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WEIGHTED COMPOSITION OPERATORS ON NACHBIN SPACES WITH OPERATOR-VALUED WEIGHTS

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Abstract. Let A be a normed space, $\mathcal{B}(A)$ the algebra of all bounded operators on A, and V a family of strongly upper semicontinuous functions from a Hausdorff completely regular space X into $\mathcal{B}(A)$. In this paper, we investigate some properties of the weighted spaces $CV(X,A)$ of all A-valued continuous functions f on X such that the mapping $x \mapsto v(x)(f(x))$ is bounded on X, for every $v \in V$, endowed with the topology generated by the seminorms $\|f\|_v = \sup\{\|v(x)(f(x))\|,\ x \in X\}$. Our main purpose is to characterize continuous, bounded, and locally equicontinuous weighted composition operators between such spaces.

1. Introduction

The study of the weighted spaces $CV(X)$ of scalar-valued continuous functions on X was initiated by L. Nachbin [20] in 1965. Since then a variety of problems related to different aspects of the general theory of Banach spaces, Banach algebras, locally convex spaces, and locally convex algebras have been investigated in $CV(X)$ by several researchers, see [3–5, 7, 10, 13, 21–23, 28–31]. Some authors have also investigated different questions in some subspaces of the weighted space $CV(X)$, such as $CV_0(X) := \{f \in CV(X) : vf \text{ vanishes at infinity for every } v \in V\}$ and, whenever X is an open subset of $\mathbb{C}^N$ for some positive integer N, the spaces $hV(X) := \{f \in CV(X) : f \text{ is harmonic on } X\}$, $HV(X) := \{f \in CV(X) : f \text{ is holomorphic on } X\}$, and their corresponding subspaces $hV_0(X) := hV(X) \cap CV_0(X)$ and $HV_0(X) := HV(X) \cap CV_0(X)$, see [3, 5, 7–9, 16, 17, 21, 22, 29].

The weighted spaces of vector-valued continuous functions and those of vector-valued holomorphic functions have also been the subject matter of a huge literature, see for instance [1,2,6,7,10,23–26,28] and the references therein.

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Several issues were considered in such spaces such as those related to approximation [19, 20, 25, 30, 31], tensor products [1, 2], inductive limits and their projective descriptions [3–7] and so on.

Different types of operators between weighted spaces, especially the multiplication and the composition operators, have been investigated by many authors, see [8, 11, 12, 23, 29] and the references therein. For weighted spaces $CV(X, A)$ with A non-locally convex, the weighted composition operators were studied mainly in [13, 15, 18, 23, 28].

For all the aforementioned authors, the Nachbin family V consists of non-negative upper semicontinuous real-valued functions. Recently, C. Shekhar and B. S. Komal introduced in [27] systems of weights with values in the set of positive operators on a Hilbert space H and investigated the generalized weighted spaces $CV(X, H)$, consisting of all H-valued continuous functions f defined on X, such that the mapping $x \mapsto v(x)(f(x))$ is bounded on X for every $v \in V$. Such spaces constitute a nice generalization of the classical weighted spaces of Nachbin. The present authors gave in [14] necessary and sufficient conditions for a multiplication operators on such weighted spaces to be continuous, bounded below, invertible or to have a dense range.

In this note, we consider Nachbin families on X, consisting of weights with values in the algebra $\mathcal{B}(A)$ of all continuous linear operators on an arbitrary normed vector space $(A, \| \|)$. This yields an interesting general framework for the study of the weighted spaces. We specially give conditions under which such spaces are complete. However, our main purpose in this note is to investigate the weighted composition operators between a subspace E of a weighted space $CV(X, A)$ into a weighted space $CU(Y, A)$ or $CU_0(Y, A)$, where Y is a Hausdorff completely regular space and V and U are (generalized) Nachbin families on X and Y respectively. Such operators, denoted by ψC_φ, are associated with a mapping $\varphi : Y \to X$ and another one $\psi : Y \to \mathcal{B}(A)$ in the following way: $\psi C_\varphi(f) : y \mapsto \psi_y(f(\varphi(y)))$, $y \in Y$ and $f \in E$. Note that the vast majority of the authors assume that $CV_0(X, A)$ is essential in the sense of [24]. This means that, for every $x \in X$, there is some $f \in CV_0(X, A)$ such that $f(x) \neq 0$. Here, the subspaces E of $CV(X, A)$ we are considering are not assumed to be essential. Therefore our results apply to a wide class of subspaces of $CV(X, A)$.

After the foregoing section, Section 2 presents basic definitions and notations to be used throughout the paper. Section 3 is devoted to the completeness of the spaces $CV(X, A)$ and $CV_0(X, A)$. Section 4 deals with the continuity of the weighted composition operators defined on a subspace E of $CV(X, A)$ with values in $CU(Y, A)$ or $CU_0(Y, A)$. The last section focuses on the conditions under which ψC_φ is bounded or locally equicontinuous.

2. Preliminaries

Throughout this paper we shall assume, unless stated otherwise, that X and Y are Hausdorff completely regular spaces and that $(A, \| \|)$ is a normed vector space over the field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). The algebra of all bounded operators from A into itself will be denoted by $\mathcal{B}(A)$. An operator $T \in \mathcal{B}(A)$ is called bounded below if there exists $M > 0$ such that, for all $a \in A$, $\|a\| \leq M\|T(a)\|$. We will denote by $\mathcal{B}_b(A)$ the set of all bounded below operators $T \in \mathcal{B}(A)$, while $\mathcal{B}_s(A)$ (resp. $\mathcal{B}_u(A)$) will stand for the topological linear space obtained by equipping $\mathcal{B}(A)$ with the strong (resp. the uniform) operator topology denoted by β (resp. σ).

For every $a \in A$, δ_a will denote the normed evaluation at a. This is $\delta_a(T) := \|T(a)\|$ for every $T \in \mathcal{B}(A)$. A $\mathcal{B}(A)$-valued mapping v on X is said to be strongly upper semicontinuous if, for every $a \in A$, the real-valued map $\delta_a \circ v$ is upper semicontinuous (u.s.c. in short) on X, i.e., the set $\{x \in X : \|v(x)a\| < \alpha\}$ is open for every $\alpha \in \mathbb{R}$.

A mapping $\nu : X \to A$ is said to vanish at infinity on X if, for every $\varepsilon > 0$, there exists a compact subset K_ε of X such that $\|\nu(x)\| < \varepsilon$ for all $x \notin K_\varepsilon$. If the mapping $x \mapsto \|\nu(x)\|$ is upper semicontinuous, then ν vanishes at infinity if and only if $\{x \in X : \|\nu(x)\| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$.

We will let $C(X,A)$ (resp. $C_b(X,A)$, $C_0(X,A)$, $\mathcal{K}(X,A)$) denote the linear space of all continuous (resp. continuous and bounded, continuous and vanishing at infinity, continuous with compact support) A-valued functions on X, while $\mathcal{F}(X,A)$ will be that of all A-valued functions on X. Whenever $A = \mathbb{K}$, we will write $C(X)$ (resp. $C_b(X)$, $C_0(X)$, $\mathcal{K}(X)$, $\mathcal{F}(X)$) instead of $C(X,A)$ (resp. $C_b(X,A)$, $C_0(X,A)$, $\mathcal{K}(X,A)$, $\mathcal{F}(X,A)$).

In [27] and subsequently in [14], it is introduced the notion of generalized Nachbin families in the framework of Hilbert spaces. Such families consist of positive operator-valued functions with some additional conditions. Here, we extend the definition of generalized Nachbin families to the framework of arbitrary normed vector spaces as follows:

Definition 1. An A-generalized Nachbin family on X is a collection V of $\mathcal{B}(A)$-valued functions on X such that:

i) Every $v \in V$ is strongly upper semicontinuous,
ii) $\forall x \in X$, $\bigcap\{\ker v(x), v \in V\} = \{0\}$,
iii) V is directed upward in the following sense: for all $v_1, v_2 \in V$ and all $\lambda > 0$, there exists $v \in V$ such that $\lambda\|v_i(x)a\| \leq \|v(x)a\|$ for all $x \in X$, all $a \in A$, and $i = 1, 2$.

Without loss of generality, we may assume that for every $v \in V$ and $\lambda > 0$, we also have $\lambda v \in V$. For every $v \in V$ and $f \in C(X,A)$, we will write vf to designate the mapping $x \mapsto v(x)(f(x))$. Therefore $v(x)f(x)$ will mean $v(x)(f(x))$. With an A-generalized Nachbin family V on X is associated the so-called generalized weighted spaces:

$$CV(X,A) := \{f \in C(X,A) : (vf)(X) \text{ is bounded in } A,\ \forall v \in V\}$$

and

$$CV_0(X,A) := \{f \in CV(X,A) : vf \text{ vanishes at infinity on } X,\ \forall v \in V\}.$$

Unlike the scalar-valued weights case, it is not clear that the mappings $x \mapsto \|v(x)f(x)\|$ are bounded on compact subsets of X so that $CV_0(X,A)$ is automatically a subspace of $CV(X,A)$. Here, we include this condition in the definition of $CV_0(X,A)$. The two definitions coincide whenever, for example, the mapping $x \mapsto \|v(x)g(x)\|$ happens to be u.s.c. on X for every $v \in V$ and every $g \in C(X,A)$; in particular, whenever each $v \in V$ is σ-continuous on X.

Both spaces $CV(X,A)$ and $CV_0(X,A)$, as well as all their subspaces, will be endowed with the locally convex topology τ_V defined by the seminorms:

$$\|f\|_v = \sup\{\|v(x)f(x)\|, x \in X\},\ v \in V.$$

This topology is Hausdorff by Definition 1(ii).

In all the following, we will drop the letter A from A-generalized Nachbin family.

Now, we provide some examples of generalized Nachbin families.

Example 2.1. Let U be a usual Nachbin family (i.e., consisting of real-valued u.s.c. non-negative functions) on X. Then, identifying $v(x)$ with the operator $a \mapsto v(x)a$, U is a generalized Nachbin family and the space $CU(X,A)$ and $CU_0(X,A)$ are exactly the classical weighted spaces.

Example 2.2. Let U be a usual Nachbin family, $T \in \mathcal{B}(A)$ a non-zero continuous operator on A, and $V := \{uT : u \in U\}$, with $uT(x) := u(x)T$. If T is injective, then V is a generalized Nachbin family on X. Moreover, $CU(X,A) \subset CV(X,A)$ and $CU_0(X,A) \subset CV_0(X,A)$ hold with continuous inclusions. In particular, if $T = I$, we are in the situation of Example 2.1.

Example 2.3. Let U be a usual Nachbin family on X and let $R : X \to \mathcal{B}(A)$ be a continuous map, $\mathcal{B}(A)$ being endowed with the topology β. If $R(x)$ is injective for every $x \in X$, then $V := \{u(\cdot)R(\cdot) : u \in U\}$ is a generalized Nachbin family on X. Furthermore, setting $N_u := \{x \in X : u(x) > 0\}$, if $R(N_u)$ is a σ-bounded subset of $\mathcal{B}(A)$ for every $u \in U$, then $CU(X,A)$ and $CU_0(X,A)$ are subsets of $CV(X,A)$ and $CV_0(X,A)$ respectively and the inclusions are continuous.

Example 2.4. To every compact subset K of X, assign a non-zero operator $T_K \in \mathcal{B}(A)$ so that, whenever K_1 and K_2 are two compact subsets of X satisfying $K_1 \subset K_2$, then $\|T_{K_1}(a)\| \leq \|T_{K_2}(a)\|$ for all $a \in A$. Now, for every such a compact K, set $v_K := 1_K T_K$, where 1_K denotes the characteristic functional of K. Since K is compact, the mapping v_K is strongly upper semicontinuous. If, in addition, we assume that for every $x \in X$, the set $\bigcap\{\ker T_K, K \subset X \text{ compact and } x \in K\}$ is reduced to $\{0\}$, then $\mathcal{K} := \{\lambda v_K; K \subset X \text{ compact and } \lambda > 0\}$ is a generalized Nachbin family on X such that $C\mathcal{K}(X,A) = C\mathcal{K}_0(X,A) = C(X,A)$ algebraically. Furthermore, whenever $C(X,A)$ is endowed with the compact open topology, the inclusion $C(X,A) \subset C\mathcal{K}(X,A)$ is continuous. The equality is a topological one whenever T_K is bounded below for every compact $K \subset X$.

Example 2.5. Let $\mathcal{T} \subset \mathcal{B}(A)$ be a separating family such that

$$\forall T_1, T_2 \in \mathcal{T},\ \exists T \in \mathcal{T}:\ \|T_i(a)\| \leq \|T(a)\|,\ \forall a \in A, i = 1, 2.$$

If v_T is the constant mapping defined on X by $v_T(x) = T$, then $\mathcal{Z} := \{\lambda v_T : T \in \mathcal{T}, \lambda > 0\}$ is a generalized Nachbin family on X such that $C_b(X, A) \subset C\mathcal{Z}(X, A)$ and $C_0(X, A) \subset C\mathcal{Z}_0(X, A)$ with a continuous injection when $C_b(X, A)$ and $C_0(X, A)$ are equipped with the uniform norm topology.

Henceforth, U will stand for a generalized Nachbin family on Y, while $\mathcal{V}_z$ will designate the filter of neighborhoods of an element z of a topological space Z.

With an arbitrary map $\varphi : Y \to X$ (resp. $\psi : X \to \mathcal{B}(A)$) is associated the composition (resp. the multiplication) operator $C_\varphi : f \mapsto f \circ \varphi$ (resp. $M_\psi : f \mapsto \psi f$) defined from $CV(X, A)$ into $\mathcal{F}(Y, A)$ (resp. into $\mathcal{F}(X, A)$) by $C_\varphi(f)(y) = f(\varphi(y))$, $y \in Y$, and $M_\psi(f)(x) := \psi(x)(f(x))$, $x \in X$.

In this note, we are interested in the linear mapping ψC_φ defined from $CV(X, A)$ into $\mathcal{F}(Y, A)$ by $\psi C_\varphi(f)(y) = \psi_y(f(\varphi(y)))$. This mapping is called the weighted composition operator associated with ψ and φ. Notice that, whenever ψ is constant with value the identity of A, ψC_φ is nothing but the composition operator C_φ, and, whenever $X = Y$ and φ is the identity of X, ψC_φ is just the multiplication operator $M_\psi : f \mapsto \psi f$.

In all the sequel, E will be a linear subspace of $CV(X, A)$ and $\text{coz}(E)$ its cozero set. This is:

$$\text{coz}(E) := \{x \in X : f(x) \neq 0 \text{ for some } f \in E\}.$$

We will also consider the sets:

$$Y_{E,\varphi} := \{y \in Y : \varphi(y) \in \text{coz}(E)\} = \text{coz}(C_\varphi(E)),$$

$$Y_{E,\varphi,\psi} := \text{coz}(\psi C_\varphi(E)).$$

The set $Y_{E,\varphi}$ (resp. $Y_{E,\varphi,\psi}$) is an open subset of Y, whenever $C_\varphi(E) \subset C(Y, A)$ (resp. $\psi C_\varphi(E) \subset C(Y, A)$) [23].

If $f \in C(X)$ and $a \in A$ are given, we will denote by $f \otimes a$ the function defined on X by $f \otimes a(x) := f(x)a$, $x \in X$.

In [23] the property "$\forall a \in A$, $\forall f \in E$, $\|f\| \otimes a \in E$", called (M), came in force in the results there. Here we will consider the following weaker properties the space E may or may not satisfy:

(P) $$\forall a \in A,\ \forall x \in \text{coz}(E),\ \exists f \in E:\ f(x) = a,$$

(P') $$\forall a \in A,\ \forall x \in \text{coz}(E),\ \exists g \in C(X):\ g(x) \neq 0 \text{ and } g \otimes a \in E,$$

(S) $\forall v \in V$, $\forall f \in E$, the mapping $x \mapsto \|v(x)f(x)\|$ is upper semicontinuous.

It is easily seen that, (M) implies (P') and that (P') implies (P). Moreover, if X is locally compact such that $\mathcal{K}(X) \otimes A \subset E$ holds, then E satisfies (P'). This is in particular the case if $E = CV(X, A)$ or $E = CV_0(X, A)$ provided the latter satisfies (S). Notice at this point that, whenever $V \subset C(X, \mathcal{B}_\sigma(A))$, every subset E of $CV(X, A)$ satisfies (S).

The condition (P') is satisfied in many situations. This is the case for example whenever $E = CU(X, A)$ in Example 2.1 or in Example 2.2. This is also the case if $E = \mathcal{K}(X, A) \cap CU(X, A)$ in Example 2.3 and if $E = C(X, A)$ in Example 2.4. Anyway, whenever every $x \in \mathrm{coz}(CV(X, A))$ possesses a neighborhood Ω_x such that $v(\Omega_x)$ is β-bounded for every $v \in V$, then $CV(X, A)$ satisfies (P'). In particular, this is true if $v(X)$ is β-bounded for every $v \in V$.

Whenever E satisfies (P), the equality $Y_{E,\varphi,\psi} = Y_{E,\varphi} \cap \mathrm{coz}(\psi)$ holds. Moreover, if E is a $C_b(X)$-module and satisfies (P'), then for all $x \in \mathrm{coz}(E)$ and all $a \in A$, one can find $f \in C(X)$ such that $0 \leq f \leq 1$, $f(x) = 1$, and $f \otimes a \in E$.

3. Completeness of $CV(X, A)$ and $CV_0(X, A)$

In this section we will investigate the completeness of $CV(X, A)$ and $CV_0(X, A)$ for every generalized Nachbin family V on X. To this purpose, let us consider, for every $v \in V$ and $r > 0$, the level set

$$N(v, r) := \{x \in X : \|v(x)a\| \geq r\|a\|, \ \forall a \in A\}.$$

As in the scalar-valued weights case, $CV_0(X, A)$ is closed in $CV(X, A)$ as shows the following proposition. Before showing it, let us denote, for simplicity, $X_0 := \mathrm{coz}(CV_0(X, A))$ and $X_1 := \mathrm{coz}(CV(X, A))$.

Proposition 3.1. *For every generalized Nachbin family V on X, $CV_0(X, A)$ is a closed subspace of $CV(X, A)$.*

Proof. Let $f \in CV(X, A)$ be in the closure $\overline{CV_0(X, A)}^{CV(X,A)}$ of $CV_0(X, A)$. Then for all $v \in V$ and $\varepsilon > 0$, there exists $g \in CV_0(X, A)$ such that $\|f - g\|_v < \frac{\varepsilon}{2}$. Since g belongs to $CV_0(X, A)$, there exists a compact subset K of X such that $\|v(x)g(x)\| < \frac{\varepsilon}{2}$ for all $x \notin K$. Therefore, for such an x, we have:

$$\|v(x)f(x)\| \leq \|v(x)(f(x) - g(x))\| + \|v(x)g(x)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This yields $f \in CV_0(X, A)$ since v is arbitrary. □

In the scalar-valued weights case, K. D. Bierstedt introduced in [1] the notion of $V_{\mathbb{R}}$-spaces as being those completely regular Hausdorff spaces X such that every real function on X whose restriction to each level set $N(v, r) := \{x \in X : v(x) \geq r\}$ is continuous, must be continuous on X, v running over V and $r > 0$. In order to extend this definition to the operator-valued weights case, we first give the following lemma which may be known. For the convenience of the reader, we include a proof of it.

Lemma 3.2. *Let Z be a Hausdorff completely regular space, B a non-trivial Hausdorff topological vector space over the field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), and $\mathcal{F}$ a collection of subsets of Z. Then the following statements are equivalents*:

i) *For every Hausdorff completely regular space Y, a function $f : Z \to Y$ is continuous provided its restriction $f_{|F}$ to every $F \in \mathcal{F}$ is.*
ii) *A function $f : Z \to \mathbb{R}$ is continuous provided its restriction to every $F \in \mathcal{F}$ is.*
iii) *A function $f : Z \to B$ is continuous provided its restriction to every $F \in \mathcal{F}$ is.*

Proof. The implication i)$\Rightarrow$ ii) is obvious because $\mathbb{R}$ is a Hausdorff completely regular space. For ii)$\Rightarrow$ iii), let $f : Z \to B$ be such that $f_{|F}$ is continuous for every $F \in \mathcal{F}$. For a continuous function $g : B \to \mathbb{R}$, $(g \circ f)_{|F}$ is continuous for every $F \in \mathcal{F}$. Therefore $g \circ f$ is continuous on X by ii). Since B is a Hausdorff completely regular space, its topology is the initial one defined by $C(B, \mathbb{R})$. Therefore f is continuous on Z. Finally, for the implication iii)$\Rightarrow$ i), assume that $f : Z \to Y$ is a mapping such that $f_{|F}$ is continuous for every $F \in \mathcal{F}$. For an arbitrary $a \in B \setminus \{0\}$, let i_a be the homeomorphism from $\mathbb{R}$ into B defined by $i_a(\lambda) = \lambda a$. For an arbitrary continuous function $g : Y \to \mathbb{R}$, the function $i_a \circ g \circ f : Z \to B$ is continuous on each $F \in \mathcal{F}$. Then it is continuous on Z by iii). As g is arbitrary in $C(Y)$ and $i_a(\mathbb{R})$ is homeomorphic to $\mathbb{R}$, f is continuous on Z. □

Definition 2. We will say that Z is an $\mathcal{F}_\mathbb{R}$-space if it satisfies one of the assertions of Lemma 3.2. In particular, X will be said to be a $V_\mathbb{R}$-space, if it is an $\mathcal{F}_\mathbb{R}$-space for

$$\mathcal{F} := \{N(v, r),\ v \in V,\ r > 0\}.$$

Since $N(v, r) = N(\frac{1}{r}v, 1)$ and $\frac{1}{r}v \in V$, X is a $V_\mathbb{R}$-space, if and only if, it is an $\mathcal{N}_\mathbb{R}$-space, where $\mathcal{N} := \{N(v, 1),\ v \in V\}$.

In [14], the authors defined a V_H-space X as a Hausdorff completely regular space such that every H-valued function defined on X is continuous provided its restriction to the level set $N(v, r)$ is continuous, for every $v \in V$ and $r > 0$. According to Lemma 3.2, the V_H-spaces of [14] are nothing but the $V_\mathbb{R}$-spaces.

Theorem 3.3. *Let A be a Banach space and X be a $V_\mathbb{R}$-space. If, for every $x \in X_1$ (resp. $x \in X_0$), there exists some $v \in V$ such that $v(x)$ is bounded below, then $CV(X, A)$ (resp. $CV_0(X, A)$) is complete.*

Proof. Let $(f_i)_{i\in I}$ be a Cauchy net in $CV(X, A)$ (resp. $CV_0(X, A)$). By our assumption, for every $x \in X_1$ (resp. $x \in X_0$), the evaluation map $\delta_x : f \mapsto f(x)$ is continuous from $CV(X, A)$ (resp. $CV_0(X, A)$) into A. Therefore $(f_i(x))_{i\in I}$ is a Cauchy net in A. Since A is complete, $(f_i(x))_{i\in I}$ converges to some $f(x) \in A$. Extend the so-defined function f over X by putting $f = 0$ identically on $X \setminus X_1$ (resp. $X \setminus X_0$). We claim that f belongs to $CV(X, A)$ (resp. $CV_0(X, A)$) and that $(f_i)_{i\in I}$ converges to f in $CV(X, A)$ (resp. $CV_0(X, A)$). Since X is a $V_\mathbb{R}$-space, in order to show that f is continuous on X, it suffices to show that its restriction to each $N(v, 1)$ is. Let then $v \in V$ and $x \in N(v, 1)$ be arbitrary. We have:

$$\|f_i(t) - f_j(t)\| \le \|v(t)(f_i(t) - f_j(t)\|,\ t \in N(v, 1).$$

Whereby

$$\|f_i(t) - f_j(t)\| \leq \|f_i - f_j\|_v,\ t \in N(v,1).$$

Since $(f_i)_{i\in I}$ is Cauchy, for $\varepsilon > 0$, there exists $i_0 \in I$ such that $\|f_i - f_j\|_v \leq \frac{\varepsilon}{3}$ whenever $i_0 \leq i$ and $i_0 \leq j$. Hence $\|f_{i_0}(t) - f_j(t)\| \leq \frac{\varepsilon}{3}$ for all $t \in N(v,1)$ and all j with $i_0 \leq j$. Passing to the limit on j, we get

$$\|f_{i_0}(t) - f(t)\| \leq \frac{\varepsilon}{3},\ \forall t \in N(v,1). \tag{1}$$

By the continuity of f_{i_0}, there exists $\Omega \in \mathcal{V}_x$ such that

$$\|f_{i_0}(t) - f_{i_0}(x)\| \leq \frac{\varepsilon}{3},\ \forall t \in \Omega. \tag{2}$$

For $t \in \Omega \cap N(v,1)$, by (1) and (2), we have

$$\begin{aligned}\|f(t) - f(x)\| &\leq \|f(t) - f_{i_0}(t)\| + \|f_{i_0}(t) - f_{i_0}(x)\| + \|f_{i_0}(x) - f(x)\| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Then $\|f(t) - f(x)\| \leq \varepsilon$ for every $t \in \Omega \cap N(v,1)$. It follows that f, restricted to $N(v,1)$, is continuous. Since $CV_0(X,A)$ is closed in $CV(X,A)$, it is enough to show that $(f_i)_{i\in I}$ converges in $CV(X,A)$ to f.

Let then $u \in V$ and $\varepsilon > 0$ be arbitrary. Since $(f_i)_{i\in I}$ is Cauchy, there exists $i_0 \in I$ such that $\|f_i - f_j\|_u < \varepsilon$ whenever $i_0 \leq i$ and $i_0 \leq j$, i.e., $\|u(t)(f_i(t) - f_j(t))\| < \varepsilon$. Since $u(t)$ is continuous, passing to the limit on j, we get $\|u(t)(f_i(t) - f(t))\| \leq \varepsilon$, $\forall t \in X$ and $i_0 \leq i$. Whereby

$$\|f_i - f\|_u \leq \varepsilon, i_0 \leq i.$$

Now, since $f = (f - f_i) + f_i$, f belongs to $CV(X,A)$. ☐

Throughout all the remainder, unless stated otherwise, we will assume that E is a $C_b(X)$-module and satisfies the reasonable conditions (P) and (S). Our purpose here is to study the relationship between the weights and some topological properties of the operator ψC_φ. We then assume that ψC_φ maps E into $C(X,A)$.

4. Continuous weighted composition operators

The following theorem characterizes the continuous operators ψC_φ from a subspace E of $CV(X,A)$, satisfying (P) and (S), into $CU(Y,A)$.

Theorem 4.1. *The operator ψC_φ maps continuously E into $CU(Y,A)$ if, and only if, the following condition holds*:

$$\forall\ u \in U,\ \exists\ v \in V : \|u(y)\psi_y(a)\| \leq \|v(\varphi(y))a\|,\ \forall a \in A, y \in Y_{E,\varphi}. \tag{3}$$

Proof. Necessity: Since $\psi C_\varphi : E \to CU(Y,A)$ is continuous, for every $u \in U$, there exists $v \in V$ such that:

$$\|\psi C_\varphi(f)\|_u \leq \|f\|_v,\ \forall f \in E.$$

Then for every $y \in Y$, one has:

$$\|u(y)\psi_y(f(\varphi(y))\| \leq \sup\{\|v(x)f(x)\|,\ x \in X\}. \tag{4}$$

Let y_0 in $Y_{E,\varphi}$ and a in A be given. Then $x_0 := \varphi(y_0)$ belongs to $\text{coz}(E)$. Therefore, since E satisfies the property (P), for every $a \in A$, there is $f \in E$ such that $f(x_0) = a$. For arbitrary integer $n > 0$, set

$$U_n := \{x \in X : \|v(x)f(x)\| < \|v(x_0)a\| + \frac{1}{n}\}.$$

Due to (S), U_n is an open neighborhood of x_0. Consider $g_n \in C_b(X)$ whose support is contained in U_n such that $g_n(x_0) = 1$ and $0 \leq g_n \leq 1$. Then $h_n := g_n f$ belongs to E and by (4),

$$\begin{aligned}\|u(y_0)\psi_{y_0}(a)\| &\leq \sup\{\|v(x)h_n(x)\|,\ x \in X\}\\ &\leq \|v(x_0)a\| + \frac{1}{n}.\end{aligned}$$

Letting n tend to infinity, we get $\|u(y_0)\psi_{y_0}(a)\| \leq \|v(\varphi(y_0))a\|$ as desired.

Sufficiency: Let $f \in E$ and $u \in U$ be given. By (3) there exists $v \in V$ such that

$$\|u(y)\psi_y(f(\varphi(y)))\| \leq \|v(\varphi(y)f(\varphi(y))\|,\ \forall\ y \in Y.$$

Therefore,

$$\begin{aligned}\|\psi C_\varphi(f)\|_u &= \sup\{\|u(y)\psi_y(f(\varphi(y)))\| : y \in Y\}\\ &\leq \sup\{\|v(\varphi(y))f(\varphi(y))\| : y \in Y\}\\ &\leq \sup\{\|v(x)f(x)\| : x \in \varphi(Y)\}\\ &\leq \|f\|_v < \infty.\end{aligned}$$

This shows at once that $\psi C_\varphi(f) \in CU(Y, A)$ and that ψC_φ is continuous. □

In case of multiplication operators (i.e., $X = Y$ and φ is the identity of X), we get:

Corollary 4.2. *M_ψ maps continuously E into $CU(X, A)$ if, and only if, the following condition holds*:

$$\forall\ u \in U, \exists\ v \in V : \|u(x)\psi_x(a)\| \leq \|v(x)a\|,\ \forall a \in A, x \in coz(E). \tag{5}$$

Similarly, in case of composition operators (i.e., ψ is the constant mapping with value the identity of A), we get:

Corollary 4.3. *C_φ maps continuously E into $CU(Y, A)$ if, and only if, the following condition holds*:

$$\forall\ u \in U, \exists\ v \in V : \|u(y)a\| \leq \|v(\varphi(y))a\|,\ \forall a \in A, y \in Y_{E,\varphi}. \tag{6}$$

Next, we will investigate the continuity of ψC_φ from E into $CU_0(Y, A)$. This condition is of course much more constraining. To this aim, let us set as in [23]

$$\text{Cst}(E) := \{K \subset X : \forall a \in A,\ \exists f \in E \text{ with } f = a \text{ identically on } K\}.$$

It is easily seen that every $K \in \mathrm{Cst}(E)$ is contained in $\mathrm{coz}(E)$ and that every $v \in V$ is β-bounded on every such a K. Therefore, if A happens to be barrelled, a fortiori if it is Banach, then $\{v(x), x \in K\}$ will be also σ-bounded.

Now, for $v \in V$ and $f \in E$, put $N(v, f) := \{x \in X : \|v(x)f(x)\| \geq 1\}$ and say that E satisfies the property (C) whenever $N(v, f)$ belongs to $\mathrm{Cst}(E)$ for every $v \in V$ and every $f \in E$. The following lemma, extending Lemma 6 of [23], gives examples where E satisfies (C).

Lemma 4.4. *Assume that E satisfies* (P') (*e.g. X is locally compact and $\mathcal{K}(X) \otimes A \subset E$*). *If $K \subset coz(E)$ is a compact set and $C \subset X$ is a closed set such that $K \cap C = \emptyset$, then, for every $a \in A$, there exists $f \in E$ such that $f = a$ on K and $f = 0$ on C.*

Proof. For any $f \in C(X)$, let us denote by $\Gamma(f)$ the mapping assigning to $x \in X$, $|f(x)|$ if $|f(x)| \leq 1$ and $\frac{1}{|f(x)|}$ otherwise. This is a continuous bounded function on X. Now, for every $a \in A$ and $x \in K$, due to (P'), there exists $g \in C(X)$ such that $g(x) = 1$ and $g \otimes a \in E$. If $\gamma := |g|^2\Gamma(g^2)$, then $\gamma \otimes a = (\overline{g}\Gamma(g^2))g \otimes a$ belongs to E. Choose then $g_x \in C_b(X)$ with $g_x(x) = 1$, $0 \leq g_x \leq 1$ and $g_x = 0$ identically on C, and set $h_x = g_x\gamma$. By a compactness argument, there exist $x_1, x_2, \ldots, x_m$ in X such that $K \subset \cup_{i=1}^m \{t \in X : h_{x_i}(t) > \frac{1}{2}\}$. It follows that the function $h := \sum_{i=1}^m h_{x_i}$ satisfies $h(t) > 1/2$ for every $t \in K$. Hence, the function $k = (2\Gamma(2h))h \otimes a$ belongs to E and enjoys the required condition. □

Theorem 4.5. *Assume that E satisfies* (C). *If ψC_φ maps continuously E into $CU_0(Y, A)$, then* (3) *holds and $\varphi^{-1}(K) \cap \{y \in Y : \|u(y)\psi_y(a)\| \geq 1\}$ is relatively compact, for all $K \in Cst(E)$, $u \in U$, and $a \in A$. The converse holds whenever, for all $v \in V$ and $f \in E$, $f(N(v, f))$ is precompact and $v(N(v, f))$ is equicontinuous on A.*

Proof. Assume that ψC_φ maps continuously E into $CU_0(Y, A)$. Then (3) follows from Theorem 4.1. Now, fix $K \in \mathrm{Cst}(E)$, $u \in U$, and $a \in A$, with $a \neq 0$. Choose $f \in E$ such that $f = a$ identically on K. As $\psi C_\varphi(f)$ belongs to $CU_0(Y, A)$, the set

$$S := \{y \in Y : \|u(y)\psi_y(f(\varphi(y)))\| \geq 1\}$$

is relatively compact and contains $\varphi^{-1}(K) \cap \{y \in Y : \|u(y)\psi_y(a)\| \geq 1\}$. Hence the latter is relatively compact.

For the converse, by Theorem 4.1, the condition (3) implies that ψC_φ maps continuously E into $CU(Y, A)$. It remains to be shown the inclusion $\psi C_\varphi(E) \subset CU_0(Y, A)$. For this, it is enough to show that, for all $f \in E$ and all $u \in U$, the set S defined above is relatively compact. Let $f \in E$ and $u \in U$ be given. By (3), there is some $v \in V$ such that:

$$\|u(y)\psi_y(a)\| \leq \|v(\varphi(y))a\|, \ \forall a \in A, y \in Y_{E,\varphi}. \tag{7}$$

This yields $\varphi(S) \subset N(v,f)$, or $S \subset \varphi^{-1}(N(v,f))$. Since $K := N(v,f)$ belongs to $\mathrm{Cst}(E)$, it suffices to show that S is contained in some union of finitely many sets of the form $C_i := \{y \in Y : \|2u(y)\psi_y(a_i)\| \geq 1\}$ for some $a_i \in A \setminus \{0\}$. But, since the set $v(K)$ is equicontinuous, one has:

$$\exists M > 0 : \|v(x)a\| \leq M\|a\|,\ a \in A,\ x \in K. \tag{8}$$

Moreover, since $f(K)$ is precompact, $f(\varphi(S))$ is also precompact in A. Therefore there are $y_1, \dots, y_n \in S$ such that:

$$f(\varphi(S)) \subset \bigcup \{f(\varphi(y_i)) + \frac{1}{2M} B_A,\ i = 1, \dots, n\},$$

where B_A denotes the unit ball of A. Thus, for $y \in S$, there is some $i \in \{1, \dots, n\}$ such that $\|f(\varphi(y)) - f(\varphi(y_i))\| < \frac{1}{2M}$. By (8), we get:

$$\|v(\varphi(y))(f(\varphi(y)) - f(\varphi(y_i)))\| < \frac{1}{2}.$$

Using (7), we obtain $\|u(y)\psi_y(f(\varphi(y)) - f(\varphi(y_i)))\| < \frac{1}{2}$. Therefore

$$\begin{aligned} 1 &\leq \|u(y)\psi_y(f(\varphi(y)))\| \\ &\leq \|u(y)\psi_y(f(\varphi(y)) - f(\varphi(y_i)))\| + \|u(y)\psi_y(f(\varphi(y_i)))\| \\ &\leq \frac{1}{2} + \|u(y)\psi_y(f(\varphi(y_i)))\|. \end{aligned}$$

Then $\|2u(y)\psi_y(f(\varphi(y_i)))\| \geq 1$. Consequently

$$S \subset \bigcup_{i=1}^{n} \{y \in Y : \|2u(y)\psi_y(a_i)\| \geq 1\},$$

with $a_i = f(\varphi(y_i))$. Since $\varphi^{-1}(K) \cap \{y \in Y : \|2u(y)\psi_y(a_i)\| \geq 1\}$ is relatively compact for each i, so is S. □

In case of multiplication operators, we get:

Corollary 4.6. *If E satisfies* (C), *$f(N(v,f))$ is precompact, and $v(N(v,f))$ is equicontinuous on A, for all $v \in V$ and all $f \in E$, then M_ψ maps continuously E into $CU_0(X,A)$ if, and only if* (5) *holds and $K \cap \{y \in Y : \|u(y)\psi_y(a)\| \geq 1\}$ is relatively compact, for all $K \in Cst(E)$, $u \in U$, and $a \in A$.*

Similarly, in case of composition operators, we get:

Corollary 4.7. *Under the same conditions as in Corollary 4.6, C_φ maps continuously E into $CU_0(X,A)$ if, and only if* (6) *holds and $\varphi^{-1}(K) \cap \{x \in X : \|u(x)a\| \geq 1\}$ is relatively compact, for all $K \in Cst(E)$, $u \in U$, and $a \in A$.*

If E enjoys (P') and the weights $v \in V$ are all continuous, the converse of Theorem 4.5 holds with compact subset K of $\mathrm{coz}(E)$ instead of all the members of $\mathrm{Cst}(E)$. At this point, let us mention that, if E contains the constant functions, then X itself is in $\mathrm{Cst}(E)$ but need not be compact.

Theorem 4.8. *Assume that $V \subset C(X, \mathcal{B}_\sigma(A))$ and that $E \subset CV_0(X, A)$ satisfies* (P'). *Then ψC_φ maps continuously E into $CU_0(Y, A)$ if, and only if,* (3) *holds and $\varphi^{-1}(K) \cap \{y \in Y : \|u(y)\psi_y(a)\| \geq 1\}$ is relatively compact, for every compact $K \subset coz(E)$, $u \in U$, and $a \in A$.*

Notice that if E satisfies (C) and if A happens to be barrelled, then $v(N(v, f))$ is automatically equicontinuous, for it is β-bounded. Furthermore, if $E \subset CV_0(X, A)$, then $f(N(v, f))$ is also automatically precompact, since $N(v, f)$ is compact, due to (S), and f is continuous. We thus get as a corollary, the following result:

Theorem 4.9. *Assume that A is barrelled and that $E \subset CV_0(X, A)$ satisfies* (C). *Then ψC_φ maps continuously E into $CU_0(Y, A)$ if, and only if,* (3) *holds and $\varphi^{-1}(K) \cap \{y \in Y : \|u(y)\psi_y(a)\| \geq 1\}$ is relatively compact, for all $K \in Cst(E)$, $u \in U$, and $a \in A$.*

5. Bounded weighted composition operators

Recall that a linear map θ is said to be bounded if it maps some 0-neighborhood into a bounded set. Whenever θ has range in a space of continuous functions on some topological space Z, it is said to be locally Z_0-equicontinuous, if θ maps every bounded set into a set which is equicontinuous on $Z_0 \subset Z$.

We obtain the following characterization of bounded weighted composition operators:

Theorem 5.1. *The operator ψC_φ is bounded from E into $CU(Y, A)$ if, and only if, there exists some $v \in V$ such that*:

(9) $$\forall\ u \in U,\ \exists\ \lambda > 0 : \|u(y)\psi_y(a)\| \leq \lambda \|v(\varphi(y))a\|,\ \forall\ a \in A, y \in Y_{E,\varphi}.$$

Proof. Necessity: Since ψC_φ is bounded from E into $CU(Y, A)$, there exists $v \in V$ such that, for every $u \in U$, there exists some $\lambda > 0$ so that

$$\|\psi C_\varphi(f)\|_u \leq \lambda \|f\|_v,\ f \in E.$$

Then for every $y \in Y$, one has

(10) $$\|u(y)\psi_y(f(\varphi(y))\| \leq \lambda \sup\{\|v(x)f(x)\|, x \in X\}.$$

Let $y_0 \in Y_{E,\varphi}$ and $a \in A$ be given, and put $x_0 := \varphi(y_0)$. As in the proof of Theorem 4.1, consider a function $f \in E$ such that $f(x_0) = a$. For any integer $n > 0$, set

$$U_n := \{x \in X : \|v(x)f(x)\| < \|v(x_0)a\| + \frac{1}{n}\}.$$

Due to (S), U_n is an open neighborhood of x_0. Choose $g_n \in C_b(X)$ vanishing outside of U_n such that $g_n(x_0) = 1$ and $0 \leq g_n \leq 1$. Then $h_n := g_n f$ belongs to E and by (10),

$$\begin{aligned} \|u(y_0)\psi_{y_0}(a)\| &\leq \lambda \sup\{\|v(x)h_n(x)\|,\ x \in X\} \\ &\leq \lambda(\|v(x_0)a\| + \frac{1}{n}). \end{aligned}$$

Letting n tend to infinity, we get:

$$\|u(y_0)\psi_{y_0}(a)\| \leq \lambda\|v(\varphi(y_0))a\|.$$

Sufficiency: Assume that, there exists $v \in V$ so that (9) holds. Let $u \in U$ be given. Then, for $f \in E$ and $y \in Y$, we have

$$\|u(y)\psi_y(f(\varphi(y)))\| \leq \lambda\|v(\varphi(y)f(\varphi(y))\|,\ y \in Y.$$

In particular, for $f \in B_v$, we get

$$\|u(y)\psi_y(f(\varphi(y)))\| \leq \lambda,\ y \in Y,$$

giving $\|\psi C_\varphi(f)\|_u \leq \lambda,\ f \in B_v$. □

In case of multiplication operators, we obtain:

Corollary 5.2. *The multiplication operator M_ψ is bounded from E into $CU(X, A)$ if, and only if, there exists $v \in V$ such that*:

$$\forall\ u \in U;\ \exists\ \lambda > 0 : \|u(x)\psi_x(a)\| \leq \lambda\|v(x)a\|,\ \forall\ a \in A,\ x \in coz(E).$$

Similarly, in case of composition operators, we obtain:

Corollary 5.3. *The composition operator C_φ is bounded from E into $CU(Y, A)$ if, and only if, there exists $v \in V$ such that*:

$$\forall\ u \in U;\ \exists\ \lambda > 0 : \|u(y)a\| \leq \lambda\|v(\varphi(y))a\|,\ \forall\ a \in A,\ y \in Y_{E,\varphi}.$$

Combining conveniently Theorem 4.8 and Theorem 5.1, we obtain the following result.

Theorem 5.4. *Assume that $V \subset C(X, \mathcal{B}_\sigma(A))$ and that $E \subset CV_0(X, A)$ satisfies* (P'). *Then ψC_φ is bounded from E into $CU_0(Y, A)$ if, and only if,* (9) *holds and $\varphi^{-1}(K) \cap \{y \in Y : \|u(y)\psi_y(a)\| \geq 1\}$ is relatively compact, for every compact $K \subset coz(E)$, $u \in U$, and $a \in A$.*

We now examine the local equicontinuity of ψC_φ. Notice that, in scalar-valued weights case, every $x \in X$ admits a neighborhood Ω such that every $v \in V$ is bounded on Ω. We will say that X is locally V-σ-bounding if:

$$\forall\ x \in X, \exists\ \Omega_x \in \mathcal{V}_x :\ \{v(t),\ t \in \Omega_x\} \text{ is bounded in } \mathcal{L}_\sigma(A),\ \forall v \in V.$$

Theorem 5.5. *Assume that X is locally V-σ-bounding and that, for all $x \in X$, $V(x) \cap \mathcal{B}_b(A) \neq \emptyset$. Then ψC_φ is locally $Y_{E,\varphi,\psi}$-equicontinuous, if, and only if, the following conditions hold*:

1. *φ is locally constant on $Y_{E,\varphi,\psi}$.*
2. *ψ is continuous from $Y_{E,\varphi,\psi}$ into $\mathcal{L}_\sigma(A)$.*

Proof. Necessity: Assume that ψC_φ is locally $Y_{E,\varphi,\psi}$-equicontinuous and suppose that, for some $y_0 \in Y_{E,\varphi,\psi}$, φ is constant on no neighborhood of y_0. Choose $f_0 \in E$ with $\psi_{y_0}(f_0(\varphi(y_0))) \neq 0$. Then every $\Omega \in \mathcal{V}_{y_0}$ contains some y_Ω with $\varphi(y_0) \neq \varphi(y_\Omega)$. Consider $f_\Omega \in C_b(X)$ such that $0 \leq f_\Omega \leq 1$, $f_\Omega(\varphi(y_\Omega)) = 0$, and $f_\Omega(\varphi(y_0)) = 1$, and put $g_\Omega := f_\Omega f_0$. Then the set $C := \{g_\Omega,\ \Omega \in \mathcal{V}_{y_0}\}$ is bounded in E and then $\psi C_\varphi(C)$ is equicontinuous at y_0. Therefore, for every $\varepsilon > 0$, there exists $\Omega_0 \in \mathcal{V}_{y_0}$ such that:

$$\|\psi_y(g_\Omega(\varphi(y))) - \psi_{y_0}(g_\Omega(\varphi(y_0)))\| \leq \varepsilon,\ y \in \Omega_0,\ \Omega \in \mathcal{V}_{y_0}.$$

Hence, for every $\Omega \subset \Omega_0$ and $y = y_\Omega$, we get $\|\psi_{y_0}(f_0(\varphi(y_0)))\| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\psi_{y_0}(f_0(\varphi(y_0))) = 0$, which is a contradiction, whereby 1 is satisfied.

2. Let y_0 be arbitrary in $Y_{E,\varphi,\psi}$ and $\varepsilon > 0$ be given. By 1, there exists a neighborhood Ω_0 of y_0 on which φ is constant with some value x_0. Since X is locally V-σ-bounding, there exists $\Omega_{x_0} \in \mathcal{V}_{x_0}$ such that, for every $v \in V$, there exists some $M_v > 0$ with $\|v(t)\| \leq M_v$, $t \in \Omega_{x_0}$. For arbitrary $b \in A$ with $\|b\| = 1$, choose a function $f_b \in E$ such that $f_b(x_0) = b$. This is possible due to (P). Then the set $U := \{x \in X : \frac{1}{2} < \|f_b(x)\| < \frac{3}{2}\}$ is open and contains x_0. Choose $g_b \in C_b(X)$ such that $g_b(x_0) = 1$, $0 \leq g_b \leq 1$, and supp $g_b \subset U \cap \Omega_{x_0}$. We claim that the set $K := \{g_b f_b, \|b\| = 1\}$ is bounded in E. Indeed, for every $v \in V$, we have:

$$\begin{aligned}\|g_b f_b\|_v &= \sup\{g_b(x)\|v(x)f_b(x)\| : x \in X\}\\ &\leq \sup\{\|v(x)f_b(x)\| : x \in U \cap \Omega_{x_0}\}\\ &\leq \frac{3}{2}M_v.\end{aligned}$$

Therefore $\psi C_\varphi(K)$ is equicontinuous at y_0. Hence there is some y_0-neighborhood Ω contained in Ω_0 such that:

$$\|\psi_y\left(g_b(\varphi(y))f_b(\varphi(y))\right) - \psi_{y_0}\left(g_b(\varphi(y_0))f_b(\varphi(y_0))\right)\| \leq \varepsilon,\ y \in \Omega,\ \|b\| = 1.$$

Whence $\|\psi_y - \psi_{y_0}\| \leq \varepsilon$ for all $y \in \Omega$, showing that ψ is σ-continuous at y_0. Since y_0 is arbitrary in $Y_{E,\varphi,\psi}$, ψ is σ-continuous on $Y_{E,\varphi,\psi}$.

Sufficiency: Given a bounded set $\mathbb{B} \subset E$, $y_0 \in Y_{E,\varphi,\psi}$, and $\varepsilon > 0$. By assumption, there is some neighborhood Ω_0 of y_0 so that φ is constant on Ω_0 with some value x_0. Choose $v \in V$ with $v(x_0)$ bounded below. Then there exists $r > 0$ such that $r\|a\| \leq \|v(x_0)a)\|$, $a \in A$. Therefore the set $B := \{f(x_0), f \in \mathbb{B}\}$ is bounded in A. Then $\|f(x_0)\| \leq M$ for every $f \in \mathbb{B}$ and some $M > 0$. Since ψ is σ-continuous at y_0, there is some other neighborhood Ω of y_0 such that $\Omega \subset \Omega_0$ and $\|\psi_y - \psi_{y_0}\| < \varepsilon$, $y \in \Omega$. Hence

$$\|\psi_y(f(x_0)) - \psi_{y_0}(f(x_0))\| \leq M\varepsilon,\ y \in \Omega,\ f \in \mathbb{B}.$$

Therefore

$$\|\psi C_\varphi(f)(y) - \psi C_\varphi(f)(y_0)\| \leq M\varepsilon,\ y \in \Omega,\ f \in \mathbb{B},$$

whereby $\psi C_\varphi(\mathbb{B})$ is equicontinuous at y_0 and then on $Y_{E,\varphi,\psi}$ since y_0 was arbitrary. □

A trivial consequence of Theorem 5.5 is the following:

Corollary 5.6. *Assume that E satisfies the conditions of Theorem 5.5. If φ is not constant on any open set* (*in particular, if X has no isolated point and φ*

is one to one), then ψC_φ is locally $Y_{E,\varphi,\psi}$-equicontinuous from E into $C(Y, A)$ if, and only if, it is identically zero.

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