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SOME RESULTS OF THE CARATHÉODORY'S INEQUALITY AT THE BOUNDARY

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Abstract. In this paper, a boundary version of the Carathéodory's inequality is investigated. We shall give an estimate below $|f'(b)|$ according to the first nonzero Taylor coefficient of about two zeros, namely $z=0$ and $z_1 \neq 0$. The sharpness of these estimates is also proved.

1. Introduction

Let f be a holomorphic function in the disc $D=\{z:|z|<1\}$, $f(0)=0$ and $|f(z)|<1$ for $|z|<1$. In accordance with the classical Schwarz lemma, for any point z in the disc D, we have $|f(z)|\leq |z|$ and $|f'(0)|\leq 1$. Equality in these inequalities (in the first one, for $z\neq 0$) occurs only if $f(z)=\lambda z$, $|\lambda|=1$ ([8], p. 329). It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point b with $|b|=1$, and if $|f(b)|=1$ and $f'(b)$ exists, then $|f'(b)|\geq 1$, which is known as the Schwarz lemma on the boundary.

Chelst, Osserman, Burns and Krantz ([3,4,20]) studied the Schwarz lemma at the boundary of the unit disk, respectively. The similar types of results which are related with the subject of the paper can be found in ([13–15]). In addition, the concerning results in more general aspects is discussed by M. Mateljević in [16] where was announced on ResearchGate. Krantz [11] explored versions of the Schwarz lemma at the boundary point of a domain, and reviewed. X. Tang, T. Liu and J. Lu [22] established a new type of the classical boundary Schwraz lemma for holomorphic self-mappings of the unit polydisk D^n in $\mathbb{C}^n$. They extended the classical Schwarz lemma at the boundary to high dimensions. Also, M. Jeong [10] got some inequalities at a boundary point for a different form of holomorphic functions and showed the sharpness of these inequalities. In addition, M. Jeong found a necessary and sufficient condition for a holomorphic map to have fixed points only on the boundary of the unit disc and compared its derivatives at fixed points to get some relations among them [9]. In the last 15 years, there have been tremendous studies on

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Schwarz lemma at the boundary (see [1,2,5–7,9,10,12,17,20,22] and references therein). Some of them are about the below boundary of modulus of the functions derivation at the points (contact points) which satisfies $|f(b)| = 1$ condition of the boundary of the unit circle. In this paper, we studied "a boundary version of the Carathéodory's inequalities" as analog the Schwarz lemma at the boundary [20].

The Carathéodory's inequality states that, if the function $f(z)$ is holomorphic in the unit disc D and $\Re f \leq A$ in D, then the inequality

$$|f(z) - f(0)| \leq \frac{2(A - \Re f(0))|z|}{1 - |z|} \tag{1.1}$$

holds for all $z \in D$, and moreover

$$|f'(0)| \leq 2(A - \Re f(0)). \tag{1.2}$$

Equality is achieved in (1.1) (for some nonzero $z \in D$) or in (1.2) if and only if f is the function of the form

$$f(z) = f(0) + \frac{2(A - \Re f(0))\, z e^{i\theta}}{1 + z e^{i\theta}},$$

where θ is a real number ([19]).

In [18], a weak version of known Carathéodory's inequality was investigated at the boundary of the unit disc. This estimation is as follows:

Let f be a holomorphic function in the unit disc D, $f(0) = 0$ and $\Re f \leq A$ for $|z| < 1$. Further assume that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b, $\Re f(b) = A$. Then

$$|f'(b)| \geq \frac{A}{2}. \tag{1.3}$$

The equality in (1.3) holds if and only if

$$f(z) = 2A\frac{z e^{i\theta}}{1 + z e^{i\theta}},$$

where θ is a real number.

In [19], we estimated a module of angular derivative of the functions, that satisfied Carathéodory's inequality, by taking into account their first nonzero two Maclaurin coefficients.

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see [21]).

Lemma 1.1 (Julia-Wolff lemma)**.** *Let f be a holomorphic function in D, $f(0) = 0$ and $f(D) \subset D$. If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial D$, $|f(b)| = 1$, then the angular derivative $f'(b)$ exists and $1 \leq |f'(b)| \leq \infty$.*

2. Main results

We have following results, which can be offered as the boundary refinement of the Carathéodory's inequality. We shall give an estimate below $|f'(b)|$ according to the first nonzero Taylor coefficient of about two zeros, namely $z=0$ and $z_1 \neq 0$. The sharpness of these estimates is also proved.

Theorem 2.1. *Let f be a holomorphic function in the unit disc D, $\Re f \leq A$ for $|z|<1$ and $f(z_1)=f(0)$ for $0<|z_1|<1$. Suppose that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b, $\Re f(b)=A$. Then we have the inequality*

$$|f'(b)| \geq \frac{A-\Re f(0)}{2}\left(1+\frac{1-|z_1|^2}{|b-z_1|^2}+\frac{2\beta|z_1|-|f'(0)|}{2\beta|z_1|+|f'(0)|}\right.$$

$$\left.\times\left[1+\frac{4\beta^2|z_1|^2+|f'(z_1)|\left(1-|z_1|^2\right)|f'(0)|-2\beta|f'(z_1)|\left(1-|z_1|^2\right)-2\beta|f'(0)|}{4\beta^2|z_1|^2+|f'(z_1)|\left(1-|z_1|^2\right)|f'(0)|+2\beta|f'(z_1)|\left(1-|z_1|^2\right)+2\beta|f'(0)|}\frac{1-|z_1|^2}{|b-z_1|^2}\right]\right), \tag{2.1}$$

where $\beta = A-\Re f(0)$.

The inequality (2.1) *is sharp, with equality for each possible values $|f'(0)| = 2\beta e$ and $|f'(z_1)| = 2\beta d$ $\left(0 \leq e \leq 2\beta|z_1|,\ 0 \leq d \leq 2\beta\frac{|z_1|}{1-|z_1|^2}\right)$.*

Proof. Let

$$q(z)=\frac{z-z_1}{1-\overline{z_1}z}.$$

Also, let $h: D \to D$ be a holomorphic function and a point $z_1 \in D$ in order to satisfy

$$\left|\frac{h(z)-h(z_1)}{1-\overline{h(z_1)}h(z)}\right| \leq \left|\frac{z-z_1}{1-\overline{z_1}z}\right| = |q(z)|$$

and

$$|h(z)| \leq \frac{|h(z_1)|+|q(z)|}{1+|h(z_1)|\,|q(z)|} \tag{2.2}$$

by Schwarz-pick lemma [8]. If $v: D \to D$ is a holomorphic function and $0<|z_1|<1$, letting

$$h(z)=\frac{v(z)-v(0)}{z\left(1-\overline{v(0)}v(z)\right)}$$

in (2.2), we obtain

$$\left|\frac{v(z)-v(0)}{z\left(1-\overline{v(0)}v(z)\right)}\right| \leq \frac{\left|\frac{v(z_1)-v(0)}{z_1\left(1-\overline{v(0)}v(z_1)\right)}\right|+|q(z)|}{1+\left|\frac{v(z_1)-v(0)}{z_1\left(1-\overline{v(0)}v(z_1)\right)}\right||q(z)|}$$

and

$$|v(z)| \leq \frac{|v(0)| + |z| \frac{|C|+|q(z)|}{1+|C||q(z)|}}{1 + |v(0)| \, |z| \frac{|C|+|q(z)|}{1+|C||q(z)|}}, \tag{2.3}$$

where

$$C = \frac{v(z_1) - v(0)}{z_1 \left(1 - \overline{v(0)} v(z_1)\right)}.$$

Without loss of generality, we will assume that $b = 1$. Let

$$\varphi(z) = \frac{f(z) - f(0)}{2\beta - (f(z) - f(0))}, \ \beta = A - \Re f(0).$$

The function $\varphi(z)$ is a holomorphic function in the unit disc D, $|\varphi(z)| < 1$ for $z \in D$.

If we take

$$v(z) = \frac{\varphi(z)}{z \frac{z - z_1}{1 - \overline{z_1} z}},$$

then

$$v(z_1) = \frac{\varphi'(z_1) \left(1 - |z_1|^2\right)}{z_1}, \ v(0) = \frac{\varphi'(0)}{-z_1}$$

and

$$C = \frac{\frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1} + \frac{\varphi'(0)}{z_1}}{z_1 \left(1 + \frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1} \frac{\varphi'(0)}{z_1}\right)},$$

where $|C| \leq 1$. Let $|v(0)| = \alpha$ and

$$\mathrm{D} = \frac{\left|\frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1}\right| + \left|\frac{\varphi'(0)}{z_1}\right|}{|z_1| \left(1 + \left|\frac{\varphi'(z_1)\left(1-|z_1|^2\right)}{z_1}\right| \left|\frac{\varphi'(0)}{z_1}\right|\right)}.$$

From (2.3), we get

$$|\varphi(z)| \leq |z| \, |q(z)| \frac{\alpha + |z| \frac{\mathrm{D}+|q(z)|}{1+\mathrm{D}|q(z)|}}{1 + \alpha \, |z| \frac{\mathrm{D}+|q(z)|}{1+\mathrm{D}|q(z)|}}$$

and

$$\begin{aligned} \frac{1 - |\varphi(z)|}{1 - |z|} &\geq \frac{1 + \alpha \, |z| \frac{\mathrm{D}+|q(z)|}{1+\mathrm{D}|q(z)|} - \alpha \, |z| \, |q(z)| - |q(z)| \, |z|^2 \frac{\mathrm{D}+|q(z)|}{1+\mathrm{D}|q(z)|}}{(1 - |z|) \left(1 + \alpha \, |z| \frac{\mathrm{D}+|q(z)|}{1+\mathrm{D}|q(z)|}\right)} \\ &= \varrho(z). \end{aligned} \tag{2.4}$$

Let $\kappa(z)=1+\alpha\left|z\right|\frac{\mathrm{D}+|q(z)|}{1+\mathrm{D}|q(z)|}$ and $\tau(z)=1+\mathrm{D}\left|q(z)\right|$. Then

$$\varrho(z)=\frac{1-\left|z\right|^{2}\left|q(z)\right|^{2}}{\left(1-\left|z\right|\right)\kappa(z)\tau(z)}+\mathrm{D}\left|q(z)\right|\frac{1-\left|z\right|^{2}}{\left(1-\left|z\right|\right)\kappa(z)\tau(z)}+\left|z\right|\mathrm{D}\alpha\frac{1-\left|q(z)\right|^{2}}{\left(1-\left|z\right|\right)\kappa(z)\tau(z)}.$$

Since

$$\lim_{z\to1}\kappa(z)=\lim_{z\to1}1+\alpha\left|z\right|\frac{\mathrm{D}+\left|q(z)\right|}{1+\mathrm{D}\left|q(z)\right|}=1+\alpha,$$
$$\lim_{z\to1}\tau(z)=\lim_{z\to1}1+\mathrm{D}\left|q(z)\right|=1+\mathrm{D}$$

and

$$1-\left|q(z)\right|^{2}=1-\left|\frac{z-z_{1}}{1-\overline{z_{1}}z}\right|^{2}=\frac{\left(1-\left|z_{1}\right|^{2}\right)\left(1-\left|z\right|^{2}\right)}{\left|1-\overline{z_{1}}z\right|^{2}},\tag{2.5}$$

passing to the angular limit in (2.4) gives

$$\left|\varphi'(1)\right|\geq\frac{2}{\left(1+\alpha\right)\left(1+\mathrm{D}\right)}\left(1+\frac{1-\left|z_{1}\right|^{2}}{\left|1-z_{1}\right|^{2}}+\mathrm{D}+\alpha\mathrm{D}\frac{1-\left|z_{1}\right|^{2}}{\left|1-z_{1}\right|^{2}}\right)$$
$$=1+\frac{1-\left|z_{1}\right|^{2}}{\left|1-z_{1}\right|^{2}}+\frac{1-\alpha}{1+\alpha}\left(1+\frac{1-\mathrm{D}}{1+\mathrm{D}}\frac{1-\left|z_{1}\right|^{2}}{\left|1-z_{1}\right|^{2}}\right).$$

Moreover, since

$$\frac{1-\alpha}{1+\alpha}=\frac{1-\left|v(0)\right|}{1+\left|v(0)\right|}=\frac{1-\left|\frac{\varphi'(0)}{z_{1}}\right|}{1+\left|\frac{\varphi'(0)}{z_{1}}\right|}=\frac{\left|z_{1}\right|-\left|\varphi'(0)\right|}{\left|z_{1}\right|+\left|\varphi'(0)\right|}$$
$$=\frac{\left|z_{1}\right|-\left|\frac{f'(0)}{2\beta}\right|}{\left|z_{1}\right|+\left|\frac{f'(0)}{2\beta}\right|}=\frac{2\beta\left|z_{1}\right|-\left|f'(0)\right|}{2\beta\left|z_{1}\right|+\left|f'(0)\right|},$$

$$\frac{1-\mathrm{D}}{1+\mathrm{D}}=\frac{1-\frac{\left|\frac{\varphi'(z_{1})\left(1-|z_{1}|^{2}\right)}{z_{1}}\right|+\left|\frac{\varphi'(0)}{z_{1}}\right|}{|z_{1}|\left(1+\left|\frac{\varphi'(z_{1})(1-|z_{1}|^{2})}{z_{1}}\right|\left|\frac{\varphi'(0)}{z_{1}}\right|\right)}}{1+\frac{\left|\frac{\varphi'(z_{1})\left(1-|z_{1}|^{2}\right)}{z_{1}}\right|+\left|\frac{\varphi'(0)}{z_{1}}\right|}{|z_{1}|\left(1+\left|\frac{\varphi'(z_{1})(1-|z_{1}|^{2})}{z_{1}}\right|\left|\frac{\varphi'(0)}{z_{1}}\right|\right)}}$$
$$=\frac{1-\frac{\left|\frac{\frac{f'(z_{1})}{2\beta}\left(1-|z_{1}|^{2}\right)}{z_{1}}\right|+\left|\frac{\frac{f'(0)}{2\beta}}{z_{1}}\right|}{|z_{1}|\left(1+\left|\frac{\frac{f'(z_{1})}{2\beta}(1-|z_{1}|^{2})}{z_{1}}\right|\left|\frac{\frac{f'(0)}{2\beta}}{z_{1}}\right|\right)}}{1+\frac{\left|\frac{\frac{f'(z_{1})}{2\beta}\left(1-|z_{1}|^{2}\right)}{z_{1}}\right|+\left|\frac{\frac{f'(0)}{2\beta}}{z_{1}}\right|}{|z_{1}|\left(1+\left|\frac{\frac{f'(z_{1})}{2\beta}(1-|z_{1}|^{2})}{z_{1}}\right|\left|\frac{\frac{f'(0)}{2\beta}}{z_{1}}\right|\right)}}$$

and

$$\frac{1-\mathrm{D}}{1+\mathrm{D}}=\frac{|z_1|\left(1+\left|\frac{\frac{f'(z_1)}{2\beta}\left(1-|z_1|^2\right)}{z_1}\right|\left|\frac{\frac{f'(0)}{2\beta}}{z_1}\right|\right)-\left|\frac{\frac{f'(z_1)}{2\beta}\left(1-|z_1|^2\right)}{z_1}\right|-\left|\frac{\frac{f'(0)}{2\beta}}{z_1}\right|}{|z_1|\left(1+\left|\frac{\frac{f'(z_1)}{2\beta}(1-|z_1|^2)}{z_1}\right|\left|\frac{\frac{f'(0)}{2\beta}}{z_1}\right|\right)+\left|\frac{\frac{f'(z_1)}{2\beta}(1-|z_1|^2)}{z_1}\right|+\left|\frac{\frac{f'(0)}{2\beta}}{z_1}\right|}$$

$$=\frac{4\beta^2|z_1|^2+\left|f'(z_1)\right|\left(1-|z_1|^2\right)\left|f'(0)\right|-2\beta\left|f'(z_1)\right|\left(1-|z_1|^2\right)-2\beta\left|f'(0)\right|}{4\beta^2|z_1|^2+|f'(z_1)|\left(1-|z_1|^2\right)|f'(0)|+2\beta|f'(z_1)|\left(1-|z_1|^2\right)+2\beta|f'(0)|},$$

we obtain

$$|\varphi'(1)|\geq 1+\frac{1-|z_1|^2}{|1-z_1|^2}+\frac{2\beta|z_1|-\left|f'(0)\right|}{2\beta|z_1|+|f'(0)|}$$
$$\times\left[1+\frac{4\beta^2|z_1|^2+\left|f'(z_1)\right|\left(1-|z_1|^2\right)\left|f'(0)\right|-2\beta\left|f'(z_1)\right|\left(1-|z_1|^2\right)-2\beta\left|f'(0)\right|}{4\beta^2|z_1|^2+|f'(z_1)|\left(1-|z_1|^2\right)|f'(0)|+2\beta|f'(z_1)|\left(1-|z_1|^2\right)+2\beta|f'(0)|}\frac{1-|z_1|^2}{|1-z_1|^2}\right].$$

From definition of $\varphi(z)$, we have

$$\varphi'(z)=\frac{2\beta f'(z)}{\left(2\beta-(f(z)-f(0))\right)^2}$$

and

$$|\varphi'(1)|=\left|\frac{2\beta f'(1)}{\left(2\beta-(f(1)-f(0))\right)^2}\right|\leq\frac{2\left|f'(1)\right|}{\beta}.$$

Thus, we obtain the inequality (2.1).

Now, we shall show that the inequality (2.1) is sharp.

Since

$$v(z)=\frac{\varphi(z)}{z\frac{z-z_1}{1-\overline{z_1}z}}$$

is a holomorphic function in the unit disc and $|v(z)|\leq 1$ for $z\in D$, we obtain

$$|\varphi'(0)|\leq|z_1|$$

and

$$|\varphi'(z_1)|\leq\frac{|z_1|}{1-|z_1|^2}.$$

We take $z_1\in(-1,0)$ and arbitrary two numbers e and f, such that $0\leq e\leq 2\beta\left|z_1\right|,\ 0\leq d\leq 2\beta\frac{|z_1|}{1-|z_1|^2}$.

Let

$$\mathtt{K}=\frac{\frac{d\left(1-|z_1|^2\right)}{z_1}+\frac{e}{z_1}}{z_1\left(1+ed\frac{1-|z_1|^2}{z_1^2}\right)}=\frac{1}{z_1^2}\frac{d\left(1-|z_1|^2\right)+e}{1+ed\frac{1-|z_1|^2}{z_1^2}}.$$

The auxiliary function

$$s(z)=z\frac{z-z_1}{1-\overline{z_1}z}\frac{\frac{-e}{z_1}+z\frac{\mathtt{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{K}\frac{z-z_1}{1-\overline{z_1}z}}}{1-\frac{e}{z_1}z\frac{\mathtt{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathtt{K}\frac{z-z_1}{1-\overline{z_1}z}}}$$

is holomorphic in D and $|s(z)|<1$ for $z\in D$. Let

$$\frac{f(z)-f(0)}{2\beta-(f(z)-f(0))}=z\frac{z-z_1}{1-\overline{z_1}z}\frac{\frac{-e}{z_1}+z\frac{\mathrm{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathrm{K}\frac{z-z_1}{1-\overline{z_1}z}}}{1-\frac{e}{z_1}z\frac{\mathrm{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathrm{K}\frac{z-z_1}{1-\overline{z_1}z}}}. \tag{2.6}$$

So, we have

$$f(z)=f(0)+2\beta\frac{z\frac{z-z_1}{1-\overline{z_1}z}\frac{\frac{-e}{z_1}+z\frac{\mathrm{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathrm{K}\frac{z-z_1}{1-\overline{z_1}z}}}{1-\frac{e}{z_1}z\frac{\mathrm{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathrm{K}\frac{z-z_1}{1-\overline{z_1}z}}}}{1+z\frac{z-z_1}{1-\overline{z_1}z}\frac{\frac{-e}{z_1}+z\frac{\mathrm{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathrm{K}\frac{z-z_1}{1-\overline{z_1}z}}}{1-\frac{e}{z_1}z\frac{\mathrm{K}+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathrm{K}\frac{z-z_1}{1-\overline{z_1}z}}}}.$$

Therefore, we take $|f'(0)|=2\beta e$ and $|f'(z_1)|=2\beta d$.

From (2.6), with the simple calculations, we obtain

$$\begin{aligned}&\frac{2\beta f'(1)}{(2\beta-(f(1)-f(0)))^2}\\&=1+\frac{1-z_1^2}{(1-z_1)^2}+\frac{\left(1+\frac{1-z_1^2}{(1-z_1)^2}\frac{1-\mathrm{K}^2}{(1+\mathrm{K})^2}\right)\left(1-\frac{e}{z_1}\right)+\frac{e}{z_1}\left(1+\frac{1-z_1^2}{(1-z_1)^2}\frac{1-\mathrm{K}^2}{(1+\mathrm{K})^2}\right)\left(1-\frac{e}{z_1}\right)}{\left(1-\frac{e}{z_1}\right)^2}\\&=1+\frac{1-z_1^2}{(1-z_1)^2}+\frac{e+z_1}{-e+z_1}\left(1+\frac{1-z_1^2}{(1-z_1)^2}\frac{z_1^2+ed\left(1-z_1^2\right)-d\left(1-z_1^2\right)-e}{z_1^2+ed\left(1-z_1^2\right)+d\left(1-z_1^2\right)+e}\right)\end{aligned}$$

and

$$|f'(1)|\geq\frac{\beta}{2}\left(1+\frac{1-z_1^2}{(1-z_1)^2}+\frac{e+z_1}{-e+z_1}\left(1+\frac{1-z_1^2}{(1-z_1)^2}\frac{z_1^2+ed\left(1-z_1^2\right)-d\left(1-z_1^2\right)-e}{z_1^2+ed\left(1-z_1^2\right)+d\left(1-z_1^2\right)+e}\right)\right).$$

Since $z_1\in(-1,0)$, the last equality show that (2.1) is sharp. □

Theorem 2.2. *Let f be a holomorphic function in the unit disc D, $\Re f\leq A$ for $|z|<1$ and $f(z_1)=f(0)$ for $0<|z_1|<1$. Assume that, for positive integers p and m, f have expansions $f(z)=f(0)+c_pz^p+c_{p+1}z^{p+1}+\cdots$, $c_p\neq 0$ and $f(z)=f(0)+a_m(z-z_1)^m+a_{m+1}(z-z_1)^{m+1}+\cdots$, $a_m\neq 0$, about the points $z=0$ and $z=z_1$, respectively. Suppose that, for some $b\in\partial D$, f has an angular limit $f(b)$ at b, $\Re f(b)=A$. Then we have the inequality*

$$\begin{aligned}|f'(b)|\geq{}&\frac{A-\Re f(0)}{2}\left(p+m\frac{1-|z_1|^2}{|b-z_1|^2}+\frac{2\beta|z_1|^m-|c_p|}{2\beta|z_1|^m+|c_p|}\right.\\&\left.\times\left[1+\frac{4\beta^2|z_1|^{p+m}+|a_m|\left(1-|z_1|^2\right)^m|c_p|-2\beta|a_m|\left(1-|z_1|^2\right)|z_1|^{m-1}-2\beta|c_p||z_1|^{p-1}}{4\beta^2|z_1|^{p+m}+|a_m|\left(1-|z_1|^2\right)^m|c_p|+2\beta|a_m|\left(1-|z_1|^2\right)|z_1|^{m-1}+2\beta|c_p||z_1|^{p-1}}\frac{1-|z_1|^2}{|b-z_1|^2}\right]\right),\end{aligned}\tag{2.7}$$

where $\beta=A-\Re f(0)$.

The inequality (2.7) *is sharp, with equality for each possible value of* $|a_m|$ *and* $|c_p|$ ($|c_p| \leq 2\beta |z_1|^p$, $|a_m| \leq 2\beta \frac{|z_1|^p}{\left(1-|z_1|^2\right)^m}$).

Proof. Consider the function

$$v(z) = \frac{\varphi(z)}{z^p \left(\frac{z-z_1}{1-\overline{z_1}z}\right)^m}.$$

$v(z)$ is a holomorphic function in the unit disc, $|v(z)| < 1$ for $|z| < 1$, $v(0) = (-1)^m \frac{c_p}{2\beta z_1^m}$ and $v(z_1) = \frac{a_m}{2\beta z_1^p}\left(1-|z_1|^2\right)^m$ ($|v(0)| \leq 1$, $|v(z_1)| \leq 1$).

Let $\varsigma = \frac{|c_p|}{2\beta |z_1^m|}$ and

$$C_1 = \frac{\left|\frac{a_m}{z_1^p}\left(1-|z_1|^2\right)^m\right| + \left|\frac{c_p}{z_1^m}\right|}{|z_1|\left(1+\left|\frac{a_m}{z_1^p}\left(1-|z_1|^2\right)^m\right|\left|\frac{c_p}{z_1^m}\right|\right)}.$$

From (2.2) and (2.3), we obtain

$$|\varphi(z)| \leq |z|^p |q(z)|^m \frac{\varsigma + |z| \frac{C_1+|q(z)|}{1+C_1|q(z)|}}{1+\varsigma |z| \frac{C_1+|q(z)|}{1+C_1|q(z)|}}$$

and

$$\mathbf{I} = \frac{1-|\varphi(z)|}{1-|z|} \geq \frac{1+\varsigma |z| \frac{C_1+|q(z)|}{1+C_1|q(z)|} - \varsigma |z|^p |q(z)|^m - |q(z)|^m |z|^{p+1} \frac{C_1+|q(z)|}{1+C_1|q(z)|}}{(1-|z|)\left(1+\varsigma |z| \frac{C_1+|q(z)|}{1+C_1|q(z)|}\right)}.$$

Let $R_1(z) = 1+\varsigma |z| \frac{C_1+|q(z)|}{1+C_1|q(z)|}$ and $R_2(z) = 1 + C_1 |q(z)|$. Therefore, we take

$$\begin{aligned}\mathbf{I} \geq \frac{1}{R_1(z)R_2(z)} &\left\{\frac{1-|z|^{p+1}|q(z)|^{m+1}}{1-|z|} + C_1 |q(z)| \frac{1-|z|^{p+1}|q(z)|^{m-1}}{1-|z|}\right. \\ &\left. + \varsigma |z| |q(z)| \frac{1-|z|^{p-1}|q(z)|^{m-1}}{1-|z|} + \varsigma |z| C_1 \frac{1-|z|^{p-1}|q(z)|^{m-1}}{1-|z|}\right\}.\end{aligned}$$

Passing to the angular limit in the last inequality and using (2.5), we obtain

$$\begin{aligned}|\varphi'(1)| \geq \frac{2}{(1+\varsigma)(1+C_1)} &\left\{p+1+\frac{1-|z_1|^2}{|1-z_1|^2}(m+1) + C_1\left[p+1+\frac{1-|z_1|^2}{|1-z_1|^2}(m+1)\right]\right. \\ &\left. + \varsigma\left[p-1+\frac{1-|z_1|^2}{|1-z_1|^2}(m-1)\right] + \varsigma C_1\left[p-1+\frac{1-|z_1|^2}{|1-z_1|^2}(m+1)\right]\right\} \\ &= p + m\frac{1-|z_1|^2}{|1-z_1|^2} + \frac{1-\varsigma}{1+\varsigma}\left[1+\frac{1-C_1}{1+C_1}\frac{1-|z_1|^2}{|1-z_1|^2}\right].\end{aligned}$$

Since

$$\frac{1-\varsigma}{1+\varsigma}=\frac{1-\frac{|c_p|}{2\beta|z_1^m|}}{1+\frac{|c_p|}{2\beta|z_1^m|}}=\frac{2\beta\left|z_1^m\right|-|c_p|}{2\beta\left|z_1^m\right|+|c_p|},$$

$$\frac{1-C_1}{1+C_1}=\frac{1-\frac{\left|\frac{a_m}{z_1^p}\left(1-|z_1|^2\right)^m\right|+\left|\frac{c_p}{z_1^m}\right|}{|z_1|\left(1+\left|\frac{a_m}{z_1^p}\left(1-|z_1|^2\right)^m\right|\left|\frac{c_p}{z_1^m}\right|\right)}}{1+\frac{\left|\frac{a_m}{z_1^p}\left(1-|z_1|^2\right)^m\right|+\left|\frac{c_p}{z_1^m}\right|}{|z_1|\left(1+\left|\frac{a_m}{z_1^p}\left(1-|z_1|^2\right)^m\right|\left|\frac{c_p}{z_1^m}\right|\right)}}$$

and

$$\frac{1-C_1}{1+C_1}=\frac{4\beta^2|z_1|^{p+m}+|a_m|\left(1-|z_1|^2\right)^m|c_p|-2\beta|a_m|\left(1-|z_1|^2\right)|z_1|^{m-1}-2\beta|c_p||z_1|^{p-1}}{4\beta^2|z_1|^{p+m}+|a_m|\left(1-|z_1|^2\right)^m|c_p|+2\beta|a_m|\left(1-|z_1|^2\right)|z_1|^{m-1}+2\beta|c_p||z_1|^{p-1}},$$

we obtain

$$|\varphi'(1)|\geq p+m\frac{1-|z_1|^2}{|1-z_1|^2}+\frac{2\beta\left|z_1^m\right|-|c_p|}{2\beta\left|z_1^m\right|+|c_p|}$$
$$\times\left[1+\frac{4\beta^2|z_1|^{p+m}+|a_m|\left(1-|z_1|^2\right)^m|c_p|-2\beta|a_m|\left(1-|z_1|^2\right)|z_1|^{m-1}-2\beta|c_p||z_1|^{p-1}}{4\beta^2|z_1|^{p+m}+|a_m|\left(1-|z_1|^2\right)^m|c_p|+2\beta|a_m|\left(1-|z_1|^2\right)|z_1|^{m-1}+2\beta|c_p||z_1|^{p-1}}\frac{1-|z_1|^2}{|1-z_1|^2}\right].$$

Thus, we obtain the inequality (2.7).

In order to show that the inequality is sharp, choose arbitrary real numbers z_1, x and y such that $0<x<2\beta|z_1|^m$, $0<y<2\beta\frac{|z_1|^p}{\left(1-|z_1|^2\right)^m}$.

Let

$$\mathbf{D}_1=\frac{\frac{y}{z_1^p}\left(1-|z_1|^2\right)^m+(-1)^{m-1}\frac{x}{z_1^m}}{z_1\left(1+(-1)^{m-1}\frac{y}{z_1^p}\left(1-|z_1|^2\right)^m\frac{x}{z_1^m}\right)},$$

$$\varphi(z)=z^p\left(\frac{z-z_1}{1-\overline{z_1}z}\right)^m\frac{(-1)^m\frac{x}{z_1^m}+z\frac{\mathbf{D}_1+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathbf{D}_1\frac{z-z_1}{1-\overline{z_1}z}}}{1+(-1)^m\frac{x}{z_1^m}z\frac{\mathbf{D}_1+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathbf{D}_1\frac{z-z_1}{1-\overline{z_1}z}}}\tag{2.8}$$

and

$$f(z)=f(0)+2\beta\frac{z^p\left(\frac{z-z_1}{1-\overline{z_1}z}\right)^m\frac{(-1)^m\frac{x}{z_1^m}+z\frac{\mathbf{D}_1+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathbf{D}_1\frac{z-z_1}{1-\overline{z_1}z}}}{1+(-1)^m\frac{x}{z_1^m}z\frac{\mathbf{D}_1+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathbf{D}_1\frac{z-z_1}{1-\overline{z_1}z}}}}{1+z^p\left(\frac{z-z_1}{1-\overline{z_1}z}\right)^m\frac{(-1)^m\frac{x}{z_1^m}+z\frac{\mathbf{D}_1+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathbf{D}_1\frac{z-z_1}{1-\overline{z_1}z}}}{1+(-1)^m\frac{x}{z_1^m}z\frac{\mathbf{D}_1+\frac{z-z_1}{1-\overline{z_1}z}}{1+\mathbf{D}_1\frac{z-z_1}{1-\overline{z_1}z}}}}.$$

From (2.8), with the simple calculations, we obtain $\frac{\varphi^{(p)}(0)}{p!} = x$, $\frac{\varphi^{(m)}(0)}{m!} = y$ and

$$\frac{2\beta f'(1)}{(2\beta - (f(1) - f(0)))^2}$$
$$= p + m\frac{1-|z_1|^2}{(1-z_1)^2} + \frac{z_1^m - (-1)^m x}{z_1^m + (-1)^m x}$$
$$\times \left[1 + \frac{1-|z_1|^2}{(1-z_1)^2} \frac{z_1^{m+p} + (-1)^{m-1} y \left(1-|z_1|^2\right)^m x - y\left(1-|z_1|^2\right)^m z_1^{m-1} - (-1)^{m-1} x z_1^{p-1}}{z_1^{m+p} + (-1)^{m-1} y \left(1-|z_1|^2\right)^m x + y\left(1-|z_1|^2\right)^m z_1^{m-1} + (-1)^{m-1} x z_1^{p-1}}\right].$$

Choosing suitable signs of the numbers x, y and z_1, we conclude from the last equality that the inequality (2.7) is sharp. □

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