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GROWTH OF SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS OF $[p, q]$-ORDER IN THE COMPLEX PLANE

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Abstract. In the paper, we study the growth and fixed point of solutions of high order linear differential equations with entire coefficients of $[p, q]$-order in the complex plane. We improve and extend some results due to T. B. Cao, J. F. Xu, Z. X. Chen, and J. Liu, J. Tu, L. Z. Shi.

1. Introduction, definitions and notations

We assume that reader is familiar with the fundamental results and the standard notations of the Nevanlinna's theory of meromorphic functions (see [7], [15]). Let us define inductively, for $r \in [0, \infty)$, $\exp_1 r = e^r$ and $\exp_{p+1} r = \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. For all sufficiently large r,we define $\log_1 r = \log r$ and $\log_{p+1} r = \log\left(\log_p r\right)$, $p \in \mathbb{N}$. We also denote $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$ and $\log_{-1} r = \exp_1 r$. In order to express the rate of growth of entire functions more precisely, we recall the following definitions (see [10], [13]):

Definition 1.1. The iterated p-order of an entire function $f(z)$ is defined by

$$\sigma_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r, f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r, f)}{\log r},$$

where $T(r, f)$ is the Nevanlinna characteristic function of f and $M(r, f) = \max_{|z|=r} |f(z)|$. For $p = 1$, this notation is called order and for $p = 2$ hyper-order.

Definition 1.2. The finiteness degree of the iterated order of an entire function $f(z)$ is defined by

$$i(f) = \begin{cases} 0 & \text{for } f \text{ polynomial,} \\ \min\{j \in \mathbb{N} : \sigma_j(f) < \infty\} & \text{for } f \text{ transcendental with } \sigma_j(f) < \infty, \\ \infty & \text{for } f \text{ with } \sigma_j(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

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For $k \geq 2$, we consider the complex linear differential equations

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = 0, \tag{1.1}$$

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_1(z) f' + A_0(z) f = F(z), \tag{1.2}$$

where $A_0(z)$, $A_1(z), \ldots, A_{k-1}(z)$ and $F(z) \not\equiv 0$ are entire functions. It is well known that all solutions of equations (1.1) and (1.2) are entire functions, and that if some coefficients of (1.1) are transcendental then (1.1) has at least one solution with infinite order $\sigma_1(f) = \infty$. We refer to [11] for the literature on the growth of entire solutions of (1.1) and (1.2).

As we know, Bernal [4] firstly introduced the idea of iterated order to express the first growth of solutions of complex linear differential equations. Since then many authors achieved many valuable results on iterated order of the solutions of the complex linear differential equations (1.1) and (1.2) with entire coefficients of finite order (see [1], [2], [3], [5], [10], [13], [14]).

In 2010, Cao, Xu and Chen [6] proved the following:

Theorem 1.1 ([6]). *Assume that $A_0, A_1, \ldots, A_{k-1}$ be meromorphic functions, and let $i(A_0) = p$ $(0 < p < \infty)$. Assume that either $i_\lambda\left(\frac{1}{A_0}\right) < p$ or $\lambda_p\left(\frac{1}{A_0}\right) < \sigma_p(A_0)$ and that either*

$$\max\{i(A_j) : j = 1, 2, \ldots, k-1\} < p$$

or

$$\max\{\sigma_p(A_j) : j = 1, 2, \ldots, k-1\} \leq \sigma_p(A_0) := \sigma \quad (0 < \sigma < \infty),$$
$$\max\{\tau_p(A_j) : \sigma_p(Aj) = \sigma_p(A_0)\} < \tau_p(A_0) := \tau \quad (0 < \tau < \infty).$$

Then every meromorphic solution $f(z) \not\equiv 0$ whose poles are of uniformly bounded multiplicities, of (1.1) satisfies $\overline{\lambda}_{p+1}(f - z) = \sigma_{p+1}(f) = \sigma_p(A_0)$.

Theorem 1.2 ([6]). *Assume that $A_0, A_1, \ldots, A_{k-1}$ be meromorphic functions. Suppose that there exist one A_s $(s \in \{0, 1, \ldots, k-1\})$ with $i(A_s) = p$ $(0 < p < \infty)$. Assume that either $i_\lambda\left(\frac{1}{A_s}\right) < p$ or $\lambda_p\left(\frac{1}{A_s}\right) < \sigma_p(A_s)$ and that either*

$$\max\{i(A_j) : j \neq s \text{ and } j = 0, 1, 2, \ldots, k-1\} < p$$

or

$$\max\{\sigma_p(A_j) : j \neq s \text{ and } j = 0, 1, 2, \ldots, k-1\} \leq \sigma_p(A_s) := \sigma \quad (0 < \sigma < \infty),$$
$$\max\{\tau_p(A_j) : \sigma_p(A_j) = \sigma_p(A_s)\} < \tau_p(A_s) := \tau \quad (0 < \tau < \infty)$$

and if $A_1(z) + zA_0(z) \not\equiv 0$ and if all solutions of (1.1) are meromorphic whose poles are of uniformly bounded multiplicities, then any transcendental meromorphic solution f with $\sigma_p(f) > \sigma_p(A_s)$ satisfies $\overline{\lambda}_p(f - z) = \sigma_p(f)$. Furthermore, there exist at least one solution f_1 satisfying $\overline{\lambda}_{p+1}(f_1 - z) = \sigma_{p+1}(f_1) = \sigma_p(A_s)$.

Theorem 1.3 ([6]). *Assume that $A_0, A_1, \ldots, A_{k-1}$ and F be meromorphic functions, and let $i(A_0) = p$ $(0 < p < \infty)$. Assume that either $i_\lambda\left(\frac{1}{A_0}\right) < p$ or $\lambda_p\left(\frac{1}{A_0}\right) < \sigma_p(A_0)$ and that either*

$$\max\{i(A_j) : j = 1, 2, \ldots, k-1\} < p$$

or

$$\max\{\sigma_p(A_j) : j = 1, 2, \ldots, k-1\} \leq \sigma_p(A_0) := \sigma \quad (0 < \sigma < \infty),$$
$$\max\{\tau_p(A_j) : \sigma_p(A_j) = \sigma_p(A_0)\} < \tau_p(A_0) := \tau \quad (0 < \tau < \infty)$$

if $F(z) - (A_1(z) + zA_0(z)) \not\equiv 0$, then every solution f whose poles are of uniformly bounded multiplicities, with $i(f) = i_\lambda\left(\frac{1}{f}\right) = p + 1$ and $\sigma_{p+1}(f) = \overline{\lambda}_{p+1}(f)$ of (1.2) *satisfies $\overline{\lambda}_{p+1}(f - z) = \sigma_{p+1}(f)$.*

In ([8], [9]) Juneja, Kapoor and Bajpai have investigated some properties of entire functions of $[p, q]$-order and obtained some results about their growth. Recently, Liu, Tu and Shi [12] introduced the concept of $[p, q]$-order for the case $p \geq q \geq 1$ to investigate the entire solutions of (1.1) and (1.2), and obtained some results.

Now we defined the $[p, q]$-order of entire functions, where p, q are positive integers satisfying $p \geq q \geq 1$ as follows:

Definition 1.3 ([12]). If $f(z)$ is a transcendental entire function, the $[p, q]$-order of $f(z)$ is defined by

$$\sigma_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r, f)}{\log_q r} = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r, f)}{\log_q r}.$$

It is easy to see that $0 \leq \sigma_{[p,q]}(f) \leq \infty$. If $f(z)$ is a polynomial, then $\sigma_{[p,q]}(f) = 0$ for any $p \geq q \geq 1$. By above definition we have that $\sigma_{[1,1]}(f) = \sigma_1(f)$, $\sigma_{[2,1]}(f) = \sigma_2(f)$ and $\sigma_{[p+1,1]}(f) = \sigma_{p+1}(f)$.

Remark 1.1. If $f(z)$ is an entire function satisfying $0 < \sigma_{[p,q]}(f) < \infty$, then

1. $\sigma_{[p-n,q]}(f) = \infty$ $(n < p)$, $\sigma_{[p,q-n]}(f) = 0$ $(n < q)$, $\sigma_{[p+n,q+n]}(f) = 1$, $(n < p)$ for $n = 1, 2, 3, \ldots$.
2. If $[p', q']$ is any pair of integers satisfying $q' = p' + q - p$ and $p' < p$, then $\sigma_{[p',q']}(f) = 0$, if $0 < \sigma_{[p,q]}(f) < 1$ and $\sigma_{[p',q']}(f) = \infty$, if $1 < \sigma_{[p,q]}(f) < \infty$.
3. $\sigma_{[p',q']}(f) = \infty$ for $q' - p' > q - p$ and $\sigma_{[p',q']}(f) = 0$ for $q' - p' < q - p$.

Definition 1.4. A transcendental entire function $f(z)$ is said to have index-pair $[p, q]$, if $0 < \sigma_{[p,q]}(f) < \infty$ and $\sigma_{[p-1,q-1]}(f)$ is not a nonzero finite number.

Remark 1.2. If $\sigma_{[p,q]}(f)$ is never greater than 1 and $\sigma_{[p',q']}(f) = 1$ for some integer $p' \geq 1$, then the index-pair of $f(z)$ is defined as $[m, m]$, where $m = \inf\left\{p' : \sigma_{[p',p']}(f) = 1\right\}$. If $\sigma_{[p,q]}(f)$ is never nonzero finite for any positive integer pair $[p, q]$ and $\sigma_{[p'',1]}(f) = 0$ for some integer $p'' \geq 1$, then the index pair of $f(z)$ is defined as $[n,1]$ where $n=\inf\left\{p'':\sigma_{[p'',1]}(f)=0\right\}$. If $\sigma_{[p,q]}(f)$ is always infinite, then the index-pair of $f(z)$ is defined to be $[\infty,\infty]$.

If $f(z)$ has the index-pair $[p,q]$, then $\sigma_{[p,q]}(f)$ is called its $[p,q]$-order. For example, set $f_1(z)=e^z,\ \ f_2(z)=e^{e^z}$, by Remark 1.2 we have the index-pair of $f_1(z)$ is $[1,1]$, and the index-pair of $f_2(z)$ is $[2,1]$.

Remark 1.3. Let $f_1(z)$ be an entire function of $[p,q]$-order σ_1 and let $f_2(z)$ be an entire function of $[p',q']$-order σ_2 and let $p\leq p'$. The following results about their comparative growth can be easily deducted:

1. If $p'-p>q'-q$, then the growth of f_1 is slower than the growth of f_2.
2. If $p'-p<q'-q$, then the growth of f_1 is faster than the growth of f_2.
3. If $p'-p=q'-q>0$, then the growth of f_1 is slower than the growth of f_2 if $\sigma_2\geq 1$ while the growth of f_1 is faster than the growth of f_2 if $\sigma_2<1$.
4. Let $p'-p=q'-q=0$, then f_1 and f_2 are of the same index-pair $[p,q]$. If $\sigma_1>\sigma_2$, then f_1 grows faster than f_2, and if $\sigma_1<\sigma_2$, then f_1 grows slow than f_2. If $\sigma_1=\sigma_2$, Definition 1.3 does not give any precise estimate about the relative growth of f_1 and f_2.

Definition 1.5. The $[p,q]$-type of an entire function $f(z)$ of $[p,q]$-order σ $(0<\sigma<\infty)$ is defined by

$$\tau_{[p,q]}(f)=\overline{\lim_{r\to\infty}}\frac{\log_{p-1}T(r,f)}{\left(\log_{q-1}r\right)^{\sigma}}=\overline{\lim_{r\to\infty}}\frac{\log_{p}M(r,f)}{\left(\log_{q-1}r\right)^{\sigma}}.$$

Definition 1.6. The $[p,q]$-exponent of convergence of the sequence of zeros of $f(z)$ is defined by

$$\lambda_{[p,q]}(f)=\overline{\lim_{r\to\infty}}\frac{\log_{p}N\left(r,\frac{1}{f}\right)}{\log_q r}.$$

Definition 1.7. The $[p,q]$-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_{[p,q]}(f)=\overline{\lim_{r\to\infty}}\frac{\log_{p}\overline{N}\left(r,\frac{1}{f}\right)}{\log_q r}.$$

By using the notion of $[p,q]$-order of entire functions, Liu, Tu and Shi [12] prove the following theorem:

Theorem 1.4 (see [12])**.** *Let* $F(z)\not\equiv 0,\ A_j(z)\ (j=0,1,\ldots,k-1)$ *be entire functions satisfying* $\max\left\{\sigma_{[p,q]}(A_j):j=0,1,2,\ldots,k-1\right\}<\sigma_{[p+1,q]}(F)$, *then we have that*

1. $\sigma_{[p+1,q]}(f)=\sigma_{[p+1,q]}(F)$ *holds for all solutions of* (1.2).
2. $\lambda_{[p+1,q]}(f)=\sigma_{[p+1,q]}(f)=\sigma_{[p+1,q]}(F)$ *holds for all solutions of* (1.2) *with at most one exceptional solution* f_0 *satisfying* $\lambda_{[p+1,q]}(f_0)<\sigma_{[p+1,q]}(F)$.

The main purpose of this paper is to consider the growth of entire solutions of equation (1.1) and (1.2) with entire coefficients of finite $[p,q]$-order in the complex plane. Some of results improve and extend earlier results of T. B. Cao, J. F. Xu, Z. X. Chen [6]; J. Liu, J. Tu, L. Z. Shi [12].

2. Main results

In this section we state the main results of the paper.

Theorem 2.1. *Let $A_0, A_1, \ldots, A_{k-1}$ be entire functions satisfying*

$$\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 1, 2, \ldots, k-1\right\} \leq \sigma_{[p,q]}\left(A_o\right)$$

and

$$\max\left\{\tau_{[p,q]}\left(A_j\right) : \sigma_{[p,q]}\left(A_j\right) = \sigma_{[p,q]}\left(A_o\right)\right\} < \tau_{[p,q]}\left(A_0\right).$$

If $A_1(z) + zA_0(z) \not\equiv 0$, then for every entire solutions $f(z) \not\equiv 0$ of (1.1) *satisfies*

$$\overline{\lambda}_{[p+1,q]}\left(f - z\right) = \sigma_{[p+1,q]}\left(f\right) = \sigma_{[p,q]}\left(A_0\right).$$

For $p > q \geq 1$ we have the following example:

Example 2.1. $f(z) = e^{e^z}$ solves the differential equation

$$f'' - f' - e^{2z} f = 0,$$

where $A_1(z) = -1$, $A_0(z) = -e^{2z}$ are entire with $\sigma_{[p,q]}(A_1) = 0, \sigma_{[p,q]}(A_0) = 0$, and $\tau_{[p,q]}(A_1) = 0$, $\tau_{[p,q]}(A_0) = \infty$.

Clearly, $\sigma_{[p,q]}(A_1) \leq \sigma_{[p,q]}(A_0)$ and $\tau_{[p,q]}(A_1) < \tau_{[p,q]}(A_0)$.

Also, $A_1(z) + zA_0(z) \not\equiv 0$.

Hence

$$\overline{\lambda}_{[p+1,q]}\left(f - z\right) = 0 = \sigma_{[p+1,q]}\left(f\right) = \sigma_{[p,q]}\left(A_0\right).$$

Theorem 2.2. *Let $A_0, A_1, \ldots, A_{k-1}$ be entire functions satisfying*

$$\max\left\{\sigma_{[p,q]}\left(A_j\right) : j \neq s \text{ and } j = 0, 1, 2, \ldots, k-1\right\} < \sigma_{[p,q]}\left(A_s\right).$$

If $A_1(z) + zA_0(z) \not\equiv 0$, then for every entire solutions $f(z)$ of (1.1) *with $\sigma_{[p,q]}(f) > \sigma_{[p,q]}(A_s)$ satisfies*

$$\overline{\lambda}_{[p,q]}\left(f - z\right) = \sigma_{[p,q]}\left(f\right).$$

Furthermore, there exist at least one solution f_1 satisfying

$$\overline{\lambda}_{[p+1,q]}\left(f_1 - z\right) = \sigma_{[p+1,q]}\left(f_1\right) = \sigma_{[p,q]}\left(A_s\right).$$

Theorem 2.3. *Let $A_0, A_1, \ldots, A_{k-1}$ and $F \not\equiv 0$ be entire functions in the plane satisfying*

$$\max\left\{\sigma_{[p,q]}\left(A_j\right) : j = 1, 2, \ldots, k-1\right\} < \sigma_{[p,q]}\left(A_0\right).$$

If $F(z) - (A_1(z) + zA_0(z)) \not\equiv 0$, then for every entire solutions $f(z)$ of (1.2) *with $\sigma_{[p+1,q]}(f) = \overline{\lambda}_{[p+1,q]}(f)$ satisfies*

$$\overline{\lambda}_{[p+1,q]}\left(f - z\right) = \sigma_{[p+1,q]}\left(f\right).$$

Example 2.2. For the differential equation $f'' - f' - \left(e^{2z} - 1\right) f = e^{e^z}$, we can easily see that this equation has a solution $f(z) = e^{e^z}$. The functions $A_1(z) = -1$, $A_0(z) = -\left(e^{2z} - 1\right)$, $F(z) = e^{e^z}$ are entire functions and $\sigma_{[1,1]}(A_1) = 0 < 1 = \sigma_{[1,1]}(A_0)$. Clearly $F(z) - (A_1(z) + zA_0(z)) \not\equiv 0$, and $\sigma_{[2,1]}(f) = 1 \neq 0 = \overline{\lambda}_{[2,1]}(f)$. Thus we get

$$\overline{\lambda}_{[2,1]}(f - z) = 0 \neq 1 = \sigma_{[2,1]}(f).$$

Example 2.3. The differential equation $f'' + e^z f = e^z + e^{2z}$ has a solution $f(z) = e^z$,where $\sigma_{[p,p]}(A_0) = 1$, $\sigma_{[p,q]}(A_1) = 0$ and $F - (A_1 + zA_0) = e^z + e^{2z} - ze^z \not\equiv 0$.

Also $\overline{\lambda}_{[p+1,q]}(f) = 0 = \sigma_{[p+1,q]}(f)$. Thus,

$$\overline{\lambda}_{[p+1,q]}(f - z) = 0 = \sigma_{[p+1,q]}(f).$$

Theorem 2.4. *Let $A_0, A_1, \ldots, A_{k-1}$ and $F \neq 0$ be entire functions satisfying*

$$\max\left\{\sigma_{[p,q]}(A_j) : j \neq s \text{ and } j = 0, 1, 2, \ldots, k-1\right\} < \sigma_{[p,q]}(A_s).$$

Suppose that g_0 is a solution of (1.2) *and $(f_1, f_2, \ldots, f_k)$ is a solution base of the corresponding homogeneous equation* (1.1) *of* (1.2), *then*

1. *If $\sigma_{[p+1,q]}(F) < \sigma_{[p,q]}(A_s)$, then there exist a f_j $(j \in \{1, 2, \ldots, k\})$ say f_1 such that the solutions in the solution subspace $G = \{f_c = cf_1 + g_0 : c \in \mathbb{C}\}$ satisfy*

$$\overline{\lambda}_{[p+1,q]}(f_c) = \lambda_{[p+1,q]}(f_c) = \sigma_{[p+1,q]}(f_c) = \sigma_{[p,q]}(A_s)$$

with at most one exception.

2. *If $\sigma_{[p+1,q]}(F) > \sigma_{[p,q]}(A_s)$, then for all solutions of* (1.2) *we have*

$$\sigma_{[p+1,q]}(f) = \sigma_{[p+1,q]}(F).$$

Let $f(z) = g(z) - z$, then the zeros of $g(z)$ is just the fixed points of $f(z)$. So obviously

$$\overline{\lambda}_{[p+1,q]}(f - z) = \overline{\lambda}_{[p+1,q]}(g)$$

and

$$\sigma_{[p+1,q]}(f) = \sigma_{[p+1,q]}(g).$$

Example 2.4. Consider the differential equation

$$f'' - 2zf' + e^z f = 4z^2 e^{e^{z^2}} e^{2z^2} + 2e^{e^{z^2}} e^{z^2} + e^{e^{z^2}} e^z,$$

we can easily see that this equation has a solution $f(z) = e^{e^{z^2}}$. The functions $A_1 = -2z$, $A_0 = e^z$ and $F(z) = 4z^2 e^{e^{z^2}} e^{2z^2} + 2e^{e^{z^2}} e^{z^2} + e^{e^{z^2}} e^z$ satisfies

$$\sigma_{[1,1]}(A_1) = 0 < \sigma_{[1,1]}(A_0) = 1,$$

and

$$\sigma_{[2,1]}(F) = 2 > \sigma_{[p,q]}(A_0) = 1.$$

Thus we get

$$\sigma_{[2,1]}(F) = 2 = \sigma_{[2,1]}(f).$$

3. Lemmas

In this section we present some lemmas which will be needed in sequel.

Lemma 3.1 ([12]). *Let $f(z)$ be an entire function of $[p,q]$-order. Then*

$$\sigma_{[p,q]}(f) = \sigma_{[p,q]}(f').$$

Lemma 3.2 ([12]). *Let $F(z) \neq 0$, $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions, let $f(z)$ be a solution of* (1.2) *satisfying* $\max\left\{\sigma_{[p,q]}(A_j), \sigma_{[p,q]}(F)\right\} < \sigma_{[p,q]}(f)$, *then we have* $\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \sigma_{[p,q]}(f)$.

Lemma 3.3 (see [12]). *Let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions satisfying* $\max\left\{\sigma_{[p,q]}(A_j) : j \neq s \text{ and } j = 0, 1, 2, \ldots, k-1\right\} < \sigma_{[p,q]}(A_s) < \infty$, *then every solution $f(z)$ of* (1.1) *satisfies $\sigma_{[p+1,q]}(f) \leq \sigma_{[p,q]}(A_s)$. Furthermore, at least one solution of* (1.1) *satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_s)$.*

Lemma 3.4 (see [12]). *Let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions satisfying*

$$\max\left\{\sigma_{[p,q]}(A_j) : \ j = 1, 2, \ldots, k-1\right\} \leq \sigma_{[p,q]}(A_0) < \infty$$

and

$$\max\left\{\tau_{[p,q]}(A_j) : \sigma_{[p,q]}(A_j) = \sigma_{[p,q]}(A_0) > 0\right\} < \tau_{[p,q]}(A_0).$$

Then every nontrivial solution $f(z)$ of (1.1) *satisfies $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.*

Lemma 3.5 (see [12]). *Let $F(z) \not\equiv 0$, $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions satisfying* $\max\left\{\sigma_{[p,q]}(A_j),\ \sigma_{[p+1,q]}(F) : j = 1, 2, \ldots, k-1\right\} < \sigma_{[p,q]}(A_0)$. *Then every solution $f(z)$ of* (1.2) *satisfies*

$$\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$$

with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \sigma_{[p,q]}(A_0)$.

4. Proofs of main theorems

Proof of Theorem 2.1. Let $f(z) \not\equiv 0$ be an entire solution of (1.1). Set $g(z) = f(z) - z$.

Obviously

$$\overline{\lambda}_{[p+1,q]}(f - z) = \overline{\lambda}_{[p+1,q]}(g)$$

and

$$\sigma_{[p+1,q]}(f) = \sigma_{[p+1,q]}(g).$$

Then from equation (1.1)

$$g^{(k)} + A_{k-1}(z)\, g^{(k-1)} + \cdots + A_1(z)\, g' + A_0(z)\, g = -\left[A_1(z) + zA_0(z)\right].$$

By Lemma 3.4, we have $\sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.

Since $A_1(z) + zA_0(z) \not\equiv 0$ is an entire function, therefore

$$\max\left\{\sigma_{[p+1,q]}(A_j), \sigma_{[p+1,q]}(-A_1(z) - zA_0(z)) : j = 0, 1, 2, \ldots, k-1\right\}$$
$$< \sigma_{[p+1,q]}(f) = \sigma_{[p+1,q]}(g).$$

With the help of Lemma 3.2, we obtain

$$\overline{\lambda}_{[p+1,q]}(g) = \sigma_{[p+1,q]}(g),$$
$$\text{i.e., } \overline{\lambda}_{[p+1,q]}(f-z) = \sigma_{[p+1,q]}(f).$$

Hence

$$\overline{\lambda}_{[p+1,q]}(f-z) = \sigma_{[p+1,q]}(f) = \sigma_{[p,q]}(A_0). \qquad \square$$

Proof of Theorem 2.2. Let $f(z)$ be an entire function of (1.1). Set $g(z) = f(z) - z$. Obviously

$$\overline{\lambda}_{[p+1,q]}(f-z) = \overline{\lambda}_{[p+1,q]}(g)$$

and

$$\sigma_{[p+1,q]}(f) = \sigma_{[p+1,q]}(g).$$

Then equation (1.1) becomes

$$g^{(k)} + A_{k-1}(z)g^{(k-1)} + \cdots + A_1(z)g' + A_0(z)g = -[A_1(z) + zA_0(z)].$$

From Lemma 3.3, we have

$$\sigma_{[p+1,q]}(f) \leq \sigma_{[p,q]}(A_s).$$

If $\sigma_{[p,q]}(f) > \sigma_{[p,q]}(A_s)$ and $A_1(z) + zA_0(z) \not\equiv 0$ is an entire function, then

$$\max\left\{\sigma_{[p,q]}(A_j), \sigma_{[p,q]}(-A_1(z) - zA_0(z)) : j = 0,1,2,\ldots,k-1\right\}$$
$$\leq \sigma_{[p,q]}(A_s) < \sigma_{[p,q]}(f).$$

By help of Lemma 3.2, we get

$$\overline{\lambda}_{[p,q]}(g) = \sigma_{[p,q]}(g),$$
$$\text{i.e., } \overline{\lambda}_{[p,q]}(f-z) = \sigma_{[p,q]}(f).$$

Again from Lemma 3.3, there exists a solution f_1 of (1.1) such that $\sigma_{[p+1,q]}(f_1) = \sigma_{[p,q]}(A_s)$, then we have

$$\max\left\{\sigma_{[p+1,q]}(A_j), \sigma_{[p+1,q]}(-A_1(z) - zA_0(z)) : j = 0,1,2,\ldots,k-1\right\}$$
$$< \sigma_{[p+1,q]}(f_1).$$

Thus

$$\overline{\lambda}_{[p+1,q]}(g_1) = \sigma_{[p+1,q]}(g_1) \quad \text{where } g_1(z) = f_1(z) - z,$$

i.e.,

$$\overline{\lambda}_{[p+1,q]}(f_1 - z) = \sigma_{[p+1,q]}(f_1).$$

Hence

$$\overline{\lambda}_{[p+1,q]}(f_1 - z) = \sigma_{[p+1,q]}(f_1) = \sigma_{[p,q]}(A_s). \qquad \square$$

Proof of Theorem 2.3. Let $f(z)$ be an entire solution of (1.2), with $\sigma_{[p+1,q]}(f) = \overline{\lambda}_{[p+1,q]}(f)$.

Set $g(z) = f(z) - z$.

Then from the equation (1.2) we have

$$g^{(k)} + A_{k-1}(z)\, g^{(k-1)} + \cdots + A_1(z)\, g' + A_0(z)\, g = F(z) - (A_1(z) + zA_0(z)).$$

By Lemma 3.5, there exists some entire solution $f(z)$ of (1.2) satisfying $\sigma_{[p+1,q]}(f) = \overline{\lambda}_{[p+1,q]}(f)$, therefore

$$\max\left\{\sigma_{[p+1,q]}(A_j), \sigma_{[p+1,q]}(F(z) - A_1(z) - zA_0(z)) : j = 0, 1, 2, \ldots, k-1\right\} < \sigma_{[p+1,q]}(f).$$

If $F(z) - A_1(z) - zA_0(z) \not\equiv 0$ is an entire function, then by Lemma 3.2, we obtain

$$\overline{\lambda}_{[p+1,q]}(g) = \sigma_{[p+1,q]}(g).$$

Hence

$$\overline{\lambda}_{[p+1,q]}(f - z) = \sigma_{[p+1,q]}(f). \qquad \square$$

Proof of Theorem 2.4. 1. Let $f(z)$ be a solution of (1.2). By the elementary theory of differential equations [15], all solutions of (1.2) are entire functions and have the form

$$f = f^* + C_1 f_1 + C_2 f_2 + \cdots + C_k f_k,$$

where $C_1, C_2, \ldots, C_k$ are complex constants, and $(f_1, f_2, \ldots, f_k)$ is a solution base of (1.1), f^* is a solution of (1.2) and has the form

$$f^* = D_1 f_1 + D_2 f_2 + \cdots + D_k f_k, \tag{4.1}$$

where $D_1, D_2, \ldots, D_k$ are certain entire functions satisfying

$$D_j' = F \cdot G_j(f_1, f_2, \ldots, f_k) \cdot W(f_1, f_2, \ldots, f_k)^{-1} \qquad (j = 1, 2, \ldots, k), \tag{4.2}$$

where $G_j(f_1, f_2, \ldots, f_k)$ are differential polynomials in $f_1, f_2, \ldots, f_k$ and their derivative with constant coefficients, and $W(f_1, f_2, \ldots, f_k)$ is the Wronskian of $f_1, f_2, \ldots, f_k$.

From Lemma 3.3 we have

$$\sigma_{[p+1,q]}(f_j) \le \sigma_{[p,q]}(A_s) \qquad (j = 1, 2, \ldots, k)$$

and there is one f_j ($j \in \{1, 2, \ldots, k\}$), say f_1 satisfies $\sigma_{[p+1,q]}(f_1) = \sigma_{[p,q]}(A_s)$.

Using Lemma 3.1, (4.1) and (4.2) we obtain

$$\sigma_{[p+1,q]}(f) \le \max\left\{\sigma_{[p+1,q]}(f_j), \sigma_{[p+1,q]}(F) : j = 1, 2, \ldots, k\right\} = \sigma_{[p,q]}(A_s). \tag{4.3}$$

Thus, all solutions f_c in $G = \{f_c = cf_1 + g_0 : c \in \mathbb{C}\}$, satisfy

$$\sigma_{[p+1,q]}(f_c) = \sigma_{[p,q]}(A_s)$$

with at most one exception.

Again, we have

$$\max\left\{\sigma_{[p+1,q]}\left(A_j\right), \sigma_{[p+1,q]}\left(F\right) : j = 0, 1, 2, \ldots, k-1\right\} < \sigma_{[p+1,q]}\left(f_c\right).$$

So, by using Lemma 3.2, we obtain

$$\overline{\lambda}_{[p+1,q]}\left(f_c\right) = \lambda_{[p+1,q]}\left(f_c\right) = \sigma_{[p+1,q]}\left(f_c\right).$$

Hence

$$\overline{\lambda}_{[p+1,q]}\left(f_c\right) = \lambda_{[p+1,q]}\left(f_c\right) = \sigma_{[p+1,q]}\left(f_c\right) = \sigma_{[p,q]}\left(A_s\right),$$

with at most one exception.

2. From the equation (4.3), we have

$$\begin{aligned}\sigma_{[p+1,q]}\left(f\right) &\leq \max\left\{\sigma_{[p+1,q]}\left(f_j\right), \sigma_{[p+1,q]}\left(F\right) : j = 1, 2, \ldots, k\right\} \\ &\leq \max\left\{\sigma_{[p+1,q]}\left(A_j\right), \sigma_{[p+1,q]}\left(F\right) : j = 0, 1, 2, \ldots, k-1\right\} \\ &= \sigma_{[p+1,q]}\left(F\right).\end{aligned}$$

Thus

$$\sigma_{[p+1,q]}\left(f\right) \leq \sigma_{[p+1,q]}\left(F\right). \tag{4.4}$$

On the other hand, by a simple order comparison from (1.2) we have

$$\sigma_{[p+1,q]}\left(F\right) \leq \max\left\{\sigma_{[p+1,q]}\left(A_j\right), \sigma_{[p+1,q]}\left(f\right) : j = 1, 2, \ldots, k-1\right\}.$$

But $\sigma_{[p+1,q]}\left(A_j\right) < \sigma_{[p+1,q]}\left(F\right)$, so we have

$$\sigma_{[p+1,q]}\left(F\right) \leq \sigma_{[p+1,q]}\left(f\right). \tag{4.5}$$

From (4.4) and (4.5) we obtain

$$\sigma_{[p+1,q]}\left(f\right) = \sigma_{[p+1,q]}\left(F\right).$$

□

5. Future aspects

It should be noted that there are still much work to be done and there is a scope of extending the present work in future. For instance, the case in which the coefficients of differential equations are meromorphic functions of $[p, q]$-order and the case in which the coefficients of differential equations of analytic functions of $[p, q]$-order in the unit disc could be considered.

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References

[1] B. Belaidi, *On the iterated order and the fixed points of entire solutions of some complex linear differential equations*, Electron. J. Qual. Theory Differ. Equ. **2006** (2006), No. 9, 11 pp.

[2] ———, *Growth and oscillation of solutions to linear differential equations with entire coefficients having the same order*, Electron. J. Differential Equations **2009** (2009), No. 70, 10 pp.

[3] ———, *On the* $[p, q]$*-order of meromorphic solutions of linear differential equations*, Acta Univ. M. Belii Ser. Math. **23** (2015), 57–69.

[4] L. G. Bernal, *On growth* k*-order of solutions of a complex homogeneous linear differential equation*, Proc. Amer. Math. Soc. **101** (1987), no. 2, 317–322.

[5] T.-B. Cao, *Complex oscillation of entire solutions of higher-order linear differential equations*, Electron. J. Differential Equations **2006** (2006), No. 81, 8 pp.

[6] T.-B. Cao, J.-F. Xu, and Z.-X. Chen, *On the meromorphic solutions of linear differential equations on the complex plane*, J. Math. Anal. Appl. **364** (2010), no. 1, 130–142.

[7] W. K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.

[8] O. P. Juneja, G. P. Kapoor, and S. K. Bajpai, *On the* (p, q)*-order and lower* (p, q)*-order of an entire function*, J. Reine Angew. Math. **282** (1976), 53–67.

[9] ———, *On the* (p, q)*-type and lower* (p, q)*-type of an entire function*, J. Reine Angew. Math. **290** (1977), 180–190.

[10] L. Kinnunen, *Linear differential equations with solutions of finite iterated order*, Southeast Asian Bull. Math. **22** (1998), no. 4, 385–405.

[11] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, De Gruyter Studies in Mathematics, **15**, Walter de Gruyter & Co., Berlin, 1993.

[12] J. Liu, J. Tu, and L.-Z. Shi, *Linear differential equations with entire coefficients of* $[p, q]$*-order in the complex plane*, J. Math. Anal. Appl. **372** (2010), no. 1, 55–67.

[13] J. Tu and Z.-X. Chen, *Growth of solutions of complex differential equations with meromorphic coefficients of finite iterated order*, Southeast Asian Bull. Math. **33** (2009), no. 1, 153–164.

[14] J. Tu and C.-F. Yi, *On the growth of solutions of a class of higher order linear differential equations with coefficients having the same order*, J. Math. Anal. Appl. **340** (2008), no. 1, 487–497.

[15] L. Yang, *Value Distribution Theory,* Springer, Berlin, 1993.

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