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NOTES ON SYMMETRIC SKEW n-DERIVATION IN RINGS

Emine Koç and Nadeem ur Rehman

Abstract. Let R be a prime ring (or semiprime ring) with center $Z(R)$, I a nonzero ideal of R, T an automorphism of R, $S : R^n \to R$ be a symmetric skew n-derivation associated with the automorphism T and Δ is the trace of S. In this paper, we shall prove that $S(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in R$ if any one of the following holds: i) $\Delta(x) = 0$, ii) $[\Delta(x), T(x)] = 0$ for all $x \in I$.

Moreover, we prove that if $[\Delta(x), T(x)] \in Z(R)$ for all $x \in I$, then R is a commutative ring.

1. Introduction

Throughout the paper R will denote an associative ring with centre $Z(R)$. A ring R is said to be prime (resp. semiprime) if $aRb = (0)$ implies that either $a = 0$ or $b = 0$ (resp. $aRa = (0)$ implies that $a = 0$). We shall write $[x, y]$ the commutator $xy - yx$. We make extensive use of basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A derivation d is inner if there exists an element $a \in R$ such that $d(x) = [a, x]$ for all $x \in R$. A mapping $S : R \times R \to R$ is said to be symmetric if $S(x, y) = S(y, x)$, for all $x, y \in R$. A mapping $\Delta : R \to R$ defined by $\Delta(x) = S(x, x)$, where $S : R \times R \to R$ is a symmetric mapping, is called the trace of S. It is obvious that in the case $S : R \times R \to R$ is a symmetric bi-additive mapping, the trace Δ of S satisfies the relation $\Delta(x+y) = \Delta(x)+\Delta(y)+2S(x, y)$ for all $x, y \in R$. A bi-additive mapping $S : R \times R \to R$ is said to be a bi-derivation if for every $x \in R$, the map $y \mapsto S(x, y)$ as well as if for every $y \in R$, the map $x \mapsto S(x, y)$ are derivations of R.

An additive mapping $d : R \to R$ is called a skew derivation (T-derivation) of R associated with the automorphism T if $d(xy) = d(x)y+T(x)d(y)$ for all $x, y \in R$. Skew derivations are one of the natural generalizations of usual derivations, when $T = I$, the identity map on R. Let $n \geq 1$ be an integer. A mapping $S : R^n \to R$ is said to be n-additive, if it is additive in each argument and it is

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called symmetric if $S(x_1,\dots,x_n) = S(x_{\pi(1)},\dots,x_{\pi(n)})$ for all $x_1,x_2,\dots,x_n \in R$ and every permutation $\pi \in S_n$, the symmetric group of degree n. An n-additive map $S : R^n \to R$ is called a skew n-derivation associated with the automorphism T if for every $k = 1,2,\dots,n$ and all $x_1,\dots,x_n \in R$, the map $x \longmapsto S(x_1,x_{k-1},x,x_{k+1},\dots,x_n)$ is a skew derivation of R associated with the automorphism T. This definition covers both the notion of skew derivations as well as the notion of skew bi-derivation. Namely, a skew 1-derivation is a skew derivation and skew 2-derivation is a skew bi-derivation.

Let S be a nonempty subset of R. A mapping F from R to R is called centralizing on S if $[F(x),x] \in Z(R)$ for all $x \in S$ and is called commuting on S if $[F(x),x] = 0$ for all $x \in S$. The study of centralizing mappings was initiated by E. C. Posner [11], which states that there existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative (Posner's second theorem). There has been an ongoing interest concerning the relationship between the commutativity of a ring and the existence of certain specific types of derivations of R (see [3], for a partial bibliography).

In [8], Maksa introduced the concept of a symmetric bi-derivation (see also [9], where an example can be found). It was shown in [8] that symmetric bi-derivations are related to general solution of some functional equations. Then, Ashraf [1] obtained the analogous result replacing d with the trace of symmetric bi-derivation. Vukman [13] and [14] also studied the symmetric biderivation on prime and semiprime rings and obtain some results concerning the traces symmetric bi-additive maps. Some results on symmetric bi-derivation in prime and semiprime rings can be found in [2, 4, 5]. In the present paper, we shall prove that R is commutative if any one of the following holds: i) $\Delta(x) = 0$, ii) $[\Delta(x),T(x)] = 0$, iii) $[\Delta(x),T(x)] \in Z(R)$ for all $x \in I$.

Example 1.1 ([10, Example 1]). Let R be a commutative ring, T be an autoporphism of R and $s : R \to R$ be a skew-derivation of R associated with the automorphism T. Then the map $S : R^n \to R$, $S(x_1,\dots,x_n) = s(x_1)s(x_2)\cdots s(x_n)$ is a skew n-derivation in R.

Example 1.2 ([10, Example 2]). Let $R = \{\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right) : x,y \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of all integers, and $T\left(\begin{smallmatrix} x & y \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & -y \\ 0 & 0 \end{smallmatrix}\right)$. Then R is a noncommutative ring and T is an automorphism of R. We define a map $S : R^n \to R$ by

$$\left(\begin{pmatrix} x_1 & y_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x_2 & y_2 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} x_n & y_n \\ 0 & 0 \end{pmatrix}\right) \to \begin{pmatrix} 0 & x_1x_2\cdots x_n \\ 0 & 0 \end{pmatrix}.$$

Then it is easy to verify that S is a skew n-derivation in R associated with the automorphism T.

2. Main results

Posner [11] proved a very striking theorem, which states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. This theorem has been extremely influential and it initiated the study of centralizing mappings. Further Vukman [14] extended the above result for bi-derivations. Recently, Jung and Park [7] considered permuting 3-derivations on prime and semiprime rings and obtained the following:

Theorem 2.1. *Let R be a noncommutative 3-torsion free semiprime ring and I be a nonzero two-sided ideal of R. Suppose that there exists a permuting 3-derivation $D : R^3 \to R$ with the trace Δ such that Δ is centralizing on I. Then Δ is commuting on I.*

Further Park [10] proved that:

Theorem 2.2. *Let $n \geq 2$ be a fixed positive integer and R be a noncommutative $n!$-torsion free semiprime ring. If there exists a symmetric n-derivation $D : R^n \to R$ such that the trace of D is centralizing on R, then the trace is commuting on R.*

Many authors ([6], [12]) partially extended the above theorems for symmetric skew n-derivations for different values of n.

Recently, Fošner [6] proved the above theorems for symmetric skew 3-derivations in prime rings. In this paper [6], author mentioned some open problems involving skew n-derivations. In the present paper, our aim is to solve these problems.

We begin our discussion with the following proposition.

Proposition 2.3. *Let R be a $n!$-torsion free prime ring, I a nonzero ideal of R, T an automorphism of R and $S : R^n \to R$ be a symmetric skew n-derivation associated with the automorphism T. If Δ is the trace of S such that $\Delta(I) = 0$, then $S(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in R$.*

Proof. We have $\Delta(x) = 0$ that is $S(x, \dots, x) = 0$ for all $x \in I$. Linearizing the identity, we have

$$0 = \Delta(x+y) = \binom{n}{0}\Delta_0 + \binom{n}{1}\Delta_1 + \cdots + \binom{n}{n}\Delta_n, \tag{2.1}$$

where $\Delta_i = S(\underbrace{x, \dots, x}_{i}, \underbrace{y, \dots, y}_{n-i})$.

Since $\Delta(x) = \Delta_n = \Delta_0 = 0$, (2.1) reduces to

$$\binom{n}{1}\Delta_1 + \binom{n}{2}\Delta_2 + \cdots + \binom{n}{n-1}\Delta_{n-1} = 0. \tag{2.2}$$

Replacing x by $x, 2x, 3x, (n-1)x$ in turn, and expressing the resulting system of $n-1$ homogeneous equations, we see that the coefficient matrix of the system is a Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 2 & 2^2 & \dots & 2^{n-1} \\ \dots & \dots & \dots & \dots \\ n-1 & (n-1)^2 & \dots & (n-1)^{n-1} \end{pmatrix}.$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less than $n-1$, and since R is $n!$-torsion free, it follows immediately that

$$\Delta_1 = \Delta_2 = \cdots = \Delta_{n-1} = 0.$$

Now $\Delta_1 = 0$ yields that $S(y, x, x, \ldots, x) = 0$ for all $x, y \in I$. Again linearizing this identity with respect to x, we can prove by the same manner that $S(y, z, x, \ldots, x) = 0$ for all $x, y, z \in I$ and hence $S(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in I$. Now replacing x_1 with $r_1 x_1$, where $r_1 \in R$, we get $0 = S(r_1 x_1, x_2, \ldots, x_n) = S(r_1, x_2, \ldots, x_n)x_1 + T(r_1)S(x_1, x_2, \ldots, x_n)$. Since $S(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in I$, we have from above relation that $S(r_1, x_2, \ldots, x_n)x_1 = 0$ for all $x_1, \ldots, x_n \in I$. Since R is prime, we conclude that $S(r_1, x_2, \ldots, x_n) = 0$ for all $x_2, \ldots, x_n \in I$ and $r_1 \in R$. Again replacing x_2 with $r_2 x_2$, where $r_2 \in R$, we have by the same arguments that $S(r_1, r_2, \ldots, x_n) = 0$ for all $x_3, \ldots, x_n \in I$ and $r_1, r_2 \in R$. Repeating the process, we obtain that $S(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in R$. □

Theorem 2.4. *Let R be a noncommutative $(n+1)!$-torsion free prime ring, I a nonzero ideal of R, T an automorphism of R and $S : R^n \to R$ be a symmetric skew n-derivation associated with the automorphism T. If Δ is the trace of S such that*

$$[\Delta(x), T(x)] = 0$$

for all $x \in I$, then $S(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in R$.

Proof. We have

$$[\Delta(x), T(x)] = 0 \tag{2.3}$$

for all $x \in I$. Replacing x with $x + y$ in above relation and then using the technique of linearizing as in Proposition 2.3, we get

$$n[S(y, x, \ldots, x), T(x)] + [\Delta(x), T(y)] = 0 \tag{2.4}$$

for all $x, y \in I$. Now we put $y = xy$ and then obtain that

$$n[\Delta(x)y + T(x)S(y, x, \ldots, x), T(x)] + [\Delta(x), T(x)T(y)] = 0 \tag{2.5}$$

that is,

$$\begin{aligned} n\Delta(x)[y, T(x)] + n[\Delta(x), T(x)]y + nT(x)[S(y, x, \ldots, x), T(x)] \\ + T(x)[\Delta(x), T(y)] + [\Delta(x), T(x)]T(y) = 0 \end{aligned} \tag{2.6}$$

for all $x, y \in I$. Now using (2.3) and (2.4), (2.6) reduces to

$$n\Delta(x)[y, T(x)] = 0 \tag{2.7}$$

for all $x, y \in I$. By using torsion free restriction on R, we can write $\Delta(x)[y, T(x)] = 0$ for all $x, y \in I$. Now putting $y = yr$, where $r \in R$, we get

$$0 = \Delta(x)[yr, T(x)] = \Delta(x)[y, T(x)]r + \Delta(x)y[r, T(x)] = \Delta(x)y[r, T(x)]$$

for all $x, y \in I$ and $r \in R$. Since R is prime, for each $x \in I$, either $\Delta(x) = 0$ or $T(x) \in Z(R)$. Now choose $x \in I$ such that $T(x) \in Z(R)$. Thus from (2.4), we can write for all $y \in I$ that $[\Delta(x), T(y)] = 0$, that is $[\Delta(x), T(I)] = 0$. Since $T(I)$ is a nonzero ideal of R, we have $\Delta(x) \in Z(R)$.

Therefore, in any case, we can write $\Delta(x) \in Z(R)$ for all $x \in I$. This implies $[\Delta(x), r] = 0$ for all $x \in I$ and $r \in R$. Again by replacing x with $x + y$ and then by using the same arguments linearization of Proposition 2.3, we have $n[S(y, x, \ldots, x), r] = 0$ for all $x, y \in I$ and $r \in R$. Since R is n-torsion free, $[S(y, x, \ldots, x), r] = 0$ for all $x, y \in I$ and $r \in R$. Putting $y = yr$ we get $0 = [S(y, x, \ldots, x)r + T(y)S(r, x, \ldots, x), r] = [S(y, x, \ldots, x), r]r + [T(y)S(r, x, \ldots, x), r]$. This implies $0 = [T(y)S(r, x, \ldots, x), r]$ for all $x, y \in I$ and $r \in R$. Putting $y = sy$, where $s \in R$, we obtain

$$\begin{aligned} 0 &= [T(s)T(y)S(r, x, \ldots, x), r] \\ &= T(s)[T(y)S(r, x, \ldots, x), r] + [T(s), r]T(y)S(r, x, \ldots, x) \\ &= [T(s), r]T(y)S(r, x, \ldots, x). \end{aligned}$$

This implies that $0 = [T(s), r]T(I)S(r, x, \ldots, x)$ for all $x \in I$ and $r, s \in R$. Since R is prime, for each $r \in R$ we conclude either $[T(s), r] = 0$ for all $s \in R$ or $S(r, x, \ldots, x) = 0$ for all $x \in I$. The sets of $r \in R$ for which these two conditions hold are additive subgroups of R whose union is R; therefore, $[T(s), r] = 0$ for all $s \in R$, for all $r \in R$ or $S(r, x, \ldots, x) = 0$ for all $x \in I$, for all $r \in R$. Since R is noncommutative, first case can not occurs, and hence $S(r_1, x, \ldots, x) = 0$ for all $x \in I$, $r_1 \in R$. Then by same argument of Proposition 2.3, we can conclude that $S(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in R$. □

Theorem 2.5. *Let R be a noncommutative $(n+1)!$-torsion free semiprime ring, I a nonzero ideal of R, T an automorphism of R and $S : R^n \to R$ be a symmetric skew n-derivation associated with the automorphism T. If Δ is the trace of S such that*

$$[\Delta(x), T(x)] \in Z(R)$$

for all $x \in I$, then $[\Delta(x), T(x)] = 0$ for all $x \in I$.

Proof. Let $x \in I$ and $t = [\Delta(x), T(x)] \in Z(R)$. Denote

$$\gamma_i(y, x) = S(\underbrace{y, \ldots, y}_{i}, \underbrace{x, \ldots, x}_{n-i}).$$

Then $\gamma_0(y, x) = S(x, \ldots, x) = \Delta(x)$ and $\gamma_n(y, x) = S(y, \ldots, y) = \Delta(y)$. Linearizing the relation $[\Delta(x), T(x)] \in Z(R)$ yields as shown in Proposition 2.3 that

$$\binom{n}{1}[\gamma_1(y, x), T(x)] + [\Delta(x), T(y)] \in Z(R),$$

$$\binom{n}{2}[\gamma_2(y, x), T(x)] + \binom{n}{1}[\gamma_1(y, x), T(y)] \in Z(R),$$

$$\binom{n}{3}[\gamma_3(y, x), T(x)] + \binom{n}{2}[\gamma_2(y, x), T(y)] \in Z(R),$$

$$\cdots\cdots\cdots$$

$$\binom{n}{n}[\Delta(y), T(x)] + \binom{n}{n-1}[\gamma_{n-1}(y,x), T(y)] \in Z(R).$$

Now putting $y = x^2$, above relations become

$$\binom{n}{1}[\gamma_1(x^2,x), T(x)] + 2tT(x) \in Z(R), \tag{2.8}$$

$$\binom{n}{2}[\gamma_2(x^2,x), T(x)] + \binom{n}{1}\Big\{[\gamma_1(x^2,x), T(x)]T(x) + T(x)[\gamma_1(x^2,x), T(x)]\Big\} \in Z(R), \tag{2.9}$$

$$\binom{n}{3}[\gamma_3(x^2,x), T(x)] + \binom{n}{2}\Big\{[\gamma_2(x^2,x), T(x)]T(x) + T(x)[\gamma_2(x^2,x), T(x)]\Big\} \in Z(R), \tag{2.10}$$

$$\cdots\cdots\cdots$$

$$\binom{n}{n}[\Delta(x^2), T(x)] + \binom{n}{n-1}\Big\{[\gamma_{n-1}(x^2,x), T(x)]T(x) + T(x)[\gamma_{n-1}(x^2,x), T(x)]\Big\} \in Z(R). \tag{2.11}$$

Commuting both sides of (2.8) with $T(x)$, we can write

$$0 = [\binom{n}{1}[\gamma_1(x^2,x), T(x)] + 2tT(x), T(x)] = \binom{n}{1}[[\gamma_1(x^2,x), T(x)], T(x)].$$

Since R is $(n+1)!$-torsion free, we conclude that $[\gamma_1(x^2,x), T(x)]$ commutes with $T(x)$. Again, commuting both sides of (2.9) with $T(x)$, we obtain by using the fact $[[\gamma_1(x^2,x), T(x)], T(x)] = 0$ that $[[\gamma_2(x^2,x), T(x)], T(x)] = 0$. In the same manner, we can prove in general that $[[\gamma_i(x^2,x), T(x)], T(x)] = 0$ for $i = 1, 2, \ldots, n-1$ and $[[\Delta(x^2), T(x)], T(x)] = 0$. Thus the relations (2.8) to (2.11) reduce to

$$\binom{n}{1}[\gamma_1(x^2,x), T(x)] + 2tT(x) \in Z(R),$$

$$\binom{n}{2}[\gamma_2(x^2,x), T(x)] + 2\binom{n}{1}[\gamma_1(x^2,x), T(x)]T(x) \in Z(R),$$

$$\binom{n}{3}[\gamma_3(x^2,x), T(x)] + 2\binom{n}{2}[\gamma_2(x^2,x), T(x)]T(x) \in Z(R),$$

$$\cdots\cdots\cdots$$

$$\binom{n}{n-1}[\gamma_{n-1}(x^2,x), T(x)] + 2\binom{n}{n-2}[\gamma_{n-2}(x^2,x), T(x)]T(x) \in Z(R),$$

$$\binom{n}{n}[\Delta(x^2), T(x)] + 2\binom{n}{n-1}[\gamma_{n-1}(x^2,x), T(x)]T(x) \in Z(R).$$

There exists a sequence of maps $\mu_i : R \to Z(R)$ such that

$$\binom{n}{1}[\gamma_1(x^2,x),T(x)] + 2tT(x) = \mu_1(x),$$

$$\binom{n}{2}[\gamma_2(x^2,x),T(x)] + 2\binom{n}{1}[\gamma_1(x^2,x),T(x)]T(x) = \mu_2(x),$$

$$\binom{n}{3}[\gamma_3(x^2,x),T(x)] + 2\binom{n}{2}[\gamma_2(x^2,x),T(x)]T(x) = \mu_3(x),$$

$$\cdots\cdots\cdots$$

$$\binom{n}{n-1}[\gamma_{n-1}(x^2,x),T(x)] + 2\binom{n}{n-2}[\gamma_{n-2}(x^2,x),T(x)]T(x) = \mu_{n-1}(x),$$

$$\binom{n}{n}[\Delta(x^2),T(x)] + 2\binom{n}{n-1}[\gamma_{n-1}(x^2,x),T(x)]T(x) = \mu_n(x).$$

Multiplying the equations $2^{n-1}T(x)^{n-1}$, $-2^{n-2}T(x)^{n-2}$, $\ldots$, $(-1)^n 2^1 T(x)^1$, $-(-1)^n.1$ respectively, we can write the equations as

$$2^{n-1}T(x)^{n-1}\binom{n}{1}[\gamma_1(x^2,x),T(x)] + 2^nT(x)^nt = 2^{n-1}T(x)^{n-1}\mu_1(x),$$

$$\begin{aligned} &-2^{n-2}T(x)^{n-2}\binom{n}{2}[\gamma_2(x^2,x),T(x)] - 2^{n-1}T(x)^{n-1}\binom{n}{1}[\gamma_1(x^2,x),T(x)] \\ = &-2^{n-2}T(x)^{n-2}\mu_2(x), \end{aligned}$$

$$\begin{aligned} &2^{n-3}T(x)^{n-3}\binom{n}{3}[\gamma_3(x^2,x),T(x)] + 2^{n-2}T(x)^{n-2}\binom{n}{2}[\gamma_2(x^2,x),T(x)] \\ = &\,2^{n-3}T(x)^{n-3}\mu_3(x), \end{aligned}$$

$$\cdots\cdots\cdots$$

$$\begin{aligned} &(-1)^n 2T(x)\binom{n}{n-1}[\gamma_{n-1}(x^2,x),T(x)] \\ &+ (-1)^n 2^2 T(x)^2\binom{n}{n-2}[\gamma_{n-2}(x^2,x),T(x)] \\ = &\,(-1)^n 2T(x)\mu_{n-1}(x), \end{aligned}$$

$$\begin{aligned} &-(-1)^n\binom{n}{n}[\Delta(x^2),T(x)] - (-1)^n 2\binom{n}{n-1}[\gamma_{n-1}(x^2,x),T(x)]T(x) \\ = &-(-1)^n\mu_n(x). \end{aligned}$$

Adding all these above equations, we obtain

$$\begin{aligned} &2^nT(x)^nt - (-1)^n[\Delta(x^2),T(x)] \\ &= 2^{n-1}T(x)^{n-1}\mu_1(x) - 2^{n-2}T(x)^{n-2}\mu_2(x) + 2^{n-3}T(x)^{n-3}\mu_3(x) \\ &\quad + \cdots + (-1)^n 2T(x)\mu_{n-1}(x) - (-1)^n\mu_n(x). \end{aligned} \tag{2.12}$$

Now by hypothesis, we have $[\Delta(x^2), T(x^2)] \in Z(R)$. Then for some $\mu_{n+1} : R \to R$, we can write $[\Delta(x^2), T(x^2)] = \mu_{n+1}(x)$. Since $[\Delta(x^2), T(x)]$ commutes with $T(x)$, we have

$$\mu_{n+1}(x) = [\Delta(x^2), T(x^2)] = 2T(x)[\Delta(x^2), T(x)].$$

Now multiplying (2.12) by $2T(x)$ in both sides and then using the fact $2T(x)[\Delta(x^2), T(x)] = \mu_{n+1}(x)$, we obtain that

$$\begin{aligned} & 2^{n+1}T(x)^{n+1}t - (-1)^n \mu_{n+1}(x) \\ (2.13) \qquad & = 2^n T(x)^n \mu_1(x) - 2^{n-1}T(x)^{n-1}\mu_2(x) + 2^{n-2}T(x)^{n-2}\mu_3(x) \\ & + \cdots + (-1)^n 2^2 T(x)^2 \mu_{n-1}(x) - (-1)^n 2T(x)\mu_n(x). \end{aligned}$$

Now commuting $T(x)^k$ with $\Delta(x)$ successively, we get

$$[\Delta(x), T(x)^k] = [\Delta(x), \underbrace{T(x).T(x).\cdots.T(x)}_{k \text{ times}}] = ktT(x)^{k-1}$$

and

$$\begin{aligned} [\Delta(x), [\Delta(x), T(x)^k]] &= kt[\Delta(x), T(x)^{k-1}] = k(k-1)t^2T(x)^{k-2} \\ &= \tfrac{k!}{(k-2)!}t^2T(x)^{k-2}. \end{aligned}$$

Thus commuting $T(x)^k$ with $\Delta(x)$ successively m-times yields

$$[\Delta(x), \ldots, [\Delta(x), T(x)^k]] = \begin{cases} \frac{k!}{(k-m)!}t^m T(x)^{k-m}, & 1 \le m \le k \\ 0, & m > k. \end{cases}$$

Using this fact, we can write, successively commuting both sides of (2.13) $(n+1)$-times with $T(x)$ and using the fact that R is $(n+1)!$-torsion free, we obtain $t^{n+2} = 0$. Since the center of semiprime ring contains no nonzero nilpotent elements, we have $t = 0$, as desired. □

Corollary 2.6. *Let R be a $(n+1)!$-torsion free prime ring, I a nonzero ideal of R, T an automorphism of R and $S : R^n \to R$ be a nonzero symmetric skew n-derivation associated with the automorphism T. If Δ is the trace of S such that*

$$[\Delta(x), T(x)] \in Z(R)$$

for all $x \in I$, then R is commutative.

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Emine Koç
Department of Mathematics
Cumhuriyet University
Sivas, Turkey
Email address: eminekoc@cumhuriyet.edu.tr

Nadeem ur Rehman
Department of Mathematics
Aligarh Muslim University
Aligarh U.P., India
Email address: nu.rehman.mm@amu.ac.in