Commun. Korean Math. Soc. **33** (2018), No. 4, pp. 1141–1158
https://doi.org/10.4134/CKMS.c170135
pISSN: 1225-1763 / eISSN: 2234-3024

STABILITY OF HAHN DIFFERENCE EQUATIONS IN BANACH ALGEBRAS

Marwa M. Abdelkhaliq and Alaa E. Hamza

Abstract. Hahn difference operator $D_{q,\omega}$ which is defined by

$$D_{q,\omega}g(t) = \begin{cases} \frac{g(qt+\omega)-g(t)}{t(q-1)+\omega}, & \text{if } t \neq \theta := \frac{\omega}{1-q}, \\ g'(\theta), & \text{if } t = \theta \end{cases}$$

received a lot of interest from many researchers due to its applications in constructing families of orthogonal polynomials and in some approximation problems. In this paper, we investigate sufficient conditions for stability of the abstract linear Hahn difference equations of the form

$$D_{q,\omega}x(t) = A(t)x(t) + f(t),\ t \in I,$$

and

$$D^2_{q,\omega}x(t) + A(t)D_{q,\omega}x(t) + R(t)x(t) = f(t),\ t \in I,$$

where $A, R : I \to \mathbb{X}$, and $f : I \to \mathbb{X}$. Here $\mathbb{X}$ is a Banach algebra with a unit element $\mathfrak{e}$ and I is an interval of $\mathbb{R}$ containing θ.

1. Introduction and preliminaries

Hahn introduced his difference operator, which is defined by

$$D_{q,\omega}f(t) = \begin{cases} \frac{f(qt+\omega)-f(t)}{t(q-1)+\omega}, & \text{if } t \neq \theta, \\ f'(\theta), & \text{if } t = \theta, \end{cases}$$

where $0 < q < 1$ and $\omega > 0$ are fixed real numbers, $\theta = \omega/(1-q)$, see [11,12]. This operator unifies and generalizes two well-known difference operators. The first is the Jackson q-difference operator defined by

$$D_q f(t) = \frac{f(qt) - f(t)}{t(q-1)},\ t \neq 0,$$

where q is fixed. Here f is supposed to be defined on a q-geometric set $A \subset \mathbb{R}$ for which $qt \in A$ whenever $t \in A$, see [1,2,4,5,10,18–20]. The second operator

Received March 31, 2017; Revised February 3, 2018; Accepted August 29, 2018.
2010 *Mathematics Subject Classification.* Primary 39A13, 39A70.
Key words and phrases. Hahn difference operator, Jackson q-difference operator, stability theory.
This work was financially supported by KRF 2003-041-C20009.

is the forward difference operator

$$\Delta_\omega f(t) = \frac{f(t+\omega) - f(t)}{\omega},$$

see [6–9, 21, 22]. Fine mathematicians applied Hahn's operator to construct families of orthogonal polynomials and to investigate some approximation problems. For more details, see [24–26]. Hahn difference operator did not receive any interest until M. H. Annaby, A. E. Hamza and K. A. Aldwoah studied this operator with a different view to establish a calculus based on Hahn difference operator. In [14], Hamza and Ahmed studied the theory of linear Hahn difference equations after proving the existence and uniqueness of solutions of Hahn difference equations. A. E. Hamza and M. M. Abdelkhaliq, studied the theory of Hahn difference equations in Banach algebras, see [13].

Recently, there has been an interest in studying the behavior of solutions of Hahn difference equations associated with Hahn difference operator. In [17], A. E. Hamza, A. S. Zaghrout and S. M. Ahmed, investigated characterizations of stability of scalar first order Hahn difference equations. They established many types of stability like (uniform, uniform exponential and ψ-) stability. Also, A. E. Hamza and S. D. Makharesh in [15] established the existence of positive solutions of non-linear Hahn difference equations.

Throughout this paper, X is a Banach space, $\mathbb{X}$ is a Banach algebra with a unit $\mathfrak{e}$ and a norm $\| \ \|$, and I is an interval including θ.

In this paper we obtain sufficient conditions for many kinds of stability like (uniform, uniform exponential and h-) stability of abstract first order Hahn difference equations in Banach algebras of the from

$$D_{q,\omega}x(t) = A(t)x(t) + f(t), \ t \in I.$$

We use these results to establish the same kinds of stability for the abstract second order Hahn difference equations of the form

$$D^2_{q,\omega}x(t) + A(t)D_{q,\omega}x(t) + R(t)x(t) = f(t), \ t \in I,$$

where $A, R : I \to \mathbb{X}$, and $f : I \to \mathbb{X}$ is continuous at θ. Every choice of the Banach algebra gives a wide family of Hahn difference equations. Therefore, we study the stability of many types of Hahn difference equations according to what Banach algebra we consider. For instance, this study allows us to consider equations with solutions with values in the Banach algebra $B(X)$, the Banach space of all bounded linear operators from a Banach space X into itself. As special cases, our study includes finite and infinite systems of Hahn difference equations.

In our study we need the function $\mu(t) = qt + \omega$, which is normally taken to be defined on the interval I. The sequence

$$\mu^k(t) = q^k t + \omega[k]_q, \ t \in I,$$

is the k-th order iteration of $\mu(t)$, which uniformly converges to θ on I, and $[k]_q$ is defined by

$$[k]_q = \frac{1-q^k}{1-q}.$$

Now, we will introduce some basic definitions and theorems that will be needed in our study.

Definition. Assume that $f : I \to X$ is a function and let $a, b \in I$. The q, ω-integral of f from a to b is defined by

$$\int_a^b f(t) d_{q,\omega} t = \int_\theta^b f(t) d_{q,\omega} t - \int_\theta^a f(t) d_{q,\omega} t,$$

where

$$\int_\theta^x f(t) d_{q,\omega} t = (x(1-q)-\omega) \sum_{k=0}^{\infty} q^k f(\mu^k(x)), \quad x \in I,$$

provided that the series converges at $x = a$ and $x = b$.

Definition. For certain $z \in \mathbb{C}$, the q, ω-exponential functions $e_z(t)$ and $E_z(t)$ are defined by

$$e_z(t) = \sum_{k=0}^{\infty} \frac{\big(z(t(1-q)-\omega)\big)^k}{(q;q)_k} = \frac{1}{\prod_{k=0}^{\infty}\big(1 - zq^k(t(1-q)-\omega)\big)}, \tag{1}$$

and

$$E_z(t) = \sum_{k=0}^{\infty} \frac{q^{\frac{1}{2}k(k-1)}\big(z(t(1-q)-\omega)\big)^k}{(q;q)_k} = \prod_{k=0}^{\infty}\big(1 + zq^k(t(1-q)-\omega)\big). \tag{2}$$

The functions $e_z(t)$ and $E_z(t)$ are the solutions of the first order Hahn difference equations

$$D_{q,\omega} y(t) = zy(t), \qquad y(\theta) = 1, \tag{3}$$

and

$$D_{q,\omega} y(t) = -zy(qt+\omega), \qquad y(\theta) = 1, \tag{4}$$

respectively, see [3]. For the proofs of the equalities in (1) and (2), see [10, Section 1.3] and [23]. Also these equalities were proved using the method of successive approximation, in [13]. Here the q-shifted factorial $(b;q)_n$ for a complex number b and $n \in \mathbb{N}_0$ is defined to be

$$(b;q)_n = \begin{cases} \prod_{j=1}^{n}(1 - bq^{j-1}), & \text{if } n \in \mathbb{N}, \\ 1, & \text{if } n = 0. \end{cases}$$

By replacing the complex fixed number z by a complex function $p(t)$ which is continuous at θ in (3) and (4), we obtain the exponential functions $e_p(t)$ and

$E_p(t)$, defined by

$$e_p(t) = \frac{1}{\prod_{k=0}^{\infty}\big(1 - p(\mu^k(t))q^k(t(1-q)-\omega)\big)},$$

$$E_p(t) = \prod_{k=0}^{\infty}\big(1 + p(\mu^k(t))q^k(t(1-q)-\omega)\big),$$

whenever the two products are convergent to a nonzero number for every $t \in I$, see [14]. It is worth noting that the two products are convergent since $\sum_{k=0}^{\infty}|p(\alpha^k(t))|q^k(t(1-q)-\omega)$ is convergent, see [27].

In [13], for a continuous operator $A : I \to \mathbb{X}$ at θ, the operator exponential functions $e_A(t)$ and $E_A(t)$ are defined to be

$$e_A(t) = \Big[\prod_{k=0}^{\infty}\Big(\mathfrak{e} - q^k(t(1-q)-\omega)A(\mu^k(t))\Big)\Big]^{-1}, \tag{5}$$

and

$$E_A(t) = \prod_{k=0}^{\infty}\Big(\mathfrak{e} + q^k(t(1-q)-\omega)A(\mu^k(t))\Big), \tag{6}$$

where the products in (5) and (6) are convergent and the first product has an inverse.

The operator exponential function $e_A(t,\tau)$ is defined to be

$$e_A(t,\tau) = e_A(t)e_A^{-1}(\tau).$$

The following lemma gives the q,ω derivative of sum, product and quotients of q,ω differentiable functions, with values in $\mathbb{X}$.

Lemma 1.1. *Let $A : I \to \mathbb{X}$ and $B : I \to \mathbb{X}$ be q,ω-differentiable at $t \in I$. Then*:

(i) $D_{q,\omega}(A+B)(t) = D_{q,\omega}A(t) + D_{q,\omega}B(t)$,

(ii) $D_{q,\omega}(AB)(t) = D_{q,\omega}(A(t))B(h(t)) + A(t)D_{q,\omega}B(t) = D_{q,\omega}(A(t))B(t) + A(h(t))D_{q,\omega}B(t)$,

(iii) *for any constant* $c \in \mathbb{X}$, $D_{q,\omega}(cA)(t) = cD_{q,\omega}(A(t))$,

(iv) $D_{q,\omega}(A^{-1})(t) = -\big(A^{-1}(h(t))\big)\big(D_{q,\omega}A(t)\big)A^{-1}(t)$ *provided that for every* $t \in I$, $\big(A^{-1}(t)\big)$ *exists*,

(v)

$$D_{q,\omega}(AB^{-1})(t) = D_{q,\omega}A(t)\big(B^{-1}(h(t))\big) - A(t)\big(B^{-1}(h(t))\big)D_{q,\omega}B(t)\big(B^{-1}(t)\big)$$

provided that for every $t \in I$, $\big(B^{-1}(t)\big)$ *exists.*

The following theorem is important and will be used later.

Theorem 1.2 ([3])**.** *Assume $f : I \to \mathbb{R}$ is continuous at θ. Then the following statements are true.*

(i) $\{f((sq^k) + \omega[k]_q)\}_{k\in\mathbb{N}}$ *converges uniformly to* $f(\theta)$ *on* I.

(ii) $\sum_{k=0}^{\infty} q^k \left|f(sq^k + \omega[k]_q)\right|$ *is uniformly convergent on* I *and consequently* f *is* q,ω*-integrable over* I.

(iii) *Define*

$$F(x) := \int_\theta^x f(t)d_{q,\omega}t, \quad x \in I.$$

Then F *is continuous at* θ. *Furthermore,* $D_{q,\omega}F(x)$ *exists for every* $x \in I$ *and*

$$D_{q,\omega}F(x) = f(x).$$

Conversely,

$$\int_a^b D_{q,\omega}f(t)d_{q,\omega}t = f(b) - f(a) \quad \textit{for all } a,b \in I.$$

2. Basic concepts of stability

In this section we introduce the concepts of many types of stability. See for instance [16]. Consider the Hahn difference equation of the form

$$D_{q,\omega}x(t) = F(t,x), \; x(\tau) = x_\tau \in \mathbb{X}, \quad t,\tau \in I, \tag{7}$$

where F is assumed to satisfy all conditions that imply Equation (7) to have a unique solution.

Definition. Equation (7) is called stable if for every $\tau \in I$ and every $\epsilon > 0$, there exists $\delta = \delta(\epsilon,\tau) > 0$ such that for any two solutions $x(t) = x(t,\tau,x_\tau)$ and $\hat{x}(t) = \hat{x}(t,\tau,\hat{x}_\tau)$ of Equation (7), we have

$$\|x_\tau - \hat{x}_\tau\| < \delta \Rightarrow \|x(t) - \hat{x}(t)\| < \epsilon \text{ for all } t \geq \tau, \; t,\tau \in I.$$

Definition. Equation (7) is called uniformly stable if for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ independent on τ such that for any two solutions $x(t) = x(t,\tau,x_\tau)$ and $\hat{x}(t) = \hat{x}(t,\tau,\hat{x}_\tau)$ of Equation (7), we have

$$\|x_\tau - \hat{x}_\tau\| < \delta \Rightarrow \|x(t) - \hat{x}(t)\| < \epsilon \text{ for all } t \geq \tau, \; t,\tau \in I.$$

Definition. Equation (7) is called asymptotically stable if it is stable and there exists $\gamma = \gamma(\tau) > 0$ such that for any two solutions $x(t) = x(t,\tau,x_\tau)$ and $\hat{x}(t) = \hat{x}(t,\tau,\hat{x}_\tau)$ of Equation (7), we have

$$\|x_\tau - \hat{x}_\tau\| < \gamma \Rightarrow \lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0.$$

Definition. Equation (7) is called uniformly asymptotically stable if it is uniformly stable and there exists $\gamma > 0$ independent of τ such that for any two solutions $x(t) = x(t,\tau,x_\tau)$ and $\hat{x}(t) = \hat{x}(t,\tau,\hat{x}_\tau)$ of Equation (7), we have

$$\|x_\tau - \hat{x}_\tau\| < \gamma \Rightarrow \lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0.$$

Definition. Equation (7) is called globally asymptotically stable if it is stable and for any two solutions $x(t) = x(t, \tau, x_\tau)$ and $\hat{x}(t) = \hat{x}(t, \tau, \hat{x}_\tau)$ of Equation (7), we have

$$\lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0.$$

Definition. Equation (7) is called exponentially stable if there exists a constant $\alpha > 0$ such that for any solution $x(t) = x(t, \tau, x_\tau)$ of Equation (7), we have

$$\|x(t)\| \le \gamma(\tau, \|x_\tau\|)e_{-\alpha}(t, \tau) \text{ for all } t \ge \tau,\ t, \tau \in I,$$

for some function $\gamma : I \times \mathbb{R}^{\ge 0} \to \mathbb{R}^+$.

Definition. Equation (7) is called uniformly exponentially stable if γ is independent on $\tau \in I$.

Definition. Let $h : I \to \mathbb{R}$ be a positive bounded function. We say that Equation (7) is h-stable if for any solution $x(t) = x(t, \tau, x_\tau)$ of Equation (7), we have

$$\|x(t)\| \le \gamma(\tau, \|x_\tau\|)h(t)h^{-1}(\tau) \text{ for all } t \ge \tau,\ t, \tau \in I,$$

for some function $\gamma : I \times \mathbb{R}^{\ge 0} \to \mathbb{R}^+$. Here $h^{-1}(t) = \frac{1}{h(t)}$.

Definition. Equation (7) is called h-uniformly stable if $\gamma > 0$ is independent on $\tau \in I$.

3. Stability of first order Hahn difference equations

In this section, we obtain some characterizations of different types of stability for linear Hahn difference equations of the form

$$CP(0) :\ D_{q,\omega}x(t) = A(t)x(t), \quad x(\tau) = x_\tau \in \mathbb{X},\ t \ge \tau,\ t, \tau \in I,$$

and

$$CP(f) :\ D_{q,\omega}x(t) = A(t)x(t) + f(t), \quad x(\tau) = x_\tau \in \mathbb{X},\ t \ge \tau,\ t, \tau \in I,$$

where $A, f : I \to \mathbb{X}$ are continuous at θ. We suppose all conditions that imply the existence of the exponential functions $e_A(t, \tau)$ and $E_A(t, \tau)$.

The initial value problems $CP(0)$ and $CP(f)$ have the unique solutions

$$x(t) = e_A(t, \tau)x_\tau$$

and

$$x(t) = e_A(t, \tau)\Big(x_\tau + \int_\tau^t f(s)e_A(\tau, \mu(s))d_{q,\omega}s\Big)$$

respectively.

For the proof of the following two theorems, see [17].

Theorem 3.1. *The following statements are equivalent.*

(i) $CP(0)$ *is stable.*

(ii) *For every* $\tau \in I$ *and every* $\epsilon > 0$*, there exists* $\delta = \delta(\epsilon, \tau)$ *such that for any solution* $x(t) = x(t, \tau, x_\tau)$ *of* $CP(0)$*, we have*

$$\|x_\tau\| < \delta \Rightarrow \|x(t)\| < \epsilon.$$

(iii) $CP(f)$ *is stable.*

(iv) *For every* $\tau \in I$*,* $\{\|e_A(t, \tau)\|\}_{t \geq \tau, t \in I}$ *is bounded.*

(v) *For every* $\tau \in I$*, there exists* $\gamma(\tau) > 0$ *such that for any solution* $x(t) = x(t, \tau, x_\tau)$ *of* $CP(0)$ *(resp.* $CP(f)$*), we have*

$$\|x(t)\| \leq \gamma(\tau)\|x_\tau\| \text{ for all } t \geq \tau, t \in I.$$

Theorem 3.2. *The following statements are equivalent.*

(i) $CP(0)$ *is uniformly stable.*

(ii) *For every* $\epsilon > 0$*, there exists* $\delta = \delta(\epsilon)$ *such that for any solution* $x(t) = x(t, \tau, x_\tau)$ *of* $CP(0)$*, we have*

$$\|x_\tau\| < \delta \Rightarrow \|x(t)\| < \epsilon.$$

(iii) $CP(f)$ *is uniformly stable.*

(iv) $\{\|e_A(t, \tau)\| : t, \tau \in I, t \geq \tau\}$ *is bounded.*

(v) *There is* $\gamma > 0$ *such that for any solution* $x(t) = x(t, \tau, x_\tau)$ *of* $CP(0)$ *(resp.* $CP(f)$*), we have*

$$\|x(t)\| \leq \gamma\|x_\tau\| \text{ for all } t \geq \tau,\ t \in I.$$

Now, we establish a necessary and sufficient condition for the global asymptotic stability of $CP(0)$.

Theorem 3.3. *The following statements are equivalent.*

(i) $CP(0)$ *is asymptotically stable.*

(ii) $\lim_{t\to\infty} \|e_A(t, \tau)x\| = 0$ *for every* $x \in \mathbb{X}$ *and every* $\tau \in I$*.*

(iii) $CP(0)$ *is globally asymptotically stable.*

Proof. (i)⇒(ii) Suppose that $CP(0)$ is asymptotically stable. Then, there exists $\gamma > 0$ such that for any solution $x(t) = x(t, \tau, x_\tau)$ of $CP(0)$, with initial value x_τ, we have

$$\|x_\tau\| < \gamma \Rightarrow \lim_{t\to\infty} \|x(t)\| = 0.$$

Let $0 \neq x \in \mathbb{X}$. Put $x_\tau = \frac{\gamma x}{2\|x\|}$. Then,

$$\lim_{t\to\infty} \left\| \frac{e_A(t, \tau)\gamma x}{2\|x\|} \right\| = 0.$$

Consequently, $\lim_{t\to\infty} \|e_A(t, \tau)x\| = 0$.

(ii)⇒(iii) By condition (ii) and the Uniform Boundedness Theorem [28], we insure the boundedness of $\{\|e_A(t, \tau)\|\}_{t\geq\tau, t\in I}$. Consequently, $CP(0)$ is stable (by Theorem 3.1). Thus by our assumption, $CP(0)$ is globally asymptotically stable.

(iii)⇒(i) is clear. □

For the proof of the next theorem, again see [17].

Theorem 3.4. *Assume that*

$$F(t) = \int_{\tau}^{t} f(s)E_{-A}(\mu(s))d_{q,\omega}s$$

is bounded for any $\tau \in I$. Then, $CP(0)$ is globally asymptotically stable if and only if $CP(f)$ is globally asymptotically stable.

We follow the proof of Theorem 2.5 in [17], to obtain the next two results.

Theorem 3.5. *The following statements are equivalent.*

(i) *$CP(0)$ is exponentially stable.*

(ii) *There exists $\alpha > 0$ such that*

$$\|e_A(t,\tau)\| \leq \beta(\tau)e_{-\alpha}(t,\tau) \text{ for all } t \geq \tau,$$

for some function $\beta : I \to \mathbb{R}^+$.

Proof. (i)⇒(ii) Let $x(t)$ be any nontrivial solution corresponding to the initial value $x_\tau \neq 0$. Then, we have $\|x(t)\| \leq \gamma(\tau, \|x_\tau\|)e_{-\alpha}(t,\tau)$ for some function $\gamma : I \times \mathbb{R}^{\geq 0} \to \mathbb{R}^+$. Consequently, we have $\|e_A(t,\tau)x_\tau\| \leq \gamma(\tau, \|x_\tau\|)e_{-\alpha}(t,\tau)$ for any initial value $x_\tau \in \mathbb{X}$. This implies that for any non zero $x_\tau \in \mathbb{X}$, we have

$$\|e_A(t,\tau)\| \leq \sup_{\|x_\tau\|=1} \gamma(\tau, \|x_\tau\|)e_{-\alpha}(t,\tau).$$

Then

$$\|e_A(t,\tau)\| \leq \beta(\tau)e_{-\alpha}(t,\tau),$$

where $\beta(\tau) = \sup_{\|x_\tau\|=1} \gamma(\tau, \|x_\tau\|)$.

(ii)⇒(i) Let $\|e_A(t,\tau)\| \leq \beta(\tau)e_{-\alpha}(t,\tau)$. Then, we have

$$\begin{aligned} \|x(t)\| &= \|e_A(t,\tau)x_\tau\| \\ &\leq \|e_A(t,\tau)\|\|x_\tau\| \\ &\leq \beta(\tau)e_{-\alpha}(t,\tau)\|x_\tau\| \text{ for all } t \geq \tau. \end{aligned}$$

Hence, $CP(0)$ is exponentially stable. □

Theorem 3.6. *The following statements are equivalent.*

(i) *$CP(0)$ is uniformly exponentially stable.*

(ii) *There exist $\alpha > 0$ and $\beta > 0$ independent on τ such that*

$$\|e_A(t,\tau)\| \leq \beta e_{-\alpha}(t,\tau) \text{ for all } t \geq \tau.$$

The following results concerning h-stability and h-unifom stability are more general than Theorems 3.5 and 3.6.

Theorem 3.7. *The following statements are equivalent.*

(i) *$CP(0)$ is h-stable.*

(ii) *There exists a function $\beta : I \to \mathbb{R}^+$ such that*

$$\|e_A(t,\tau)\| \leq \beta(\tau)h(t)h^{-1}(\tau) \text{ for all } t \geq \tau.$$

Proof. (i)⇒(ii) Assume that $CP(0)$ is h-stable. There exists $\gamma : I \to \mathbb{R}^+$ such that any solution $x(t) = x(t, \tau, x_\tau)$ $CP(0)$ with an initial value $x_\tau \in \mathbb{X}$, satisfies

$$\|x(t)\| = \|e_A(t,\tau)x_\tau\| \leq \gamma(\tau, \|x_\tau\|)h(t)h^{-1}(\tau) \text{ for all } t \geq \tau.$$

Consequently, we have

$$\|e_A(t,\tau)\| \leq \sup_{\|x_\tau\|=1} \gamma(\tau, \|x_\tau\|)h(t)h^{-1}(\tau).$$

Then

$$\|e_A(t,\tau)\| \leq \beta(\tau)h(t)h^{-1}(\tau),$$

where $\beta(\tau) = \sup_{\|x_\tau\|=1} \gamma(\tau, \|x_\tau\|)$.

(ii)⇒(i) Assume that $\|e_A(t,\tau)\| \leq \beta(\tau)h(t)h^{-1}(\tau)$ for some function $\beta : I \to \mathbb{R}^+$. Then, we have

$$\begin{aligned}\|x(t)\| &= \|e_A(t,\tau)x_\tau\| \\ &\leq \|e_A(t,\tau)\|\|x_\tau\| \\ &\leq \beta(\tau)\|x_\tau\|h(t)h^{-1}(\tau) \text{ for } t \geq \tau.\end{aligned}$$

Hence, $CP(0)$ is h-stable. □

We can prove the following theorem similarly, so the proof will be omitted.

Theorem 3.8. *The following statements are equivalent.*

(i) $CP(0)$ *is h-uniformly stable.*

(ii) *There exists $\gamma > 0$ independent on τ such that*

$$\|e_A(t,\tau)\| \leq \gamma h(t)h^{-1}(\tau) \text{ for all } t \geq \tau.$$

Now, we study some different types of q, ω- stability of the non-homogeneous first order Hahn difference equations. As usual I denotes an interval which contains θ. For $s \in I$, we define the class $[s]_{q,\omega}$ by

$$[s]_{q,\omega} =: \{\mu^k(s) = sq^k + \omega[k]_q : k \in \mathbb{Z}\} \bigcap I \bigcup \{\theta\}.$$

It is well known that the following facts are true:

(1) For $s > \theta$, we have $\mu^k(s) \to \theta$ as $k \to \infty$ and $\mu^{-k}(s) \to \infty$.

(2) For $s < \theta$, we have $\mu^k(s) \to \theta$ as $k \to \infty$ and $\mu^{-k}(s) \to -\infty$.

See [3], [14].

Definition. We say that Equation (7) is q, ω-exponentially stable if there exists a constant $\alpha > 0$ such that for any $s \in I$, for any $\tau \in [s]_{q,\omega}$ and for any solution $x(t) = x(t, \tau, x_\tau)$ of Equation (7) with initial value x_τ, we have

$$\|x(t)\| \leq \gamma(\tau, \|x_\tau\|)e_{-\alpha}(t,\tau) \text{ for all } t \geq \tau,\ t \in [s]_{q,\omega},$$

for some function $\gamma : I \times \mathbb{R}^{\geq 0} \to \mathbb{R}^+$.

Definition. Equation (7) is called q, ω-uniformly exponentially stable if γ is independent on $\tau \in I$.

Definition. Let $h : I \to \mathbb{R}$ be a positive bounded function. We say that Equation (7) is q, ω-h-stable if for any $s \in I$, for any $\tau \in [s]_{q,\omega}$ and for any solution $x(t) = x(t, \tau, x_\tau)$ of Equation (7), we have

$$\|x(t)\| \leq \gamma(\tau, \|x_\tau\|)h(t)h^{-1}(\tau) \text{ for all } t \geq \tau,\ t \in [s]_{q,\omega},$$

for some function $\gamma : I \times \mathbb{R}^{\geq 0} \to \mathbb{R}^+$. Here $h^{-1}(t) = \frac{1}{h(t)}$.

Definition. Equation (7) is called q, ω-h-uniformly stable if γ is independent on $\tau \in I$.

The proofs of the following results concerning the q, ω-exponentially stability and q, ω-h-stability are similar to the proofs of Theorems 3.5-3.8. So they will be omitted.

Theorem 3.9. *The following statements are equivalent.*

(i) $CP(0)$ *is* q, ω*-exponentially stable.*

(ii) *There exists* $\alpha > 0$ *such that for any* $s \in I$, $\tau \in [s]_{q,\omega}$, *we have*

$$\|e_A(t, \tau)\| \leq \beta(\tau)e_{-\alpha}(t, \tau) \text{ for all } t \geq \tau,\ t \in [s]_{q,\omega},$$

for some function $\beta : I \to \mathbb{R}^+$.

Theorem 3.10. *The following statements are equivalent.*

(i) $CP(0)$ *is* q, ω*-uniformly exponentially stable.*

(ii) *There exists* $\alpha > 0$ *such that for any* $s \in I$, *we have*

$$\|e_A(t, \tau)\| \leq \beta e_{-\alpha}(t, \tau) \text{ for all } t \geq \tau,\ t, \tau \in [s]_{q,\omega},$$

for some constant $\beta > 0$.

Theorem 3.11. *The following statements are equivalent.*

(i) $CP(0)$ *is* q, ω*-*h*-stable.*

(ii) *There exists a function* $\beta : I \to \mathbb{R}^+$ *such that for any* $s \in I$, $\tau \in [s]_{q,\omega}$, *we have*

$$\|e_A(t, \tau)\| \leq \beta(\tau)h(t)h^{-1}(\tau) \text{ for all } t \geq \tau, t \in [s]_{q,\omega}.$$

Theorem 3.12. *The following statements are equivalent.*

(i) $CP(0)$ *is* q, ω*-uniformly* h*-stable.*

(ii) *There exists a constant* $\beta > 0$ *such that for any* $s \in I$, $\tau \in [s]_{q,\omega}$, *we have*

$$\|e_A(t, \tau)\| \leq \beta h(t)h^{-1}(\tau) \text{ for all } t \geq \tau,\ t \in [s]_{q,\omega}.$$

Theorem 3.13. *Assume that there exists a constant* $\alpha > 0$, *and there are functions* $\gamma : I \to \mathbb{R}^+$ *which is continuous at* θ, *and* $\beta : I \to \mathbb{R}^+$ *such that for any* $s \in I$ *and for any* $\tau \in [s]_{q,\omega}$,*the following conditions hold*

(i)

$$\|e_A(t, \tau)\| \leq \gamma(\tau)e_{-\alpha}(t, \tau) \text{ for all } t \geq \tau,\ t \in [s]_{q,\omega}.$$

(ii)

$$\int_{\tau}^{t} \gamma(\mu(s))e_{-\alpha}(\tau,\mu(s))\|f(s)\|d_{q,\omega}s < \beta(\tau),\ t\geq\tau,\ t\in[s]_{q,\omega}.$$

Then $CP(f)$ is q,ω-exponentially stable and every solution $x(t)$ with an initial value x_τ satisfies the following inequality

$$\|x(t)\| \leq \gamma(\tau)\big(\|x_\tau\| + \beta(\tau)\big)e_{-\alpha}(t,\tau),\ t,\tau\in[s]_{q,\omega},$$

for every $s\in I$.

Proof. Let $x(t)$ be a solution of $CP(f)$ with an initial value x_τ. Then, we have

$$\begin{aligned}\|x(t)\| &\leq \|e_A(t,\tau)\|\Big(\|x_\tau\| + \int_{\tau}^{t}\|f(s)\|\|e_A(\tau,\mu(s))\|d_{q,\omega}s\Big)\\ &\leq \gamma(\tau)e_{-\alpha}(t,\tau)\Big[\|x_\tau\| + \int_{\tau}^{t}\|f(s)\|\gamma(\mu(s))e_{-\alpha}(\tau,\mu(s))\|d_{q,\omega}s\Big]\\ &\leq \gamma(\tau)(\|x_\tau\|+\beta(\tau))e_{-\alpha}(t,\tau), t,\tau\in[s]_{q,\omega}.\end{aligned}$$

Therefore, Equation $CP(f)$ is q,ω-exponentially stable. □

Theorem 3.14. *Assume that there exists $\alpha > 0$, and there are constants $\gamma,\beta > 0$, such that for any $s\in I$ and for any $\tau\in[s]_{q,\omega}$, the following conditions hold*

(i)

$$\|e_A(t,\tau)\| \leq \gamma e_{-\alpha}(t,\tau) \text{ for all } t\geq\tau,\ t\in[s]_{q,\omega}.$$

(ii)

$$\int_{\tau}^{t} e_{-\alpha}(\tau,\mu(s))\|f(s)\|d_{q,\omega}s < \beta,\quad t\geq\tau,\ t\in[s]_{q,\omega}.$$

Then $CP(f)$ is q,ω-uniformly exponentially stable and every solution $x(t)$ with an initial value x_τ satisfies the following inequality

$$\|x(t)\| \leq \gamma\big(\|x_\tau\|+\beta\big)e_{-\alpha}(t,\tau),\quad \tau,t\in[s]_{q,\omega},$$

for every $s\in I$.

Proof. The proof is similar to the proof of Theorem 3.13 and will be omitted. □

The following results concerning h-stability and h-unifom stability are more general than Theorems 3.13 and 3.14.

Theorem 3.15. *Assume that there exists $\gamma: I\to\mathbb{R}^+$ continuous at θ and $\beta: I\to\mathbb{R}^+$ such that for any $s\in I$ and for any $\tau\in[s]_{q,\omega}$,the following conditions hold*

(i)

$$\|e_A(t,\tau)\| \leq \gamma(\tau)h(t)h^{-1}(\tau) \text{ for all } t\geq\tau,\ t\in[s]_{q,\omega}.$$

(ii)

$$\int_{\tau}^{t} \gamma(\mu(s))h(\tau)h^{-1}(\mu(s))\|f(s)\|d_{q,\omega}s < \beta(\tau),\ t \in [s]_{q,\omega},\ t \geq \tau.$$

Then $CP(f)$ is q,ω-h-stable and every solution $x(t)$ with an initial value x_τ satisfies the following inequality

$$\|x(t)\| \leq \gamma(\tau)\big(\|x_\tau\| + \beta(\tau)\big)h(t)h^{-1}(\tau),\ t,\tau \in [s]_{q,\omega},$$

for every $s \in I$.

Proof. Let $x(t)$ be a solution of $CP(f)$ with initial value x_τ, $\tau \in [s]_{q,\omega}$. Then, we have

$$\begin{aligned}\|x(t)\| &\leq \|e_A(t,\tau)\|\Big(\|x_\tau\| + \int_{\tau}^{t} \|f(s)\|\|e_A(\tau,\mu(s))\|d_{q,\omega}s\Big)\\ &\leq \gamma(\tau)h(t)h^{-1}(\tau)\Big[\|x_\tau\| + \int_{\tau}^{t} \|f(s)\|\gamma(\mu(s))h(\tau)h^{-1}(\mu(s))d_{q,\omega}s\Big]\\ &\leq \gamma(\tau)\Big(\|x_\tau\| + \beta(\tau)\Big)h(t)h^{-1}(\tau),\ t \in [s]_{q,\omega}.\end{aligned}$$

Therefore, Equation $CP(f)$ is q,ω-h-stable. □

Theorem 3.16. *Assume that the following conditions hold.*

(i) *There exists a constant $\gamma > 0$, such that*

$$\|e_A(t,\tau)\| \leq \gamma h(t)h^{-1}(\tau).$$

(ii) *There exists a constant $\beta > 0$, such that*

$$\int_{\tau}^{t} \|f(s)\|h(\mu(s))h^{-1}(\tau)d_{q,\omega}s < \beta \quad \text{for all } t \geq \tau.$$

Then $CP(f)$ is uniform q,ω-h-stable and every solution $x(t)$ with initial value x_τ satisfies the following inequality

$$\|x(t)\| \leq \gamma\big(\|x_\tau\| + \beta\big)h(t)h^{-1}(\tau).$$

Proof. The proof is similar to the proof of Theorem 3.15 and will be omitted. □

4. Second order Hahn difference equations

In this section, we study the stability of the second order Hahn difference equations of the form

$$D^2_{q,\omega}x(t) + A(t)D_{q,\omega}x(t) + R(t)x(t) = 0,\ t \in I \tag{8}$$

and

$$D^2_{q,\omega}x(t) + A(t)D_{q,\omega}x(t) + R(t)x(t) = f(t),\ t \in I \tag{9}$$

with initial conditions $D_{q,\omega}x(\tau) = x_{1\tau}$ and $x(\tau) = x_{0\tau}$, where $A, R, f : I \to \mathbb{X}$ are continuous at θ. Let $z : I \to \mathbb{X}$ be a particular solution of the corresponding Riccati equation

$$D_{q,\omega}z(t) - \mathcal{F}(t)z(t) = R(t),\ t \in I, \tag{10}$$

where $\mathcal{F}(t) = z(\mu(t)) - A(t)$.

We need the following lemma. The proof is simple and will be omitted.

Lemma 4.1. *If x is a solution of Equation* (9) *or* (8), *then $g(t) = D_{q,\omega}x(t) + z(t)x(t)$ is a solution of*

$$D_{q,\omega}g(t) - \mathcal{F}(t)g(t) - f(t) = 0 \tag{11}$$

or

$$D_{q,\omega}g(t) - \mathcal{F}(t)g(t) = 0 \tag{12}$$

respectively.

Now we study the different kinds of stability for Equation (9), and the results can be applied on Equation (8) by replacing $f(t)$ with zero.

Theorem 4.2. *If the functions $\|e_{\mathcal{F}}(t,\tau)\|$, $\|e_{-z}(t,\tau)\|$, and $\int_\tau^t \|e_{-z}(t,\mu(s))\| d_{q,\omega}s$ are bounded for every $\tau \in I$, then Equation* (9) *is stable.*

Proof. We denote by $L = \sup_{t\geq\tau} \|e_{-z}(t,\tau)\|$, $K = \sup_{t\geq\tau} \int_\tau^t \|e_{-z}(t,\mu(s))\| d_{q,\omega}s$, and $M = \sup_{t\geq\tau} \|e_{\mathcal{F}}(t,\tau)\|$. From Theorem 3.1, $D_{q,\omega}g - \mathcal{F}g - f = 0$ is stable, since $\{\|e_{\mathcal{F}}(t,\tau)\| : t \geq \tau\}$ is bounded. Then for every $\epsilon > 0$, there is $\delta_1(\epsilon,\tau) > 0$ such that for any two solutions $g(t) = g(t,\tau,g_\tau)$ and $\hat{g}(t) = \hat{g}(t,\tau,\hat{g}_\tau)$ with initial values g_τ and $\hat{g}_\tau$, respectively, we have

$$\|g_\tau - \hat{g}_\tau\| < \delta_1 \Longrightarrow \|g(t) - \hat{g}(t)\| < \frac{\epsilon}{2K}.$$

Choose $\delta > 0$ such that

$$\delta \leq \min\Big(\frac{\delta_1}{\max\big(\|z(\tau)\|, 1\big)}, \frac{\epsilon}{2L}\Big).$$

Let $x(t) = x(t,\tau,x_{0\tau},x_{1\tau})$ and $\hat{x}(t) = \hat{x}(t,\tau,\hat{x}_{0\tau},\hat{x}_{1\tau})$ be two solutions of Equation (9) with initial values $X(\tau) = (x_{0\tau}, x_{1\tau})$ and $\hat{X}(\tau) = (\hat{x}_{0\tau}, \hat{x}_{1\tau})$ such that

$$\|X(\tau) - \hat{X}(\tau)\| < \delta.$$

Hence,

$$g(t) = D_{q,\omega}x(t) + z(t)x(t) \quad \text{and} \quad \hat{g}(t) = D_{q,\omega}\hat{x}(t) + z(t)\hat{x}(t)$$

are solutions of Equation (11) corresponding respectively to the initial conditions

$$g_\tau = D_{q,\omega}x(\tau) + z(\tau)x(\tau) \quad \text{and} \quad \hat{g}_\tau = D_{q,\omega}\hat{x}(\tau) + z(\tau)\hat{x}(\tau).$$

We see that $\|g_\tau - \hat{g}_\tau\| < \delta_1$. Consequently, $\|g(t) - \hat{g}(t)\| < \frac{\epsilon}{2K}$, $\forall t \geq \tau, t \in I$. The solutions $x(t)$ and $\hat{x}(t)$ of Equation (9) are given by

$$x(t) = e_{-z}(t,\tau)\big(x_{0\tau} + \int_\tau^t e_{-z}(\tau,\mu(s))g(s)d_{q,\omega}s\big)$$

and

$$\hat{x}(t) = e_{-z}(t,\tau)\big(\hat{x}_{0\tau} + \int_\tau^t e_{-z}(\tau,\mu(s))\hat{g}(s)d_{q,\omega}s\big).$$

This implies that $\|x(t) - \hat{x}(t)\| \leq \epsilon$. Therefore, Equation (9) is stable. □

Similarly, we can prove the following theorem.

Theorem 4.3. *If the functions $\|e_{\mathcal{F}}(t,\tau)\|$, $\|e_{-z}(t,\tau)\|$ and $\int_\tau^t \|e_{-z}(t,\mu(s))\|d_{q,\omega}s$ are uniformly bounded with respect to $\tau \in I$, then Equation* (9) *is uniformly stable.*

Now, we establish the characterizations of the q,ω-exponential stability and the uniform q,ω-exponential stability.

Theorem 4.4. *Assume there exist constants $\alpha > 0$ and $\alpha_1 > 0$, and there are functions $\gamma : I \to \mathbb{R}^+$ which is continuous at θ, and $\beta, L : I \to \mathbb{R}^+$ such that for any $s \in I$ and for any $\tau \in [s]_{q,\omega}$, the following conditions hold*

(i)

$$\|e_{\mathcal{F}}(t,\tau)\| \leq \gamma(\tau)e_{-\alpha}(t,\tau)$$

and

$$\|e_{-z}(t,\tau)\| \leq \gamma(\tau)e_{-\alpha_1}(t,\tau), \quad t \in [s]_{q,\omega} \text{ for all } t \geq \tau.$$

(ii)

$$\int_\tau^t \gamma(\mu(s))e_{-\alpha}(\tau,\mu(s))\|f(s)\|d_{q,\omega}s < \beta(\tau), \quad t \in [s]_{q,\omega},\ t \geq \tau;$$

and

$$\int_\tau^t \gamma(\mu(s))e_{-\alpha_1}(\tau,\mu(s))e_{-\alpha}(s,\tau)d_{q,\omega}s < L(\tau), \quad t \in [s]_{q,\omega},\ t \geq \tau.$$

Then Equation (9) *is q,ω-exponentially stable.*

Proof. From Theorem 3.13, $D_{q,\omega}g - \mathcal{F}g - f = 0$ is q,ω-exponentially stable. Then any solution $g(t) = g(t,\tau,g_\tau)$ with initial value g_τ, satisfies

$$\|g(t)\| \leq \gamma(\tau)e_{-\alpha}(t,\tau)\Big[\|g_\tau\| + \beta(\tau)\Big] \quad \text{for all } t \geq \tau,\ t \in [s]_{q,\omega}.$$

Set

$$\gamma_1(\tau,r) = \gamma(\tau)\Big[\|r\| + \beta(\tau)\Big].$$

This gives

$$\|g(t)\| \leq \gamma_1(\tau,\|g_\tau\|)e_{-\alpha}(t,\tau).$$

Let $x(t)$ be a solution of Equation (9) with initial value $X(\tau) = (x_{0\tau}, x_{1\tau})$. Then $g(t) = D_{q,\omega}x(t) + z(t)x(t)$ is a solution of Equation (11) with initial value $g(\tau) = x_{1\tau} + z(\tau)x_{0\tau}$. The solution $x(t)$ is given by

$$x(t) = e_{-z}(t,\tau)\Big[x_{0\tau} + \int_{\tau}^{t} e_{-z}(\tau,\mu(s))g(s)d_{q,\omega}s\Big].$$

Hence

$$\begin{aligned}\|x(t)\| &\leq \gamma(\tau)e_{-\alpha_1}(t,\tau)\Big[\|X(\tau)\| \\ &\qquad + \int_{\tau}^{t} \gamma(\mu(s))e_{-\alpha_1}(\tau,\mu(s))\gamma_1(\tau,\|g_\tau\|)e_{-\alpha}(s,\tau)d_{q,\omega}s\Big] \\ &\leq \gamma(\tau)e_{-\alpha_1}(t,\tau)\Big[\|X(\tau)\| + L\gamma_1(\tau,\|g_\tau\|)\Big].\end{aligned}$$

Therefore, Equation (9) is q,ω-exponentially stable. □

Theorem 4.5. *Assume there exist positive numbers α, α_1, γ, β and L such that for any $s \in I$ and for any $\tau \in [s]_{q,\omega}$, the following conditions hold*

(i)

$$\|e_{\mathcal{F}}(t,\tau)\| \leq \gamma e_{-\alpha}(t,\tau)$$

and

$$\|e_{-z}(t,\tau)\| \leq \gamma e_{-\alpha_1}(t,\tau) \quad \text{for all } t \geq \tau,\ t \in [s]_{q,\omega}.$$

(ii)

$$\int_{\tau}^{t} \gamma e_{-\alpha}(\tau,\mu(s))\|f(s)\|d_{q,\omega}s < \beta, \quad t \in [s]_{q,\omega},\ t \geq \tau;$$

and

$$\int_{\tau}^{t} \gamma e_{-\alpha_1}(\tau,\mu(s))e_{-\alpha}(s,\tau)d_{q,\omega}s < L, \quad t \in [s]_{q,\omega},\ t \geq \tau.$$

Then Equation (9) is q,ω-uniformly exponentially stable.

Proof. The proof is similar to the proof of Theorem 4.4, and will be omitted. □

Theorem 4.6. *Assume that there exist two positive bounded functions $h, h_1 : I \to \mathbb{R}$, and there are functions $\gamma, \beta, L : I \to \mathbb{R}^+$ such that γ is continuous at θ. If for any $s \in I$ and for any $\tau \in [s]_{q,\omega}$, the following conditions*

(i)

$$\|e_{\mathcal{F}}(t,\tau)\| \leq \gamma(\tau)h(t)h^{-1}(\tau)$$

and

$$\|e_{-z}(t,\tau)\| \leq \gamma(\tau)h_1(t)h_1^{-1}(\tau) \text{ for all } t \geq \tau,\ t \in [s]_{q,\omega}.$$

(ii)

$$\int_{\tau}^{t} \gamma(\mu(s))h(\tau)h^{-1}(\mu(s))\|f(s)\|d_{q,\omega}s < \beta(\tau),\ t \in [s]_{q,\omega};$$

and

$$\int_{\tau}^{t} \gamma(\mu(s))h_1(\tau)h_1^{-1}(\mu(s))h(s)h^{-1}(\tau)d_{q,\omega}s < L(\tau),\ t \in [s]_{q,\omega},$$

hold, then Equation (9) *is* q,ω-*h*-*stable.*

Proof. The equation $D_{q,\omega}g - \mathcal{F}g - f = 0$ is q,ω-h-stable, by Theorem 3.15 and any solution $g(t) = g(t,\tau,g_\tau)$ with initial value g_τ, satisfies

$$\|g(t)\| \leq \gamma(\tau)h(t)h^{-1}(\tau)\Big[\|g_\tau\| + \beta(\tau)\Big] \quad \text{for all } t \geq \tau,\ t \in [s]_{q,\omega}.$$

Set

$$\gamma_1(\tau, r) = \gamma(\tau)\Big[\|r\| + \beta(\tau)\Big].$$

This gives

$$\|g(t)\| \leq \gamma_1(\tau, \|g_\tau\|)h(t)h^{-1}(\tau).$$

Let $x(t)$ be a solution of Equation (9) with initial value $X(\tau) = (x_{0\tau}, x_{1\tau})$. Then $g(t) = D_{q,\omega}x(t) + z(t)x(t)$ is a solution of Equation (11) with initial value $g(\tau) = x_{1\tau} + z(\tau)x_{0\tau}$. The solution $x(t)$ is given by

$$x(t) = e_{-z}(t,\tau)\Big[x_{0\tau} + \int_{\tau}^{t} e_{-z}(\tau, \mu(s))g(s)d_{q,\omega}s\Big].$$

Hence

$$\begin{aligned}\|x(t)\| \leq\ & \gamma(\tau)h_1(t)h_1^{-1}(\tau)\Big[\|X(\tau)\| \\ & \qquad + \int_{\tau}^{t} \gamma(\mu(s))h_1(\tau)h_1^{-1}(\mu(s))\gamma_1(\tau, \|g_\tau\|)h(s)h^{-1}(\tau)d_{q,\omega}s\Big] \\ \leq\ & \gamma(\tau)h_1(t)h_1^{-1}(\tau)\Big[\|X(\tau)\| + L(\tau)\gamma_1(\tau, \|g_\tau\|)\Big].\end{aligned}$$

Therefore, Equation (9) is q,ω-h-stable. □

Theorem 4.7. *Assume there exist positive bounded functions* $h, h_1 : I \to \mathbb{R}$ *and there are positive constants* γ, β *and* L *such that for any* $s \in I$ *and for any* $\tau \in [s]_{q,\omega}$*, the following conditions hold*

(i)

$$\|e_{\mathcal{F}}(t,\tau)\| \leq \gamma h(t)h^{-1}(\tau) \quad \textit{and} \quad \|e_{-z}(t,\tau)\| \leq \gamma h_1(t)h_1^{-1}(\tau), \quad t \in [s]_{q,\omega}.$$

(ii)

$$\int_{\tau}^{t} \gamma h(\tau)h^{-1}(\mu(s))\|f(s)\|d_{q,\omega}s < \beta, \quad t \in [s]_{q,\omega};$$

and

$$\int_{\tau}^{t} \gamma h_1(\tau)h_1^{-1}(\mu(s))h(s)h^{-1}(\tau)d_{q,\omega}s < L, \quad t \in [s]_{q,\omega}.$$

Then Equation (9) *is uniform* q,ω-h-*stable.*

Proof. The proof is similar to the proof of Theorem 4.6 and will be omitted. □

References

[1] M. H. Abu-Risha, M. H. Annaby, M. E. H. Ismail, and Z. S. Mansour, *Linear q-difference equations*, Z. Anal. Anwend. **26** (2007), no. 4, 481–494.

[2] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications, **71**, Cambridge University Press, Cambridge, 1999.

[3] M. H. Annaby, A. E. Hamza, and K. A. Aldwoah, *Hahn difference operator and associated Jackson-Nörlund integrals*, J. Optim. Theory Appl. **154** (2012), no. 1, 133–153.

[4] M. H. Annaby and Z. S. Mansour, *q-Taylor and interpolation series for Jackson q-difference operators*, J. Math. Anal. Appl. **344** (2008), no. 1, 472–483.

[5] ———, *q-fractional calculus and equations*, Lecture Notes in Mathematics, **2056**, Springer, Heidelberg, 2012.

[6] M. T. Bird, *On generalizations of sum formulas of the Euler-MacLaurin type*, Amer. J. Math. **58** (1936), no. 3, 487–503.

[7] G. D. Birkhoff, *General theory of linear difference equations*, Trans. Amer. Math. Soc. **12** (1911), no. 2, 243–284.

[8] R. D. Carmichael, *Linear difference equations and their analytic solutions*, Trans. Amer. Math. Soc. **12** (1911), no. 1, 99–134.

[9] ———, *On the theory of linear difference equations*, Amer. J. Math. **35** (1913), no. 2, 163–182.

[10] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, second edition, Encyclopedia of Mathematics and its Applications, **96**, Cambridge University Press, Cambridge, 2004.

[11] W. Hahn, *Über Orthogonalpolynome, die q-Differenzengleichungen genügen*, Math. Nachr. **2** (1949), 4–34.

[12] ———, *Ein Beitrag zur Theorie der Orthogonalpolynome*, Monatsh. Math. **95** (1983), no. 1, 19–24.

[13] A. E. Hamza and M. M. Abdelkhaliq, *Hahn difference equations in Banach algebras*, Adv. Difference Equ. **2016** (2016), Paper No. 161, 25 pp.

[14] A. E. Hamza and S. M. Ahmed, *Existence and uniqueness of solutions of Hahn difference equations*, Adv. Difference Equ. **2013** (2013), 316, 15 pp.

[15] A. E. Hamza and S. D. Makharesh, *Positive solutions of nonlinear Hahn difference equations*, Adv. Dyn. Syst. Appl. **11** (2016), no. 2, 113–123.

[16] A. E. Hamza and K. M. Oraby, *Stability of abstract dynamic equations on time scales*, Adv. Difference Equ. **2012** (2012), 143, 15 pp.

[17] A. E. Hamza, A. S. Zaghrout, and S. M. Ahmed, *Characterization of stability of first order Hahn difference equations*, J. Adv. in Math. **5** (2013), 678–687.

[18] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Encyclopedia of Mathematics and its Applications, **98**, Cambridge University Press, Cambridge, 2005.

[19] F. H. Jackson, *On q-functions and a certain difference operator*, Transactions of the Royal Society Edinburgh **46** (1908), 253–281.

[20] ———, *Basic integration*, Quart. J. Math., Oxford Ser. (2) **2** (1951), 1–16.

[21] D. L. Jagerman, *Difference Equations with Applications to Queues*, Monographs and Textbooks in Pure and Applied Mathematics, **233**, Marcel Dekker, Inc., New York, 2000.

[22] C. Jordan, *Calculus of Finite Differences*, Third Edition. Introduction by Harry C. Carver, Chelsea Publishing Co., New York, 1965.

[23] T. H. Koornwinder, *Special functions and q-commuting variables*, in Special functions, q-series and related topics (Toronto, ON, 1995), 131–166, Fields Inst. Commun., 14, Amer. Math. Soc., Providence, RI, 1997.
[24] K. H. Kwon, D. W. Lee, S. B. Park, and B. H. Yoo, *Hahn class orthogonal polynomials*, Kyungpook Math. J. **38** (1998), no. 2, 259–281.
[25] P. A. Lesky, *Charakterisierung der q-Orthogonalpolynome in x*, Monatsh. Math. **144** (2005), no. 4, 297–316.
[26] J. Petronilho, *Generic formulas for the values at the singular points of some special monic classical $H_{q,\omega}$-orthogonal polynomials*, J. Comput. Appl. Math. **205** (2007), no. 1, 314–324.
[27] E. C. Titchmarsh, *The Theory of Functions*, second edition, Oxford University Press, Oxford, 1939.
[28] K. Yosida, *Functional Analysis*, sixth edition, Grundlehren der Mathematischen Wissenschaften, **123**, Springer-Verlag, Berlin, 1980.

Marwa Abdelkhaliq
Basic Science Department
Pyramids Higher Institute for Engineering and Technology
6-October, Egypt
Email address: marwaabdelkhaliq@yahoo.com

Alaa Hamza
Department of Mathematics
Faculty of Science
University of Jeddah
Jeddah 21589, Saudi Arabia
and
Department of Mathematics
Faculty of Science
Cairo University
Giza, Egypt
Email address: hamzaaeg2003@yahoo.com