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WEIGHTED COMPOSITION OPERATORS FROM THE KIM CLASS AND THE SMIRNOV CLASS TO WEIGHTED BLOCH TYPE SPACES

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Abstract. In this paper, we prove that boundedness with respect to metric balls of weighted composition operators from the Kim class and the Smirnov class to weighted Bloch type spaces is equivalent to metrical compactness of weighted composition operators between these spaces.

1. Introduction and preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the space of all holomorphic functions on $\mathbb{D}$, $\mathbb{T}$ the boundary of $\mathbb{D}$ and $d\sigma$ denote the normalized Lebesgue measure on $\mathbb{T}$. The *Nevanlinna class* N is defined as the set of all holomorphic functions f on $\mathbb{D}$ such that

$$\sup_{0\le r<1}\int_{\mathbb{T}}\log(1+|f(r\zeta)|)\,d\sigma(\zeta)<\infty.$$

It is well-known fact that every holomorphic function f in the class N has a finite non-tangential limit, denoted by f^*, almost everywhere on $\mathbb{T}$.

The *Smirnov class* N^+ is a subclass of the class N such that f satisfies

$$\sup_{0\le r<1}\int_{\mathbb{T}}\log(1+|f(r\zeta)|)\,d\sigma(\zeta)=\int_{\mathbb{T}}\log(1+|f^*(\zeta)|)\,d\sigma(\zeta).$$

With respect to the translation-invariant metric d_{N^+} define as $d_{N^+}(f,g)=\|f-g\|_{N^+}$ for $f,g\in N^+$, where

$$\|f\|_{N^+}=\int_{\mathbb{T}}\log(1+|f^*(\zeta)|)\,d\sigma(\zeta),$$

the Smirnov class N^+ becomes an F-space. Moreover, the subharmonicity of $\log(1+|f|)$ implies that $f\in N^+$ has the following growth estimation:

$$\log(1+|f(z)|)\le\frac{4\|f\|_{N^+}}{1-|z|^2}\tag{1.1}$$

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for $z \in \mathbb{D}$. Thus the convergence with respect to the metric d_{N^+} is stronger than the uniform convergence on compact subsets of $\mathbb{D}$.

The class M which is a subclass of N^+ is the set of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_M := \int_{\mathbb{T}} \log(1 + Mf(\zeta))\, d\sigma(\zeta) < \infty,$$

where $Mf(\zeta) = \sup_{0 \le r < 1} |f(r\zeta)|$ is the radial maximal function of f. This class M had been introduced by H. O. Kim [6] and we call it the *Kim class*. For more about these type of spaces, we refer [1], [2], [3], [5], [6] and [13].

Let ν be a strictly positive continuous function (*weight*) on $\mathbb{D}$ such that $\nu(z) = \nu(|z|)$ for every $z \in \mathbb{D}$, ν is nonincreasing with respect to $|z|$ and $\nu(z) \to 0$ as $|z| \to 1$. For such a weight ν, the *weighted Bloch-type space* $\mathcal{B}_\nu$ on $\mathbb{D}$ is the space of all holomorphic functions f on $\mathbb{D}$ such that

$$\sup_{z \in \mathbb{D}} \nu(z)|f'(z)| < \infty.$$

The *little weighted Bloch-type space* $\mathcal{B}_{\nu,0}$ consists of all $f \in \mathcal{B}_\nu$ such that

$$\lim_{|z| \to 1} \nu(z)|f'(z)| = 0.$$

Both spaces $\mathcal{B}_\nu$ and $\mathcal{B}_{\nu,0}$ are Banach spaces with the norm

$$\|f\|_{\mathcal{B}_\nu} = |f(0)| + \sup_{z \in \mathbb{D}} \nu(z)|f'(z)|,$$

and $\mathcal{B}_{\nu,0}$ is a closed subspace of $\mathcal{B}_\nu$. For a closed subset $L \subset \mathcal{B}_{\nu,0}$, the compactness of it can be characterized as follows.

Lemma 1.1. *A closed set L in $\mathcal{B}_{\nu,0}$ is compact if and only if it is bounded with respect to the norm $\|\cdot\|_{\mathcal{B}_\nu}$ and satisfies*

$$\lim_{|z| \to 1} \sup_{f \in L} \nu(z)|f'(z)| = 0.$$

This result for the case $\nu(z) = (1 - |z|^2)$ was proved by Madigan and Matheson [7]. By a modification of their proof, we can prove the above lemma.
We also need *weighted type space* $\mathcal{A}_\nu$ of all holomorphic functions f on $\mathbb{D}$ such that

$$\sup_{z \in \mathbb{D}} \nu(z)|f(z)| < \infty$$

and *little weighted type space* $\mathcal{A}_{\nu,0}$ consists of all $f \in \mathcal{A}_\nu$ such that

$$\lim_{|z| \to 1} \nu(z)|f(z)| = 0.$$

Once again, both spaces $\mathcal{A}_\nu$ and $\mathcal{A}_{\nu,0}$ are Banach spaces with the norm

$$\|f\|_{\mathcal{A}_\nu} = \sup_{z \in \mathbb{D}} \nu(z)|f(z)|.$$

Let $\psi \in H(\mathbb{D})$ and φ a holomorphic self-map of $\mathbb{D}$, the *weighted composition operator* $W_{\psi,\varphi}$ is a linear operator on $H(\mathbb{D})$ defined by $W_{\psi,\varphi} f = \psi \cdot f \circ \varphi$

for $f \in H(\mathbb{D})$. Some operator theoretic properties of $W_{\psi,\varphi}$ acting on various holomorphic function spaces have been studied by many mathematicians. In recent progresses of studies on the operator $W_{\psi,\varphi}$, some authors have investigated the case $W_{\psi,\varphi}$ acting between different function spaces. It is of interest to provide function-theoretic characterizations involving ψ and φ of boundedness and compactness of $W_{\psi,\varphi}$ acting between different function spaces. Let X and Y be two linear topological vector spaces. Recall that a linear operator $T : X \to Y$ is bounded if the image of any topological bounded set in X under T is also a topological bounded set in Y. When X and Y are metric spaces with respect to suitable metrics d_X and d_Y, we can define a metrically bounded operator from X into Y. A linear operator $T : X \to Y$ is *metrically bounded* if there exists a constant $C > 0$ such that $d_Y(Tf, 0) \le C d_X(f, 0)$ for $f \in X$. In general, the boundedness of T and the metrical boundedness of T do not coincide. However, if X and Y are Banach spaces, then the metrical boundedness of T coincides with the boundedness of T. For more information on the metrical boundedness, we can refer to papers [2], [3] and [9]. Also recall that a linear operator $T : X \to Y$ is *bounded with respect to metric balls* if it takes every metric ball in X into a metric ball in Y. Since a metric ball is also a bounded set, so boundedness with respect to metric balls also coincides with topological boundedness and metrical boundedness, if X and Y are Banach spaces. The topological boundedness of operators on the Smirnov class and on the Kim class is not equivalent to the metrical boundedness or the boundedness with respect to metric balls, see [6] and [13]. To consider the boundedness for operators on these classes, thus, we need boudedness with respect to metric balls. For the compactness of linear operators, for example composition operators or weighted composition operators on these spaces, we use the metrical compactness. Namely, $T : X \to Y$ is *metrically compact* if it takes every metric ball in X into a relatively compact subset in Y. These operators on Nevanlinna type spaces, the Smirnov class and the Kim class have been studied by several authors (see [1–3, 8–12]). Motivated by the work cited above, in this paper, we characterize boundedness with respect to metric balls and the metrical compactness of $W_{\psi,\varphi}$ acting from the Smirnov class and the Kim class into weighted Bloch-type spaces.

2. Boundedness and compactness

We first give sufficient conditions for $W_{\psi,\varphi}$ to be bounded with respect to metric balls.

Proposition 2.1. *Let $\psi \in H(\mathbb{D})$ and φ be a holomorphic self map of $\mathbb{D}$ such that*

$$\sup_{z\in\mathbb{D}} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} < \infty \tag{2.1}$$

and

$$\sup_{z\in\mathbb{D}} \nu(z)|\psi'(z)| \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} < \infty \tag{2.2}$$

for every $c > 0$. Then $W_{\psi,\varphi} : N^+$ or $M \to \mathcal{B}_\nu$ is bounded with respect to metric balls.

Proof. For $f \in N^+$ the inequality (1.1) gives

$$|f(z)| \leq \exp\left\{\frac{4\|f\|_{N^+}}{1-|z|^2}\right\}$$

for every $z \in \mathbb{D}$. Thus by Cauchy's integral formula, we obtain that

$$(1-|z|^2)|f'(z)| \leq \frac{2}{\pi}\int_{\mathbb{T}} |f(z+(1-|z|)\zeta/2)|\,|d\zeta| \leq 4\exp\left\{\frac{16\|f\|_{N^+}}{1-|z|^2}\right\} \tag{2.3}$$

for each $z \in \mathbb{D}$. Hence we have that

$$\begin{aligned}\|W_{\psi,\varphi}f\|_{\mathcal{B}_\nu} &= |\psi(0)f(\varphi(0))| + \sup_{z\in\mathbb{D}} \nu(z)|\psi'(z)f(\varphi(z)) + \psi(z)\varphi'(z)f'(\varphi(z))| \\ &\leq |\psi(0)|\exp\left\{\frac{4\|f\|_{N^+}}{1-|\varphi(0)|^2}\right\} + \sup_{z\in\mathbb{D}} \nu(z)|\psi'(z)|\exp\left\{\frac{4||f||_{N^+}}{1-|\varphi(z)|^2}\right\} \\ &\quad + 4\sup_{z\in\mathbb{D}} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\exp\left\{\frac{16||f||_{N^+}}{1-|\varphi(z)|^2}\right\}.\end{aligned}$$

Combining above inequality and the conditions (2.1) and (2.2), we see that $W_{\psi,\varphi}$ takes every metric ball in N^+ into a metric ball in $\mathcal{B}_\nu$, namely $W_{\psi,\varphi} : N^+ \to \mathcal{B}_\nu$ is bounded with respect to metric balls. Moreover, $M \subseteq N^+$ and $||f||_{N^+} \leq ||f||_M$. Thus every metric ball in M is also a metric ball in N^+. Hence $W_{\psi,\varphi} : M \to \mathcal{B}_\nu$ is also bounded with respect to metric balls. □

Proposition 2.2. *Let $\psi \in H(\mathbb{D})$ and φ be a holomorphic self map of $\mathbb{D}$ satisfying the conditions* (2.1) *and* (2.2) *for every $c > 0$. Then $W_{\psi,\varphi} : N^+ \to \mathcal{B}_\nu$ is bounded.*

Proof. Suppose that the conditions (2.1) and (2.2) hold. Then proceeding as in the proof of Proposition 2.1, we see that $W_{\psi,\varphi}(N^+) \subset \mathcal{B}_\nu$. Since N^+ is an F-space, hence by the closed graph theorem, we have that $W_{\psi,\varphi} : N^+ \to \mathcal{B}_\nu$ is also bounded. □

The main result of this paper, characterizes the metrical compactness of $W_{\psi,\varphi} : N^+$ or $M \to \mathcal{B}_\nu$. In fact, we prove that boundedness of $W_{\psi,\varphi} : N^+$ or $M \to \mathcal{B}_\nu$ with respect to metric balls is equivalent to metrical compactness of $W_{\psi,\varphi} : N^+$ or $M \to \mathcal{B}_\nu$.

To prove the main result, we need the following lemma.

Lemma 2.3. *Let $X = N^+$ or M. Suppose that $\psi \in H(\mathbb{D})$ and φ a holomorphic self-map of $\mathbb{D}$ such that $W_{\psi,\varphi}(X) \subset \mathcal{B}_\nu$. Then $W_{\psi,\varphi} : X \to \mathcal{B}_\nu$ is metrically compact if and only if for any sequences $\{f_j\}$ in X with $\|f_j\|_X \leq K$*

and converge to zero uniformly on compact subsets of $\mathbb{D}$, $\{W_{\psi,\varphi} f_j\}$ *converges to zero in* $\mathcal{B}_\nu$.

Proof. This is an extension of a well-known result on the compactness of weighted composition operators on holomorphic function spaces. By (1.1), we see that any metrical bounded sequence in X form a normal family. Hence an argument by using the Montel theorem proves this lemma proceeding on the same lines as the proof of Proposition 3.11 in [4]. □

Theorem 2.4. *Let* $\psi \in H(\mathbb{D})$ *and* φ *be a holomorpic self-map of* $\mathbb{D}$. *Then the following conditions are equivalent*;

(i) $W_{\psi,\varphi} : N^+ \to \mathcal{B}_\nu$ *is bounded with respect to metric balls,*
(ii) $W_{\psi,\varphi} : M \to \mathcal{B}_\nu$ *is bounded with respect to metric balls,*
(iii) $W_{\psi,\varphi} : N^+ \to \mathcal{B}_\nu$ *is metrically compact,*
(iv) $W_{\psi,\varphi} : M \to \mathcal{B}_\nu$ *is metrically compact,*
(v) $\psi \in \mathcal{B}_\nu$, $\psi\varphi' \in \mathcal{A}_\nu$

$$\lim_{|\varphi(z)|\to 1} \nu(z)|\psi'(z)| \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} = 0, \tag{2.4}$$

$$\lim_{|\varphi(z)|\to 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} = 0 \tag{2.5}$$

for any $c > 0$.

Proof. Since $\|f\|_{N^+} \le \|f\|_M$, so every metric ball in M is also a metric ball in N^+. Thus implications (i) ⇒ (ii) and (iii) ⇒ (iv) are trivial. A relatively compact set in $\mathcal{B}_\nu$ is a bounded set in it. Since $\mathcal{B}_\nu$ is a Banach space, each bounded set is also in a metric ball in $\mathcal{B}_\nu$. Hence the metrically compactness of $W_{\psi,\varphi}$ mapping into $\mathcal{B}_\nu$ gives the boundedness with respect to metric balls. So implications (iii) ⇒ (i) and (iv) ⇒ (ii) are obvious.

Now we will prove (v) ⇒ (iii). Assume that the conditions (2.4) and (2.5) hold for every $c \ge 0$ and take an $\varepsilon > 0$ arbitrary. Then we can choose an $0 < r_0 < 1$ such that

$$\sup_{|\varphi(z)|>r_0} \nu(z)|\psi'(z)| \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} < \varepsilon$$

and

$$\sup_{|\varphi(z)|>r_0} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} < \varepsilon$$

for any $c > 0$. Choose any sequence $\{f_j\}$ in N^+ with $\|f_j\|_{N^+} \le K$ for all j converging to zero uniformly on compact subsets of $\mathbb{D}$. Since $\psi \in \mathcal{B}_\nu$ and (2.4) imply (2.2), and $\psi\varphi' \in \mathcal{A}_\nu$ and (2.5) imply (2.1), so by Proposition 2.2, we see that $W_{\psi,\varphi}(N^+) \subset \mathcal{B}_\nu$. The assumptions $\psi \in \mathcal{B}_\nu$ and $\psi\varphi' \in \mathcal{A}_\nu$ also imply that

$$\sup_{|\varphi(z)|\le r_0} \nu(z)|\psi'(z)||f_j(\varphi(z))| \le \sup_{z\in\mathbb{D}} \nu(z)|\psi'(z)| \cdot \max_{|w|\le r_0} |f_j(w)| \to 0$$

and

$$\sup_{|\varphi(z)|\le r_0} \nu(z)|\psi(z)\varphi'(z)||f_j'(\varphi(z))| \le \sup_{z\in\mathbb{D}} \nu(z)|\psi(z)\varphi'(z)| \cdot \max_{|w|\le r_0} |f_j'(w)| \to 0$$

as $j \to \infty$. Therefore, we have that

$$\sup_{|\varphi(z)|\le r_0} \nu(z)|(W_{\psi,\varphi} f_j)'(z)| \to 0$$

as $j \to \infty$. On the other hand, it follows from (1.1) and (2.3) that

$$\begin{aligned}\sup_{|\varphi(z)|> r_0} \nu(z)|(W_{\psi,\varphi} f)'(z)| \le \sup_{|\varphi(z)|> r_0} \nu(z)|\psi'(z)| \exp\left\{\frac{4K}{1-|\varphi(z)|^2}\right\} \\ + 4 \sup_{|\varphi(z)|> r_0} \frac{\nu(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{16K}{1-|\varphi(z)|^2}\right\} < \varepsilon.\end{aligned}$$

Thus we see that

$$\limsup_{j\to\infty} \|W_{\psi,\varphi} f_j\|_{\mathcal{B}_\nu} < \varepsilon.$$

Since ε is arbitrary, we get $\|W_{\psi,\varphi} f_j\|_{\mathcal{B}_\nu} \to 0$ as $j \to \infty$. Lemma 2.3 shows that $W_{\psi,\varphi} : N^+ \to \mathcal{B}_\nu$ is metrically compact.

Finally we prove the implication (ii) $\Rightarrow$ (v). By taking the constant function $f(z) = 1$ in M, we have that $\psi \in \mathcal{B}_\nu$. Again by taking $f(z) = z$ in M and using the fact that $\psi \in \mathcal{B}_\nu$, we have that $\psi\varphi' \in \mathcal{A}_\nu$. Fix $c > 0$ and put $w = \varphi(z)$. We define the following functions:

$$f_w(v) = \left\{3\frac{(1-|w|^2)}{(1-\overline{w}v)^2} - 2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\} \exp\left\{c\left\{3\frac{(1-|w|^2)}{(1-\overline{w}v)^2} - 2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}\right\}$$

and

$$g_w(v) = \log\left\{3\frac{(1-|w|^2)}{(1-\overline{w}v)^2} - 2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\} + \left\{c\left\{3\frac{(1-|w|^2)}{(1-\overline{w}v)^2} - 2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}\right\}.$$

Then g_w belongs to the Hardy space H^1 and $\|g_w\|_{H^1} \le 7(c+1)$. By an application of the nontangential complex maximal theorem for H^1 functions (see [5, Theorem II.3.1]), we have that

$$\int_{\mathbb{T}} \log(1+N_\alpha f_w(\zeta))\, d\sigma(\zeta) \le \int_{\mathbb{T}} N_\alpha g_w(\zeta)\, d\sigma(\zeta) \le C\|g_w\|_{H^1} = 7C(c+1), \tag{2.6}$$

where $N_\alpha f$ denotes the nontangential maximal function:

$$N_\alpha f(\zeta) = \sup\{|f(z)| : z \in \Gamma_\alpha(\zeta)\}$$

and $\Gamma_\alpha(\zeta)$ $(0 < \alpha < 1)$ is the open convex hull of the set $\{\zeta\} \cup \alpha\mathbb{D}$. Inequalities (2.6) show that $\{f_w\}$ forms a metric ball in M. Also we have that

$$\begin{aligned}f_w'(v) = 6\overline{w}\left\{1+c\left\{3\frac{(1-|w|^2)}{(1-\overline{w}v)^2} - 2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}\right\}\left\{\frac{(1-|w|^2)}{(1-\overline{w}v)^3} - \frac{(1-|w|^2)^2}{(1-\overline{w}v)^4}\right\} \\ \times \exp\left\{c\left\{3\frac{(1-|w|^2)}{(1-\overline{w}v)^2} - 2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}\right\}.\end{aligned}$$

Since $\{W_{\psi,\varphi}f_w\}$ is a metric ball in $\mathcal{B}_\nu$, there is a positive constant C which independent of $w = \varphi(z)$ such that $\|W_{\psi,\varphi}f_w\|_{\mathcal{B}_\nu} \leq C$. Thus using the fact that $f'_w(w) = 0$, we obtain that

$$\begin{aligned} C &\geq \nu(z)|(W_{\psi,\varphi}f_w)'(z)| \\ &= \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\}, \end{aligned}$$

and so

$$\nu(z)|\psi(z)\varphi'(z)|\exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} \leq C(1-|\varphi(z)|^2).$$

Taking limit as $|\varphi(z)| \to 1$ on both sides of the above inequality, we get (2.4). Again fix $c > 0$, put $w = \varphi(z)$ and consider the following function

$$h_w(v) = \left\{\frac{1-|w|^2}{(1-\overline{w}v)^2} - \frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}\exp\left\{c\left\{3\frac{1-|w|^2}{(1-\overline{w}v)^2} - 2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}\right\}$$

and

$$\tau_w(v) = \log\left\{\frac{1-|w|^2}{(1-\overline{w}v)^2} - \frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\} + c\left\{3\frac{1-|w|^2}{(1-\overline{w}v)^2} - 2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}.$$

Then τ_w belongs to the Hardy space H^1 and $\|\tau_w\|_{H^1} \leq 7c+3$. Thus proceeding as above, we can show that $\{h_w\}$ forms a metric ball in M and $h_w(w) = 0$. Also we have that

$$\begin{aligned} h'_w(v) = \Bigg[&\overline{w}\left\{2\frac{1-|w|^2}{(1-\overline{w}v)^3} - 3\frac{(1-|w|^2)^2}{(1-\overline{w}v)^4}\right\} + 6c\overline{w}\left\{\frac{1-|w|^2}{(1-\overline{w}v)^2} - \frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\} \\ &\left\{\frac{1-|w|^2}{(1-\overline{w}v)^3} - \frac{(1-|w|^2)^2}{(1-\overline{w}v)^4}\right\}\Bigg]\exp\left\{c\left\{3\frac{1-|w|^2}{(1-\overline{w}v)^2} - 2\frac{(1-|w|^2)^2}{(1-\overline{w}v)^3}\right\}\right\}. \end{aligned}$$

Therefore, we have that

$$h'_w(w) = \frac{-\overline{w}}{(1-|w|^2)^2}\exp\left\{\frac{c}{(1-|\varphi(z)|^2)}\right\}.$$

Then τ_w belongs to the Hardy space H^1 and $\|\tau_w\|_{H^1} \leq 7c+3$. Thus proceeding as above, we can show that $\{h_w\}$ forms a metric ball in M. Since $\{W_{\psi,\varphi}h_w\}$ is a metric ball in $\mathcal{B}_\nu$, there is a positive constant C which independent of $w = \varphi(z)$ such that $\|W_{\psi,\varphi}h_w\|_{\mathcal{B}_\nu} \leq C$. Thus we obtain that

$$\begin{aligned} C &\geq \nu(z)|(W_{\psi,\varphi}f_w)'(z)| \\ &= 2c\frac{\nu(z)|\psi'(z)||\varphi(z)|}{(1-|\varphi(z)|^2)^2}\exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\}, \end{aligned}$$

and so

$$\frac{\nu(z)|\psi'(z)|}{1-|\varphi(z)|^2}\exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} \leq \frac{C(1-|\varphi(z)|^2)}{2c|\varphi(z)|}.$$

Taking limit as $|\varphi(z)| \to 1$ on both sides of the above inequality, we get (2.5). This completes the proof. □

Next we will investigate operators $W_{\psi,\varphi} : N^+$ or $M \to \mathcal{B}_{\nu,0}$.

Proposition 2.5. *If $\psi \in H(\mathbb{D})$ and φ a holomorphic self-map of $\mathbb{D}$ satisfies*

$$\lim_{|z|\to 1} \nu(z)|\psi'(z)| \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} = 0, \tag{2.7}$$

$$\lim_{|z|\to 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} = 0 \tag{2.8}$$

for all $c > 0$, then $W_{\psi,\varphi} : N^+$ or $M \to \mathcal{B}_{\nu,0}$ is bounded with respect to metric balls. Also we obtain that $W_{\psi,\varphi}(N^+) \subset \mathcal{B}_{\nu,0}$ or $W_{\psi,\varphi}(M) \subset \mathcal{B}_{\nu,0}$.

Proof. Since $W_{\psi,\varphi} : N^+$ or $M \to \mathcal{B}_\nu$ is bounded with respect to balls by Proposition 2.1, we only prove that $W_{\psi,\varphi}(L) \subset \mathcal{B}_{\nu,0}$ for any metric balls L in N^+. However the inequalities (1.1) and (2.3) show that

$$\begin{aligned}\nu(z)|(W_{\psi,\varphi}f)'(z)| \le \nu(z)|\varphi'(z)| \exp\left\{\frac{4\|f\|_{N^+}}{1-|\varphi(z)|^2}\right\}\\ + 4\frac{\nu(z)|\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{16\|f\|_{N^+}}{1-|\varphi(z)|^2}\right\}\end{aligned}$$

which holds for each $f \in L$. Thus the conditions (2.7) and (2.8) imply that $\nu(z)|(W_{\psi,\varphi}f)'(z)| \to 0$ as $|z| \to 1$. The case $W_{\psi,\varphi} : M \to \mathcal{B}_{\nu,0}$ can be verified using the relation $\|f\|_{N^+} \le \|f\|_M$. □

Theorem 2.6. *Let $\psi \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then the following conditions are equivalent*;

(i) $W_{\psi,\varphi} : N^+ \to \mathcal{B}_{\nu,0}$ *is bounded with respect to metric balls,*
(ii) $W_{\psi,\varphi} : N^+ \to \mathcal{B}_\nu$ *is bounded with respect to metric balls, $\psi \in \mathcal{B}_{\nu,0}$ and $\psi\varphi' \in \mathcal{A}_{\nu,0}$,*
(iii) $W_{\psi,\varphi} : N^+ \to \mathcal{B}_{\nu,0}$ *is metrically compact,*
(iv) $W_{\psi,\varphi} : M \to \mathcal{B}_{\nu,0}$ *is bounded with respect to metric balls,*
(v) $W_{\psi,\varphi} : M \to \mathcal{B}_\nu$ *is bounded with respect to metric balls, $\psi \in \mathcal{B}_{\nu,0}$ and $\psi\varphi' \in \mathcal{A}_{\nu,0}$,*
(vi) $W_{\psi,\varphi} : M \to \mathcal{B}_{\nu,0}$ *is metrically compact,*
(vii) ψ *and* φ *satisfy the conditions* (2.7) *and* (2.8).

Proof. By the same reasons as in the proof of Theorem 2.4, implications (i) ⇒ (iv), (iii) ⇒ (vi), (iii) ⇒ (i) and (vi) ⇒ (iv) are true. Also we can easily see that (i) ⇒ (ii) and (iv) ⇒ (v) are hold. In fact, we may consider the function $f(z) = 1$ in N^+ or M. This one satisfies $\|f\|_{N^+} \le \|f\|_M \le \log 2$, and so f is in some metric balls in N^+ or M. This shows that $\psi \in \mathcal{B}_{\nu,0}$. Again by consider the function $f(z) = z$ in N^+ or M. Once again $\|f\|_{N^+} \le \|f\|_M \le \log 2$, and so f is in some metric balls in N^+ or M. Thus using $\psi \in \mathcal{B}_{\nu,0}$, we can shows that $\psi\varphi' \in \mathcal{A}_{\nu,0}$.

To prove (ii) $\Rightarrow$ (vii), we take a sequence $\{z_j\}$ in $\mathbb{D}$ with $|z_j| \to 1$ as $j \to \infty$. Then

$$\limsup_{|z|\to 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} \tag{2.9}$$
$$= \lim_{j\to\infty} \frac{\nu(z_j)|\psi(z_j)\varphi'(z_j)|}{1-|\varphi(z_j)|^2} \exp\left\{\frac{c}{1-|\varphi(z_j)|^2}\right\}.$$

If $\sup_{j\geq 1}|\varphi(z_j)| < 1$, then the assumption $\psi' \in \mathcal{B}_{\nu,0}$ implies that the right limit in the equation (2.9) is equal to 0, and so we obtain the condition (2.7). If $\sup_{j\geq 1}|\varphi(z_j)| = 1$, then we can choose a subsequence $\{z_{j_k}\} \subset \{z_j\}$ such that $|\varphi(z_{j_k})| \to 1$ as $k \to \infty$. Since $W_{\psi,\varphi} : N^+ \to \mathcal{B}_\nu$ is bounded with respect to metric balls, by Theorem 2.4, ψ and φ satisfies

$$\lim_{|\varphi(z)|\to 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} = 0 \tag{2.10}$$

for any $c > 0$. By (2.9) and (2.10) we have that

$$\begin{aligned}
&\limsup_{|z|\to 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} \\
&= \lim_{k\to\infty} \frac{\nu(z_{j_k})|\psi(z_j)\varphi'(z_{j_k})|}{1-|\varphi(z_{j_k})|^2} \exp\left\{\frac{c}{1-|\varphi(z_{j_k})|^2}\right\} \\
&\leq \limsup_{|\varphi(z)|\to 1} \frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{c}{1-|\varphi(z)|^2}\right\} = 0.
\end{aligned}$$

This implies that (2.7) holds. Similarly, we can show that (2.8) also holds. (v) $\Rightarrow$ (vii) is also verified by the above arguments.

Finally we will prove the implication (vii) $\Rightarrow$ (iii). Take any metric ball L_{N^+} in N^+. Then there is a constant $K > 0$ such that $\|f\|_{N^+} \leq K$ for any $f \in L_{N^+}$. For any $f \in L_{N^+}$ and $z \in \mathbb{D}$ we have that

$$\begin{aligned}
\nu(z)|(W_{\psi,\varphi}f)'(z)| \leq \nu(z)|\psi'(z)| \exp\left\{\frac{4K}{1-|\varphi(z)|^2}\right\} \\
+ 4\frac{\nu(z)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2} \exp\left\{\frac{16K}{1-|\varphi(z)|^2}\right\}.
\end{aligned}$$

Combining this with (2.7) and (2.8), we obtain

$$\lim_{|z|\to 1} \sup_{f\in L_{N^+}} \nu(z)|(W_{\psi,\varphi}f)'(z)| = 0,$$

and so Lemma 1.1 shows that $W_{\psi,\varphi}(L_{N^+})$ is compact in $\mathcal{B}_{\nu,0}$ for any metric balls L_{N^+}. This means that $W_{\psi,\varphi} : N^+ \to \mathcal{B}_{\nu,0}$ is metrically compact. The proof is accomplished. $\square$

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