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SOME PROPERTIES OF EXTENDED τ-HYPERGEOMETRIC FUNCTION

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Abstract. Recently, Parmar [5] introduced a new extension of the τ-Gauss hypergeometric function ${}_2R_1^\tau(z)$. The main object of this paper is to study this extended τ-Gauss hypergeometric function and obtain its properties including connection with modified Bessel function of third kind and extended generalized hypergeometric function, several contiguous relations, differential relations, integral transforms and elementary integrals. Various special cases of our results are also discussed.

1. Introduction

The Gauss hypergeometric function [6] is defined as

$$(1)\quad {}_2F_1(z) = F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \qquad (|z|<1,\ c \neq 0,-1,-2,\ldots),$$

where $(\lambda)_v$ $(\lambda \in \mathbb{C}, v \in \mathbb{Z}_+)$ denotes the Pochhammer symbol defined by

$$(2)\qquad (\lambda)_v := \frac{\Gamma(\lambda+v)}{\Gamma(\lambda)} = \begin{cases} 1 & (v=0;\ \lambda \in \mathbb{C}\backslash\{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+v-1) & (v \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}$$

it being assumed conventionally that $(0)_0 := 1$ and understood tacitly that the Γ-quotient exists (see, for details, [11]).

The classical generalized hypergeometric function [4] is defined by

$$(3)\quad {}_pF_q(z) = {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p; \\ b_1,\ldots,b_q; \end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!},\ (p=q+1,\ |z|<1),$$

where no denominator parameter is zero or negative integer.

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In 2001, Virchenko et al. [12] defined the following extension of hypergeometric function

$$(4)\quad {}_2R_1^{\tau}(z)={}_2R_1(a,b;c;\tau;z)=\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!},\quad (\tau>0,\ |z|<1)$$

and Rao et al. [7, 8] studied various properties of it.

In 2014, Srivastava [10] introduced and studied the following family of generalized hypergeometric functions:

$$(5)\quad {}_rF_s\left[\begin{array}{r}(a_1,p),a_2,\ldots,a_r;\\ b_1,\ldots,b_s;\end{array}\ z\right]=\sum_{n=0}^{\infty}\frac{(a_1;p)_n(a_2)_n\cdots(a_r)_n}{(b_1)_n\cdots(b_s)_n}\frac{z^n}{n!}$$

in terms of the generalized Pochhammer symbol [10]

$$(6)\quad (\lambda;p)_v=\begin{cases}\dfrac{\Gamma_p(\lambda+v)}{\Gamma(\lambda)} & (\Re(p)>0,\ \lambda,v\in\mathbb{C})\\ (\lambda)_v & (p=0,\ \lambda,v\in\mathbb{C}),\end{cases}$$

where no denominator parameter is zero or negative integer in (5) and provided that the series on the right-hand side of (5) converges.

The generalized gamma function $\Gamma_p(z)$ involved in the definition (6) is introduced by Chaudhry and Zubair [2] (see also [3]) as

$$(7)\quad \Gamma_p(z)=\begin{cases}\displaystyle\int_0^{\infty}t^{z-1}\exp\left(-t-\frac{p}{t}\right)dt & (\Re(p)>0,\ z\in\mathbb{C})\\ \Gamma(z) & (p=0,\ z\in\mathbb{C}).\end{cases}$$

Recently, Parmar [5] introduced the extension of τ-hypergeometric function (4) as

$$(8)\quad {}_2R_1^{\tau}((a,p),b;c;z)=\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a;p)_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!},$$
$$(\Re(p)\geqslant 0;\ \tau>0,\ |z|<1;\ \Re(c)>\Re(b)>0 \text{ when } p=0)\,.$$

Motivated by the aforecited investigations of the extended τ-hypergeometric function ${}_2R_1^{\tau}$ defined by (8), we discuss several other interesting properties of this function. First, we show the connection of ${}_2R_1^{\tau}$ with the Bessel function of third kind and the extended generalized hypergeometric function. Then many contiguous relations and differential properties, integral transforms namely Beta transform, Laplace transform and Whittaker transform and some elementary integrals are derived.

2. Connection of ${}_2R_1^\tau$ with other special functions

Theorem 1 (Connection with Bessel function of third kind)**.** *If* $a, b, c, p \in \mathbb{C}$, $\Re(p) > 0$, *then*

$$ {}_2R_1^\tau[(a,p),b;c;z] = \frac{2p^{\alpha/2}\Gamma(c)}{\Gamma(b)\Gamma(a)} \sum_{n=0}^{\infty} K_{a+n}\left(2\sqrt{p}\right) \frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{(z\sqrt{p})^n}{n!}, \tag{9} $$

where $K_\alpha(x)$ *is the modified Bessel function of third kind or Macdonald function which has the following integral representation* [1] (*see also* [4])*.*

$$ K_\alpha(x) = \frac{\left(\frac{x}{2}\right)^\alpha}{2} \int_0^\infty \exp\left(-t - \frac{x^2}{4t}\right) t^{-\alpha-1} dt \qquad (\Re(x) > 0). \tag{10} $$

Proof. Using the definitions (6) and (7) in (8), one can write

$$ {}_2R_1^\tau[(a,p),b;c;z] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(a)} \sum_{n=0}^{\infty} \left(\int_0^\infty t^{a+n-1} \exp\left(-t - \frac{p}{t}\right) dt \right) \frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!}. $$

Formula (10) gives us

$$ \int_0^\infty t^{a+n-1} \exp\left(-t - \frac{p}{t}\right) dt = 2p^{\frac{a+n}{2}} K_{a+n}\left(2\sqrt{p}\right). \tag{11} $$

By putting this in (2) and taking into account that $K_\alpha(x) = K_{-\alpha}(x)$ (see [4]), this leads to the required result (9). □

Theorem 2 (Connection with the extended generalized hypergeometric function)**.** *If* $q \in \mathbb{N}$, *then*

$$ {}_2R_1^q\left[\begin{matrix}(a;p),b; \\ c;\end{matrix}\ z\right] = {}_{q+1}F_q\left[\begin{matrix}(a;p),\Delta(q;b); \\ \Delta(q;c);\end{matrix}\ z\right], \tag{12} $$

where function ${}_{q+1}F_q$ *on the right is the extended and generalized hypergeometric function* (5) *and* $\Delta(q;b)$ *is the abbreviation for the array of* q *parameters* $\frac{b}{q}, \frac{b+1}{q}, \ldots, \frac{b+q-1}{q}$.

Proof. Using the property of Pochammer symbol

$$ (\alpha)_{qn} = q^{nq} \left(\frac{\alpha}{q}\right)_n \left(\frac{\alpha+1}{q}\right)_n \cdots \left(\frac{\alpha+q-1}{q}\right)_n \qquad (q \in \mathbb{N}), $$

from (2) and (8), we have

$$ \begin{aligned} {}_2R_1^q\left[\begin{matrix}(a;p),b; \\ c;\end{matrix}\ z\right] &= \sum_{n=0}^{\infty} \frac{(a;p)_n (b)_{qn}}{(c)_{qn}} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(a;p)_n\, q^{qn} \prod_{i=1}^{q} \left(\frac{b+i-1}{q}\right)_n}{q^{qn} \prod_{j=1}^{q} \left(\frac{c+j-1}{q}\right)_n} \frac{z^n}{n!} \end{aligned} $$

$$= {}_{q+1}F_q \left[\begin{matrix} (\alpha;p), \frac{b}{q}, \frac{b+1}{q}, \ldots, \frac{b+q-1}{q}; \\ \frac{c}{q}, \frac{c+1}{q}, \ldots, \frac{c+q-1}{q}; \end{matrix} \; z \right],$$

which is precisely the assertion (12). □

3. Contiguous relations

We now state and prove the following Lemma which is useful to prove some contiguous relations given below.

Lemma. *If* $\alpha \in \mathbb{C}\backslash\{0,1\}$; $p, v \in \mathbb{C}$; $\Re(p) > 0$, *then the following difference formula for the generalized Pochhammer symbol* (6) *holds true*

$$(\alpha+1;p)_v = \frac{\alpha+v}{\alpha}(\alpha;p)_v + \frac{p}{\alpha(\alpha-1)}(\alpha-1;p)_v. \tag{13}$$

Proof. To prove the assertion (13), we use the difference formula [3] for the generalized gamma function $\Gamma_p(z)$

$$\Gamma_p(\alpha+1) = \alpha\,\Gamma_p(\alpha) + p\,\Gamma_p(\alpha-1). \tag{14}$$

From the definition (6), we have

$$\begin{aligned} (\alpha+1;p)_v &= \frac{\Gamma_p(\alpha+1+v)}{\Gamma(\alpha+1)} \\ &= \frac{(\alpha+v)\Gamma_p(\alpha+v) + p\Gamma_p(\alpha-1+v)}{\Gamma(\alpha+1)} \\ &= \frac{\alpha+v}{\alpha}\frac{\Gamma_p(\alpha+v)}{\Gamma(\alpha)} + \frac{p\Gamma_p(\alpha-1+v)}{\alpha(\alpha-1)\Gamma(\alpha-1)}, \end{aligned}$$

which leads to the desired result (13). □

Theorem 3. *If* $a, b, c, p \in \mathbb{C}$; $a \neq 1$, $\Re(p) > 0$, $\tau > 0$; $|z| < 1$, *then*

$$\begin{aligned} & a\tau\,{}_2R_1^\tau \left[\begin{matrix} (a+1;p), b; \\ c; \end{matrix} \; z \right] - b\,{}_2R_1^\tau \left[\begin{matrix} (a;p), b+1; \\ c; \end{matrix} \; z \right] \\ &= (a\tau - b)\,{}_2R_1^\tau \left[\begin{matrix} (a;p), b; \\ c; \end{matrix} \; z \right] + \frac{\tau p}{a-1}\,{}_2R_1^\tau \left[\begin{matrix} (a-1;p), b; \\ c; \end{matrix} \; z \right]. \end{aligned} \tag{15}$$

Proof. Upon substituting for the ${}_2R_1^\tau$ from (8) and using (13), we have

$$\begin{aligned} & a\tau{}_2R_1^\tau \left[\begin{matrix} (a+1;p), b; \\ c; \end{matrix} \; z \right] - b\,{}_2R_1^\tau \left[\begin{matrix} (a;p), b+1; \\ c; \end{matrix} \; z \right] \\ &= a\tau \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \left[\frac{a+n}{a}(a;p)_n + \frac{p}{a(a-1)}(a-1;p)_n \right] \frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!} \\ &\quad - \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} (a;p)_n \frac{(b+\tau n)\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!} \end{aligned}$$

$$= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} [\tau(a+n) - (b+\tau n)] \frac{(a;p)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!} + \frac{\tau p}{(a-1)} \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a-1;p)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!},$$

this immediately leads to the assertion (15). □

Similar method of proof of Theorem 3 can be applied in order to derive relation (16) below.

Theorem 4. *If* $a, b, c, p \in \mathbb{C}$; $a \neq 1$, $\Re(p) > 0$, $\tau > 0$; $|z| < 1$, *then*

$$a\tau \, {}_2R_1^{\tau} \left[\begin{matrix} (a+1;p), b; \\ c; \end{matrix} \; z \right] - (c-1) \, {}_2R_1^{\tau} \left[\begin{matrix} (a+1;p), b; \\ c-1; \end{matrix} \; z \right] = (a\tau - c + 1) \, {}_2R_1^{\tau} \left[\begin{matrix} (a;p), b; \\ c; \end{matrix} \; z \right] + \frac{\tau p}{a-1} \, {}_2R_1^{\tau} \left[\begin{matrix} (a-1;p), b; \\ c; \end{matrix} \; z \right]. \tag{16}$$

Theorem 5. *If* $b, c, p \in \mathbb{C}$; $a \in \mathbb{C} \backslash \{0, 1\}$, $\Re(p) > 0$, $\tau > 0$; $|z| < 1$, *then*

$$\Gamma(b)\Gamma(c+\tau) \, {}_2R_1^{\tau} \left[\begin{matrix} (a+1;p), b; \\ c; \end{matrix} \; z \right] - z\Gamma(c)\Gamma(b+\tau) \, {}_2R_1^{\tau} \left[\begin{matrix} (a+1;p), b+\tau; \\ c+\tau; \end{matrix} \; z \right] = \Gamma(b)\Gamma(c+\tau) \left\{ {}_2R_1^{\tau} \left[\begin{matrix} (a;p), b; \\ c; \end{matrix} \; z \right] + \frac{p}{a(a-1)} \, {}_2R_1^{\tau} \left[\begin{matrix} (a-1;p), b; \\ c; \end{matrix} \; z \right] \right\}. \tag{17}$$

Proof. Using (13), we have

$$\Gamma(b)\Gamma(c+\tau) \, {}_2R_1^{\tau} \left[\begin{matrix} (a+1;p), b; \\ c; \end{matrix} \; z \right] - z\Gamma(c)\Gamma(b+\tau) \, {}_2R_1^{\tau} \left[\begin{matrix} (a+1;p), b+\tau; \\ c+\tau; \end{matrix} \; z \right] = \Gamma(c)\Gamma(c+\tau) \sum_{n=0}^{\infty} \left[\frac{a+n}{a} (a;p)_n + \frac{p}{a(a-1)} (a-1;p)_n \right] \frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!} - \Gamma(c)\Gamma(c+\tau) \sum_{n=0}^{\infty} \frac{(a+1;p)}{a} \frac{\Gamma(b+\tau(n+1))}{\Gamma(c+\tau(n+1))} \frac{z^{n+1}}{n!}. \tag{18}$$

Now the generalized Pochhammer symbol $(\lambda;p)_v$ has the property [10] $(\lambda;p)_{v+\mu} = (\lambda)_v(\lambda+v;p)_\mu$, from which, one can get

$$(\lambda+1;p)_n = \frac{(\lambda;p)_{n+1}}{\lambda}. \tag{19}$$

Applying (19) on equation (18), one get

$$\Gamma(b)\Gamma(c+\tau) \, {}_2R_1^{\tau} \left[\begin{matrix} (a+1;p), b; \\ c; \end{matrix} \; z \right] - z\Gamma(c)\Gamma(b+\tau) \, {}_2R_1^{\tau} \left[\begin{matrix} (a+1;p), b+\tau; \\ c+\tau; \end{matrix} \; z \right] = \Gamma(c)\Gamma(c+\tau) \sum_{n=0}^{\infty} \left[\frac{a+n}{a} (a;p)_n + \frac{p}{a(a-1)} (a-1;p)_n \right] \frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!}$$

$$- \Gamma(c)\Gamma(c+\tau)\sum_{n=0}^{\infty}\frac{(a+1;p)}{a}\frac{\Gamma(b+\tau(n+1))}{\Gamma(c+\tau(n+1))}\frac{z^{n+1}}{n!}$$

$$= \Gamma(c)\Gamma(c+\tau)\left\{\sum_{n=0}^{\infty}\frac{a+n}{a}\frac{(a;p)_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}+\sum_{n=0}^{\infty}\frac{p}{a(a-1)}\frac{(a-1;p)_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}\right.$$
$$\left.-\sum_{n=0}^{\infty}\frac{n}{a}\frac{(a;p)_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}\right\}$$
$$= \Gamma(c)\Gamma(c+\tau)\left\{\sum_{n=0}^{\infty}\frac{(a;p)_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}+\frac{p}{a(a-1)}\sum_{n=0}^{\infty}\frac{(a-1;p)_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}\right\},$$

this can easily leads to the assertion (17). □

Theorem 6. *If* $b, c, p \in \mathbb{C}$; $a \in \mathbb{C}\backslash\{0,1\}$, $\Re(p) > 0$, $\tau > 0$; $|z| < 1$, *then*

(20)
$$\Gamma(b+1)\Gamma(c+\tau)\Gamma(c+2\tau)\,{}_2R_1^{\tau}\left[\begin{matrix}(a+1;p), b+1; \\ c;\end{matrix}\; z\right]$$
$$- \Gamma(b+\tau)\Gamma(c)\Gamma(c+2\tau)(a\tau+b+\tau)z\,{}_2R_1^{\tau}\left[\begin{matrix}(a+1;p), b+\tau; \\ c+\tau;\end{matrix}\; z\right]$$
$$- \Gamma(b+2\tau)\Gamma(c)\Gamma(c+\tau)(a+1)\tau z^2\,{}_2R_1^{\tau}\left[\begin{matrix}(a+2;p), b+2\tau; \\ c+2\tau;\end{matrix}\; z\right]$$
$$= \Gamma(b+1)\Gamma(c+\tau)\Gamma(c+2\tau)\left\{{}_2R_1^{\tau}+\frac{p}{a(a-1)}\,{}_2R_1^{\tau}\left[\begin{matrix}(a-1;p), b+1; \\ c;\end{matrix}\; z\right]\right\}.$$

Proof. From the definition (8) of ${}_2R_1^{\tau}$ and using the property (13) of the generalized Pochhammer symbol, the left hand side (say S) of equation (20) is equal to

(21)
$$S = \Gamma(c)\Gamma(c+\tau)\Gamma(c+2\tau)\left\{\sum_{n=0}^{\infty}\left[\frac{a+n}{a}(a;p)_n+\frac{p}{a(a-1)}(a-1;p)_n\right]\right.$$
$$\frac{\Gamma(b+1+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}-(a\tau+b+\tau)\sum_{n=0}^{\infty}\frac{(a;p)_{n+1}}{a}\frac{\Gamma(b+\tau(n+1))}{\Gamma(c+\tau(n+1))}\frac{z^{n+1}}{n!}$$
$$\left.-(a+1)\tau\sum_{n=0}^{\infty}\frac{(a;p)_{n+2}}{(a)_2}\frac{\Gamma(b+\tau(n+2))}{\Gamma(c+\tau(n+2))}\frac{z^{n+2}}{n!}\right\}$$
$$= \Gamma(c)\Gamma(c+\tau)\Gamma(c+2\tau)\left\{\sum_{n=0}^{\infty}\frac{a+n}{a}\frac{(b+\tau n)\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}-(a\tau+b+\tau)\right.$$
$$\left.\sum_{n=0}^{\infty}\frac{n\,(a;p)_n}{a}\frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}-(a+1)\tau\sum_{n=0}^{\infty}\frac{n(n-1)(a;p)_n}{(a)_2}\frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}\right\}$$

$$+\Gamma(c+\tau)\Gamma(c+2\tau)\Gamma(b+1)\left\{\frac{\Gamma(c)}{\Gamma(b+1)}\frac{p}{a(a-1)}\sum_{n=0}^{\infty}\frac{(a-1;p)_n\Gamma(b+1+\tau n)}{\Gamma(c+\tau n)}\frac{z^n}{n!}\right\}.$$

From (21), assertion (20) of Theorem 6 follows readily. ☐

Applying the techniques we have used to prove aforementioned theorems, one can prove Theorem 7 and Theorem 8 below.

Theorem 7. *If* $b, c, p \in \mathbb{C}$; $a \in \mathbb{C}\backslash\{0,1\}$, $\Re(p) > 0$, $\tau > 0$; $|z| < 1$, *then*

$$\begin{aligned}(22)\quad &\Gamma(b)\Gamma(c+1+\tau)\Gamma(c+1+2\tau)\\ &\left\{{}_2R_1^\tau\left[\begin{matrix}(a+1;p), b;\\ c;\end{matrix}\ z\right]-\frac{p}{a(a-1)}\,{}_2R_1^\tau\left[\begin{matrix}(a-1;p), b;\\ c;\end{matrix}\ z\right]\right\}\\ &-\Gamma(b+\tau)\Gamma(c)\Gamma(c+1+2\tau)[\tau(a+1)+c]z\,{}_2R_1^\tau\left[\begin{matrix}(a+1;p), b+\tau;\\ c+1+\tau;\end{matrix}\ z\right]\\ &-\Gamma(b+2\tau)\Gamma(c)\Gamma(c+1+\tau)[(a+1)\tau]z^2\,{}_2R_1^\tau\left[\begin{matrix}(a+2;p), b+2\tau;\\ c+1+2\tau;\end{matrix}\ z\right]\\ =\ &\Gamma(b)\Gamma(c+1+\tau)\Gamma(c+1+2\tau)\,{}_2R_1^\tau\left[\begin{matrix}(a;p), b;\\ c+1;\end{matrix}\ z\right].\end{aligned}$$

Theorem 8. *If* $a, b, c, p \in \mathbb{C}$; $\Re(p) > 0$, $\tau > 0$; $|z| < 1$, *then*

$$\begin{aligned}(23)\quad &{}_2R_1^\tau\left[\begin{matrix}(a;p), b;\\ c-\tau;\end{matrix}\ z\right]-{}_2R_1^\tau\left[\begin{matrix}(a;p), b-1;\\ c-\tau;\end{matrix}\ z\right]\\ &= a\tau z\,\frac{\Gamma(c-\tau)\Gamma(b-1+\tau)}{\Gamma(b)\Gamma(c)}\,{}_2R_1^\tau\left[\begin{matrix}(a+1;p), b-1+\tau;\\ c;\end{matrix}\ z\right].\end{aligned}$$

4. Differential properties

Theorem 9. *If* $b, c, p \in \mathbb{C}$; $a \in \mathbb{C}\backslash\{0,1\}$, $\Re(p) > 0$, $\tau > 0$; $|z| < 1$, *then*

$$\begin{aligned}(24)\quad &\frac{d}{dz}\left\{z^a\,{}_2R_1^\tau\left[\begin{matrix}(a;p), b;\\ c;\end{matrix}\ z\right]\right\}\\ &= z^{a-1}\left\{a\,{}_2R_1^\tau\left[\begin{matrix}(a+1;p), b;\\ c;\end{matrix}\ z\right]-\frac{p}{a-1}\,{}_2R_1^\tau\left[\begin{matrix}(a-1;p), b;\\ c;\end{matrix}\ z\right]\right\}.\end{aligned}$$

Proof. From (13), we have

$$(25)\quad (a+n)(a;p)_n = a(a+1;p)_n - \frac{p}{a-1}(a-1;p)_n.$$

Upon differentiating and then using the property (25), we have

$$\frac{d}{dz}\left\{z^a\,{}_2R_1^\tau\left[\begin{matrix}(a;p), b;\\ c;\end{matrix}\ z\right]\right\}$$

$$
\begin{aligned}
&= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a;p)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{(n+a) z^{n+a-1}}{n!} \\
&= z^{a-1} \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \left[a(a+1;p)_n - \frac{p}{a-1}(a-1;p)_n \right] \frac{\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!} \\
&= z^{a-1} \left\{ \frac{a\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a+1;p)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!} \right. \\
&\qquad \left. - \frac{p}{a-1} \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a-1;p)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!} \right\} \\
&= z^{a-1} \left\{ a\, {}_2R_1^{\tau} \left[\begin{matrix} (a+1;p), b; \\ c; \end{matrix} \; z \right] - \frac{p}{a-1}\, {}_2R_1^{\tau} \left[\begin{matrix} (a-1;p), b; \\ c; \end{matrix} \; z \right] \right\}.
\end{aligned}
$$

This completes the proof of Theorem 9. □

Theorem 10. *If* $b, c, p \in \mathbb{C}$; $a \in \mathbb{C}\backslash\{0,1\}$, $\Re(p) > 0$, $\tau > 0$; $|z| < 1$, *then*

$$
\begin{aligned}
\text{(26)} \qquad & \left(z\frac{d}{dz} + a \right) {}_2R_1^{\tau} \left[\begin{matrix} (a;p), b; \\ c; \end{matrix} \; z \right] \\
&= a\, {}_2R_1^{\tau} \left[\begin{matrix} (a+1;p), b; \\ c; \end{matrix} \; z \right] - \frac{p}{a-1}\, {}_2R_1^{\tau} \left[\begin{matrix} (a-1;p), b; \\ c; \end{matrix} \; z \right].
\end{aligned}
$$

Proof. Starting with the left hand side

$$
\begin{aligned}
& \left(z\frac{d}{dz} + a \right) {}_2R_1^{\tau} \left[\begin{matrix} (a;p), b; \\ c; \end{matrix} \; z \right] \\
&= z \left(\sum_{n=0}^{\infty} \frac{(a;p)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{n z^{n-1}}{n!} \right) + a \sum_{n=0}^{\infty} \frac{(a;p)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(n+a)(a;p)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!}.
\end{aligned}
$$

Now repeating the steps of the proof of Theorem 9, we get the assertion (26). □

5. Integral transforms

Theorem 11 (Euler (Beta) transform)**.** *If* $a, b, c, \alpha, \beta \in \mathbb{C}$, $\tau \in \mathbb{R}_+$, $\Re(b) > 0$, $\Re(c) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(p) > 0$, *then*

(27)

$$
\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\, {}_2R_1^{\tau} \left[\begin{matrix} (a;p), b; \\ c; \end{matrix} \; xz^{\tau} \right] dz = B(\alpha,\beta)\, {}_3R_2^{\tau} \left[\begin{matrix} (a;p), b, \alpha; \\ c, \alpha+\beta; \end{matrix} \; x \right].
$$

Proof. Using (8) we get

$$
\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\, {}_2R_1^{\tau} \left[\begin{matrix} (a;p), b; \\ c; \end{matrix} \; xz^{\tau} \right] dz
$$

$$= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a;p)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{x^n}{n!} \int_0^1 z^{\alpha+\tau n-1}(1-z)^{\beta-1}\, dz$$

$$= \frac{\Gamma(c)\Gamma(\beta)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a;p)_n \Gamma(b+\tau n)\Gamma(\alpha+\tau n)}{\Gamma(c+\tau n)\Gamma(\alpha+\beta+\tau n)} \frac{x^n}{n!}$$

$$= B(\alpha,\beta)\, {}_3R_2^{\tau}\left[\begin{matrix} (a;p), b, \alpha; \\ c, \alpha+\beta; \end{matrix}\; x\right].$$

Hence the proof. □

Remark 1. In the special case of Theorem 11 when we have $\alpha = c$, equation (27) becomes

$$\text{(28)} \quad \int_0^1 z^{c-1}(1-z)^{\beta-1}\, {}_2R_1^{\tau}\left[\begin{matrix} (a;p), b; \\ c; \end{matrix}\; xz^{\tau}\right] dz = B(c,\beta)\, {}_2R_1^{\tau}\left[\begin{matrix} (a;p), b; \\ c+\beta; \end{matrix}\; x\right].$$

Theorem 12 (Laplace transform)**.** *If $a, b, c, \alpha \in \mathbb{C}$; $\tau, \sigma \in \mathbb{R}_+$; $\Re(b) > 0$, $\Re(c) > 0$, $\Re(\alpha) > 0$, $\Re(s) > 0$, $\Re(p) > 0$ and $\left|\frac{x}{s^{\sigma}}\right| < 1$, then*

$$\text{(29)} \quad \int_0^{\infty} e^{-sz} z^{\alpha-1}\, {}_2R_1^{\tau}\left[\begin{matrix} (a;p), b; \\ c; \end{matrix}\; xz^{\sigma}\right] dz = s^{-\alpha}\,\Gamma(\alpha)\, {}_3R_1^{\tau}\left[\begin{matrix} (a;p), b, \alpha; \\ c; \end{matrix}\; \frac{x}{s^{\sigma}}\right].$$

Proof. Starting with the left hand side

$$\int_0^{\infty} e^{-sz} z^{\alpha-1}\, {}_2R_1^{\tau}\left[\begin{matrix} (a;p), b; \\ c; \end{matrix}\; xz^{\tau}\right] dz$$

$$= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a;p)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{x^n}{n!} \int_0^{\infty} e^{-sz} z^{\alpha+\tau n-1}\, dz$$

$$= \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a;p)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{\Gamma(\alpha+\tau n)}{s^{\alpha+\tau n}} \frac{x^n}{n!}$$

$$= s^{-\alpha}\,\Gamma(\alpha)\, {}_3R_1^{\tau}\left[\begin{matrix} (a;p), b, \alpha; \\ c; \end{matrix}\; \frac{x}{s^{\sigma}}\right].$$

This completes the proof of Theorem 12. □

Theorem 13 (Whittaker transform)**.** *If $a, b, c, \rho, \sigma \in \mathbb{C}$; $\Re(b) > 0$, $\Re(c) > 0$, $\Re(\rho) > 0$, then*

$$\text{(30)} \quad \int_0^{\infty} e^{-\frac{1}{2}\sigma t} t^{\rho-1}\, W_{\lambda,\mu}(\sigma t)\, {}_2R_1^{\tau}\left[\begin{matrix} (a;p), b; \\ c; \end{matrix}\; \omega t^{\tau}\right] dt$$

$$= \frac{\sigma^{-\rho}\,\Gamma\left(\frac{1}{2}+\mu+\rho\right)\Gamma\left(\frac{1}{2}-\mu+\rho\right)}{\Gamma(1-\lambda+\rho)}\, {}_4R_2^{\tau}\left[\begin{matrix} (a;p), b, \left(\frac{1}{2}+\mu+\rho\right), \left(\frac{1}{2}-\mu+\rho\right); \\ c, (1-\lambda+\rho); \end{matrix}\; \frac{\omega}{\sigma^{\tau}}\right],$$

where $W_{\lambda,\mu}(t)$ is the Whittaker function [13] *defined as*

$$W_{\lambda,\mu}(t) = \frac{e^{-\frac{t}{2}} t^{\lambda}}{\Gamma\left(\frac{1}{2}-\lambda+\mu\right)} \int_0^{\infty} \zeta^{-\lambda-\frac{1}{2}+\mu} \left(1+\frac{\zeta}{t}\right)^{\lambda-\frac{1}{2}+\mu} e^{-\zeta} d\zeta, \;\; \left(\Re\left(\lambda-\tfrac{1}{2}+\mu\right) \leqslant 0\right).$$

Proof. Taking the substitution $\sigma t = v$ in the integral on the left of equation (30), we have

$$\int_0^\infty e^{-\frac{1}{2}\sigma t} t^{\rho-1} W_{\lambda,\mu}(\sigma t)\, {}_2R_1^\tau \left[\begin{matrix} (a;p), b; \\ c; \end{matrix} \; \omega t^\tau \right] dt$$
$$= \frac{\Gamma(c)}{\sigma^\rho \Gamma(b)} \sum_{n=0}^\infty \frac{(a;p)_n \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \left(\frac{\omega}{\sigma^\tau}\right)^n \frac{1}{n!} \int_0^\infty e^{\left(-\frac{v}{2}\right)} v^{\rho+\tau n-1} W_{\lambda,\mu}(v)\, dv\,.$$

Now, using the formula

$$\int_0^\infty e^{\left(-\frac{t}{2}\right)} v^{v-1}\, W_{\lambda,\mu}(t)\, dt = \frac{\Gamma\left(\frac{1}{2}+\mu+v\right)\Gamma\left(\frac{1}{2}-\mu+v\right)}{\Gamma(1-\lambda+v)},$$

where $\Re(v \pm \mu) > -\frac{1}{2}$.

We get the final assertion (30) of Theorem 13. □

6. Elementary integrals for ${}_2R_1^\tau((a,p),b;c;z)$

We have the basic integral representation of ${}_2R_1^\tau[(a,p),b;c;z]$ [5] is as following

(31)
$${}_2R_1^\tau((a,p),b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}\,(1-t)^{c-b-1}\, {}_1F_0[(a,p);-;zt^\tau]\, dt.$$

This integral can be transformed into other elementary integrals for ${}_2R_1^\tau((a,p), b;c;z)$. By taking $s = \frac{-t}{t-1}$, $t = \frac{1}{s}$ and $t = e^{-s}$ in (31), we derive the integrals

(32)
$${}_2R_1^\tau[(a,p),b;c;z] = \frac{1}{B(b,c-b)} \int_0^\infty s^{b-1}\,(1+s)^{-c}\, {}_1F_0 \left[\begin{matrix} (a,p) \\ - \end{matrix} \; ; \frac{zs^\tau}{(1+s)^\tau} \right] ds$$
$$(\Re(c) > \Re(b) > 0,\ |\arg(1-z)| < \pi),$$

(33)
$${}_2R_1^\tau \left[\begin{matrix} (a,p), b \\ c \end{matrix} \; ; \frac{1}{z} \right] = \frac{1}{B(b,c-b)} \int_1^\infty (s-1)^{c-b-1} s^{-c}\, {}_1F_0 \left[\begin{matrix} (a,p) \\ - \end{matrix} \; ; \frac{1}{zs^\tau} \right] ds$$
$$(\Re(c) > \Re(b) > 0,\ |\arg(z-1)| < \pi),$$

and

(34)
$${}_2R_1^\tau \left[\begin{matrix} (a,p), b \\ c \end{matrix} \; ; z \right] = \frac{1}{B(b,c-b)} \int_0^1 e^{-bs}\,(1-e^{-s})^{c-b-1}\, {}_1F_0[(a,p);-;ze^{-s\tau}]\, ds$$
$$(\Re(c) > \Re(b) > 0).$$

Remark 2. When $p = 0$, (32), (33) and (34) reduce to the following integrals for the τ-hypergeometric function (4)

(35)
$${}_2R_1^\tau \left[\begin{matrix} (a,p), b \\ c \end{matrix} \; ; z \right] = \frac{1}{B(b,c-b)} \int_0^\infty s^{b-1}\,(1+s)^{-c-\tau a}\,((1+s)^\tau - zs^\tau)^{-a} ds,$$

$$ {}_2R_1^{\tau}(a,b;c;1/z) = \frac{1}{B(b,c-b)} \int_1^{\infty} (s-1)^{c-b-1} s^{a\tau-c} \left(s^{\tau}-1/z\right)^{-a} ds \tag{36} $$
$$ (1+\Re(a) > \Re(c) > \Re(b) > 0,\ |\arg(z-1)| < \pi), $$

and

$$ {}_2R_1^{\tau}[a,b;c;z] = \frac{1}{B(b,c-b)} \int_0^{1} e^{-bs} \left(1-e^{-s}\right)^{c-b-1} \left(1-ze^{-s\tau}\right)^{-a} ds. \tag{37} $$

Furthermore, if also $\tau = 1$, these integrals reduces for the Gauss hypergeometric function (see [9]).

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