n - p\|^2 \alpha_n \left(k_n^2 - 1\right).
\end{aligned}$$

Letting $m \to \infty$, as in Case (i) we have

$$\sum_{n=1}^{\infty} \alpha_n \beta_n k_n^2 (1 - \beta_n) g(\|x_n - T^n x_n\|) < \infty.$$

Since $\alpha_n \downarrow \alpha > 0$ and $\beta_n \downarrow \beta < 1$. From condition (B) we have

$$\sum_{n=1}^{\infty} \alpha_n \beta_n k_n^2 (1 - \beta_n) = \infty.$$

Put $c_n = \|x_n - T^n x_n\|$ then from Lemma 2.2, $\{c_{n_k}, c_{n_k+1}\}$ converges to zero.

Case (iii). Let α_n and β_n satisfy (C).

Using the condition $0 \leq \alpha_n \leq b < 1$ in (5),

$$\begin{aligned}
&\alpha_n (1 - b) g(\|x_n - T^n(y_n)\|) \\
\leq &\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \|x_n - p\|^2 \left\{\alpha_n \beta_n k_n^2 (k_n^2 - 1) + \alpha_n (k_n^2 - 1)\right\}.
\end{aligned}$$

Summing the first m terms,

$$\begin{aligned}
&\sum_{n=1}^{m} \alpha_n (1 - b) g(\|x_n - T^n y_n\|) \\
\leq &\sum_{n=1}^{m} (\|x_n - p\|^2 - \|x_{n+1} - p\|^2) + \sum_{n=1}^{m} \|x_n - p\|^2 \left\{\alpha_n \beta_n k_n^2 (k_n^2 - 1) + \alpha_n (k_n^2 - 1)\right\}
\end{aligned}$$

$$= \|x_1 - p\|^2 - \|x_{m+1} - p\|^2 + \sum_{n=1}^{m} \|x_n - p\|^2 \left\{\alpha_n \beta_n k_n^2 (k_n^2 - 1) + \alpha_n (k_n^2 - 1)\right\}.$$

Since $\{\|x_n - p\|\}$ is bounded, allowing $m \to \infty$ we have

$$\sum_{n=1}^{\infty} \alpha_n (1-b) g(\|x_n - T^n y_n\|) < \infty.$$

Let $d_n = \|x_n - T^n y_n\|$, $c_n = \|x_n - T^n x_n\|$. Proceeding as in Case (i), we conclude that $\{c_{n_k}, c_{n_k+1}\}$ converges to zero.

In all the three cases we have proved that $\{c_{n_k}, c_{n_k+1}\}$ converges to zero. Applying in Lemma 1.8, we obtain a subsequence $\{x_{n_k}\}$ such that $\|x_{n_k} - Tx_{n_k}\| = 0$ and this completes the proof of the theorem. □

Theorem 2.5. *Let E be uniformly convex, C a nonempty closed convex and bounded subset of E. Let $T\colon C \to C$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences satisfying anyone of the conditions* (A), (B) *and* (C) *in Theorem 2.4. Define*

$$x_{n+1} = \alpha_n T^n y_n + (1-\alpha_n) x_n,$$
$$y_n = \beta_n T^n x_n + (1-\beta_n) x_n.$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Define $\{x_n\}$ as above, then by Theorem 2.4 and since T is completely continuous, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$ and $x = Tx$. But $\lim_{n\to\infty} \|x_n - p\|$ exists whenever p is a fixed point of T. Therefore $x_n \to x$. □

Corollary 2.6. *Let E be uniformly convex, C a nonempty closed convex and bounded subset of E. Let $T\colon C \to C$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$ and $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Suppose*

$$x_{n+1} = \alpha_n T^n x_n + (1-\alpha_n) x_n.$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Put $\beta_n = 0$ in condition (A). □

Corollary 2.7. *Let E be uniformly convex, C a nonempty closed convex and bounded subset of E. Let $T\colon C \to C$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Suppose*

$$x_{n+1} = \frac{1}{n} T^n x_n + (1 - \frac{1}{n}) x_n.$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Put $\alpha_n = \frac{1}{n}$ and $\beta_n = 0$ in condition (A). □

Corollary 2.8. *Let E be uniformly convex, C a nonempty closed convex and bounded subset of E. Let $T\colon C \to C$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Suppose*

$$x_{n+1} = (1 - \frac{1}{n})T^n y_n + \frac{1}{n}x_n,$$
$$y_n = \frac{1}{n}T^n x_n + (1 - \frac{1}{n})x_n.$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Put $\alpha_n = 1 - \frac{1}{n}$ and $\beta_n = \frac{1}{n}$ either in (A) or (B) or (C). □

Corollary 2.9. *Let E be uniformly convex, C a nonempty closed convex and bounded subset of E. Let $T\colon C \to C$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Suppose*

$$x_{n+1} = \frac{1}{n}T^n y_n + (1 - \frac{1}{n})x_n,$$
$$y_n = \frac{1}{n}T^n x_n + (1 - \frac{1}{n})x_n.$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Put $\alpha_n = \frac{1}{n}$ and $\beta_n = \frac{1}{n}$ in condition (A). □

Corollary 2.10. *Let E be uniformly convex, C a nonempty closed convex and bounded subset of E. Let $T\colon C \to C$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Suppose*

$$x_{n+1} = (1 - \frac{1}{n+1})T^n x_n + \frac{1}{n+1}x_n.$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Put $\alpha_n = 1 - \frac{1}{n+1}$ and $\beta_n = 0$ in condition (A). □

Corollary 2.11. *Let E be uniformly convex, C a nonempty closed convex and bounded subset of E. Let $T\colon C \to C$ be a completely continuous asymptotically nonexpansive mapping with sequence $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Suppose*

$$x_{n+1} = T^n y_n,$$
$$y_n = \frac{1}{n}T^n x_n + (1 - \frac{1}{n})x_n.$$

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Put $\alpha_n = 1$ and $\beta_n = \frac{1}{n}$ in condition (B). □

Example 2.12. Let $E = \mathbb{R}^2$ be the euclidean space and let $C = \mathbb{B}[0 : 9/10] \subseteq \mathbb{R}^2$ where $\mathbb{B}[0 : 9/10]$ is the closed ball centered at 0 with radius 9/10. Define $T\colon C \to C$ by

$$T(x_1, x_2) = (x_1^2, \sin x_2).$$

Here $T^n(x_1, x_2) = (x_1^{2n}, \sin^{(n)} x_2)$ where $\sin^{(n)} x_2$ is the composition of sine function over n times at x_2.

Consider (x_1, x_2) and (y_1, y_2) in C

$$\begin{aligned}\|T^n(x_1,x_2) - T^n(y_1,y_2)\| &= \left\|(x_1^{2n}, \sin^{(n)} x_2) - (y_1^{2n}, sin^{(n)} y_2)\right\|_2 \\ &= \left(\left(x_1^{2n} - y_1^{2n}\right)^2 + \left(\sin^{(n)} x_2 - \sin^{(n)} y_2\right)^2\right)^{1/2}.\end{aligned}$$

Let $x_1 < y_1$, then

$$\begin{aligned}(x_1^{2n} - y_1^{2n}) &= (x_1 - y_1)(x_1^{2n-1} + y_1 x_1^{2n-2} + y_1^2 x_1^{2n-3} + \cdots + x_1 y_1^{2n-2} + y_1^{2n-1}) \\ &\leq (x_1 - y_1) 2n y_1^{2n-1}.\end{aligned}$$

By mean value theorem we have $|\sin(\sin x_2) - \sin(\sin y_2)| \leq |\sin x_2 - \sin y_2| \leq |x_2 - y_2|$. Inductively for each $n \in \mathbb{N}$

$$|\sin^{(n)} x_2 - \sin^{(n)} y_2| \leq | x_2 - y_2 | .$$

Thus

$$\|T^n(x_1,x_2) - T^n(y_1,y_2)\| = \left((x_1 - y_1)^2 4n^2 y_1^{2(2n-1)} + (x_2 - y_2)^2\right)^{1/2}.$$

Since $4n^2 y_1^{2(2n-1)} < 1$ for sufficiently large n it follows that T is asymptotically nonexpansive. But T is not non expansive for

$$\|T(0.8,0) - T(0.7,0)\| = \|(0.64,0) - (0.49,0)\| = 0.15 > \|(0.8,0) - (0.7,0)\|.$$

Let $x_1 = (x_{1_1}, x_{1_2}) \in C$ and suppose $x_n = (x_{n_1}, x_{n_2})$ is defined as in Corollary 2.10 by

$$x_{n+1} = \left(\frac{x_{n_1} + n x_{n_1}^{2n}}{n+1}, \frac{x_{n_2} + sin^{(n)} x_{n_2}}{n+1}\right).$$

Then $\{x_n\}$ converges to the fixed point $(0, 0)$.

3. Rate of convergence

In this section, we present a convergence result for modified Mann's iterative sequences which establishes that for a certain class of operators the iteration defined in Corollary 2.10, converges faster than the usual modified Mann's iteration.

Definition 3.1 ([1]). An operator T is called a Zamfirescu operator if there exist real numbers α, β and γ, $0 \leq \alpha < 1$, $0 \leq \beta, \gamma < 0.5$, such that for any $x, y \in E$ at least one of the following conditions hold:

(a) $\|Tx - Ty\| \leq \alpha \|x - y\|$,

(b) $\|Tx - Ty\| \leq \beta[\|x - Tx\| + \|y - Ty\|]$,
(c) $\|Tx - Ty\| \leq \gamma[\|x - Ty\| + \|y - Tx\|]$.

Definition 3.2. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two positive real sequences converging to 0. We say $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$ if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Theorem 3.3. *Let E be uniformly convex, C a nonempty closed convex and bounded subset of E. Let $T\colon C \to C$ be a Zamfirescu operator. Let $x_1 = y_1 = a \in C$. Suppose we define the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ by*

$$x_{n+1} = (1 - \frac{1}{n+1})T^n x_n + \frac{1}{n+1} x_n, \tag{7}$$

$$y_{n+1} = \alpha_n T^n y_n + (1-\alpha_n) y_n, \qquad \epsilon \leq \alpha_n \leq 1 - \epsilon, \tag{8}$$

where $\{y_n\}_{n=1}^{\infty}$ is the usual modified Mann's iteration. Then both $\{x_n\}$ and $\{y_n\}$ converge to the unique fixed point p of T and $\{x_n\}$ converges faster to p than $\{y_n\}$.

Proof. It is known from Theorem 2.4 of [1] that any Zamfirescu operator possess a unique fixed point and satisfies

$$\|Tx - Ty\| \leq \delta.\|x - y\| + 2\delta.\|x - Tx\| \tag{9}$$

for all x,$y \in C$, where

$$\delta = \max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\} < 1.$$

Let y_n be the sequence defined by (8). Take $y = y_n$ and $x = p$ in (9), then

$$\|Ty_n - p\| \leq \delta.\|y_n - p\| \tag{10}$$

and thus

$$\|T^n y_n - p\| \leq \delta.\|T^{n-1} y_n - p\| \leq \delta^2.\|T^{n-2} y_n - p\| \leq \cdots \leq \delta^n \|y_n - p\|. \tag{11}$$

From (11), we obtain

$$\begin{aligned}
\|y_{n+1} - p\| &= \|\alpha_n T^n y_n + (1-\alpha_n) y_n - p\| \\
&\leq \alpha_n \|T^n y_n - p\| + (1-\alpha_n)\|y_n - p\| \\
&\leq \alpha_n \delta^n \|y_n - p\| + (1-\alpha_n)\|y_n - p\| \\
&\leq (1 - \alpha_n + \alpha_n \delta^n)\|y_n - p\|,
\end{aligned}$$

which implies that

$$\|y_{n+1} - p\| \leq \prod_{k=1}^{n} (1 - \alpha_k + \alpha_k \delta^k)\|y_1 - p\|. \tag{12}$$

Similarly for the sequence x_n defined by (7), we obtain

$$\|x_{n+1} - p\| \leq \prod_{k=1}^{n} \left(\frac{1}{k+1} + \left(1 - \frac{1}{k+1}\right) \delta^k \right) \|x_1 - p\|. \tag{13}$$

Let $a_n = \prod_{k=1}^{n} \left(\frac{1}{k+1} + \left(1 - \frac{1}{k+1}\right) \delta^k \right)$ and $b_n = \prod_{k=1}^{n}(1 - \alpha_k + \alpha_k \delta^k)$. Since $x_1 = y_1 = a$, it is enough to show that $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. Let $c_n = \frac{a_n}{b_n}$, then applying ratio test we see that

$$\frac{c_{n+1}}{c_n} = \frac{\left(\frac{1}{n+2} + \left(1 - \frac{1}{n+2}\right) \delta^{n+1} \right)}{(1 - \alpha_{n+1} + \alpha_{n+1} \delta^{n+1})}.$$

Since $\epsilon \leq (1 - \alpha_{n+1}) \leq 1 - \epsilon$ and $\delta^n \to 0$, we get $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0$ and therefore we conclude that $\sum_{n=1}^{\infty} c_n$ converges which implies $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. $\square$

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