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JORDAN TYPES OF COMMUTING NILPOTENT MATRICES

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Abstract. Let A and B be matrices which are polynomials in r pairwise commuting nilpotent matrices over a field. We give a sufficient condition for the null space of A^i to equal that of B^i for all i, in particular, for A and B to be similar.

1. Introduction

Let X be a nilpotent $d \times d$ matrix of nilpotency n, that is $X^n = 0$, $X^{n-1} \neq 0$, and let $[a]$ denote the $a \times a$ Jordan block with 0's on the main diagonal and 1's on the superdiagonal. We write $J_X = m_1[a_1] \oplus \cdots \oplus m_t[a_t]$ with $n \geq a_1 > \cdots > a_t \geq 1$, whenever the Jordan canonical form of X consists of m_i many $[a_i]$'s for $m_i \geq 1$ for $i = 1, \ldots, t$ and refer to J_X as the *Jordan type* of X. In other words, J_X represents the similarity class of X. We say that X is of *maximal Jordan type* whenever $J_X = m[n]$ (here $m = d/n$ necessarily). If X is a matrix representing the action of $g-1$ on a finitely generated $k[\langle g \rangle]$-module V, where the field k is of characteristic p, J_X is referred as the *Jordan type of the module* V, that determines the decomposition of V as a direct sum of indecomposable $k[\langle g \rangle]$-modules. Note that V is of maximal Jordan type if and only if V is isomorphic to $(k[\langle g \rangle])^m$ for some integer $m \geq 1$. The Jordan type is used to define the class of *modules of constant Jordan type* in [1]. In [5], by examining the Jordan types of powers of a nilpotent matrix we verified several conjectures stated in [1] for the subclass of restricted modules of constant Jordan type, we also stated many similar conjectures. Our Theorem below emerged from that framework and it is of interest by itself. It is a generalization of the main results of [2] and [3].

The null space of a matrix C, the solution space of $Cv = 0$, is denoted by null(C), the dimension of C is referred as the nullity of C. The following Lemma is crucial in the proof of Theorem. It is proved easily by induction and it is of interest by itself.

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Lemma. *Suppose C and D are commuting matrices with* $\mathrm{null}(C) = \mathrm{null}(D)$. *Then* $\mathrm{null}(C^i) = \mathrm{null}(D^i)$ *for $i \geq 2$. In particular, if C and D are commuting nilpotent matrices with* $\mathrm{null}(C) = \mathrm{null}(D)$*, then C and D are similar.*

Theorem. *Let $X_1, \dots, X_r$ be r pairwise commuting nilpotent matrices over a field k and n_i be the nilpotency of X_i with $2 \leq n_1 \leq \cdots \leq n_r$. If A, B are $f(X_1, \dots, X_r)$, $g(X_1, \dots, X_r)$ respectively, for the polynomials f, g in $k[x_1, \dots, x_r]$ having at least one linear term and no constant term, and $f \equiv g \pmod{I^s}$ where $I = (x_1, \dots, x_r)$ and $s = n_2 + \cdots + n_r - r + 2$, then* $\mathrm{null}(A^i) = \mathrm{null}(B^i)$ *for all $i \geq 1$. In particular, A and B are similar.*

The special case of Theorem with $k \supseteq \mathbb{F}_p$, $r = 2$, $n_1 = n_2 = p$, hence $s = p$, is Corollary 3 in [3]. The proofs in this article are in spirit parallel to the ones in [3]. The Main Theorem in [2] is a variation of this Theorem with a much weaker conclusion, namely, A is of maximal Jordan type if and only if B is of maximal Jordan type. Besides, the proof of that theorem in [2] is module theoretic.

Remark 1.1. The integer s in Theorem is the smallest lower bound to guarantee the conclusion as the following example shows. Let

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $XY = YX = 0$, X and Y are of nilpotencies 2 and 4 respectively, and hence $s = 4$. Let $f(x, y) = x$, $g(x, y) = x + y^3$, with no constant term, at least one linear term and $f \equiv g \pmod{I^3}$ where $I = (x, y) \subset k[x, y]$ and $A = f(X, Y)$, $B = g(X, Y)$. However $\mathrm{rank}(A) = 1$, $\mathrm{rank}(B) = \mathrm{rank}(X + Y^3) = 2$. Hence $\mathrm{null}(A) \neq \mathrm{null}(B)$. Hence, congruence modulo $I^3 = I^{s-1}$ is not sufficient.

2. Proof of Lemma

We repeatedly use the hypothesise $CD = DC$ and $\mathrm{null}(C) = \mathrm{null}(D)$ in the proof. Note that $\mathrm{null}(C) = \mathrm{null}(D)$ is equivalent to the statement that $Cw = 0$ if and only if $Dw = 0$ for any $d \times 1$ vector w. Suppose that $C^2 v = 0$. We have; $C(Cv) = 0$ if and only if $0 = D(Cv) = C(Dv)$ if and only if $D(Dv) = D^2 v = 0$. Therefore the statement is true for $i = 2$. Suppose $\mathrm{null}(C^l) = \mathrm{null}(D^l)$ for some $l \geq 2$. Then, $C^{l+1} v = C(C^l v) = 0$ if and only if $D(C^l v) = C^l(Dv) = 0$ if and only if $D^l(Dv) = D^{l+1} v = 0$. Hence the statement $\mathrm{null}(C^i) = \mathrm{null}(D^i)$ holds for $i \geq 2$ by induction.

Suppose further that C and D are nilpotent. As shown in [3] for nilpotent matrix C of nilpotency n, the Jordan type $J_C = m_1[a_1] \oplus \cdots \oplus m_t[a_t]$ with $n \geq a_1 > \cdots > a_t \geq 1$ is determined completely by the ranks of the powers of the matrix, namely, $m_i = \operatorname{rank}(C^{i-1}) - 2\operatorname{rank}(C^i) + \operatorname{rank}(C^{i+1})$. Thus, if $\operatorname{null}(C) = \operatorname{null}(D)$, then the nullities and hence the ranks of C and D are the same. By the first part of Lemma we obtain that the ranks of C^i and D^i are the same for all $i \geq 1$. Hence $J_C = J_D$, that is, C and D are similar.

3. Proof of Theorem

The arguments in the proof of Theorem are in spirit parallel to the ones in [3], but they are much more tedious even though we introduced "generic elements" to make the proof more clear. Reading the proof of Theorem 1 in [3] in advance may shed light into the following one.

Suppose $X_1, \dots, X_r$ are nilpotent and commute pairwise with X_i of nilpotency n_i and $2 \leq n_1 \leq \cdots \leq n_r$. Suppose further that $f, g \in k[x_1, \dots, x_r]$ are polynomials having no constant term, having at least one linear term and $f \equiv g \pmod{I^s}$ for $I = (x_1, \dots, x_r)$ and $\underline{s = n_2 + \cdots + n_r - r + 2}$, $A = f(X_1, \dots, X_r)$, $B = g(X_1, \dots, X_r)$. By Lemma, it remains to show that $\operatorname{null}(A) = \operatorname{null}(B)$. Since the situation is symmetric with respect to A and B, it suffices to prove only one inclusion, say, $\operatorname{null}(A) \subseteq \operatorname{null}(B)$.

At first, we make several observations while introducing some notation. Let J be the ideal $(X_1, \dots, X_r)$ in the ring $\operatorname{Mat}_d(k)$ of $d \times d$ matices. The ideal J^t, for $t \geq 1$, consists of all k-linear combinations of $X_1^{t_1} \cdots X_r^{t_r}$ with integers $n_i - 1 \geq t_i \geq 0$ and $t_1 + \cdots + t_r \geq t$. Hence, for $\underline{\tau = (n_1 - 1) + \cdots + (n_r - 1)}$ we have

$$J^\tau = kX_1^{n_1-1} \cdots X_r^{n_r-1} \quad \text{and} \quad J^{\tau+1} = 0. \tag{1}$$

On the other hand, by the hypothesise on f and g, we can write $B = A + w(X_1, \dots, X_r)$ with $A = a_1X_1 + \cdots + a_rX_r + c(X_1, \dots, X_r)$ with $a_j \neq 0$ for some $j \in \{1, \dots, r\}$, where the polynomial $c(x_1, \dots, x_r) = 0$ or it consists of terms of degree at least 2, and the polynomial $w(x_1, \dots, x_r)$ consists of terms of degrees at least s. Without loss of generality assume that $a_q \neq 0$.

For the purposes of the proof it is sufficient to work with 'generic elements' instead of specific ones. Let the symbol $[\cdots]^{M-j}$ represent

$$X_1^{n_1-1-m_1} \cdots \widehat{X_q^{n_q-1-m_q}} \cdots X_r^{n_r-1-m_r} \in J^{M-j}$$

and refer to it as a *generic element* where $\underline{M = n_1 + \cdots + \widehat{n_q} + \cdots + n_r - r + 1}$ $= \tau - n_q + 1$ and $j = m_1 + \cdots + \widehat{m_q} + \cdots + m_r$ with $m_i \in \{0, 1, \dots, n_i - 2\}$. Note that the exponent of X_i in $[\cdots]^{M-j}$ is at least one for $i \neq q$. When $j = 0$ the only possibility for any m_i is 0. Thus, $[\cdots]^M$ is the uniquely determined element $X_1^{n_1-1} \cdots \widehat{X_q^{n_q-1}} \cdots X_r^{n_r-1} \in J^M$ and it is annihilated by $X_1, \dots, \widehat{X_q}, \dots, X_r$. However $[\cdots]^{M-j}$ is not uniquely determined for $j \geq 1$, it represents as many elements as the number of possibilities for positive integers $m_1, \dots, \widehat{m_q}, \dots, m_r$ with $m_i \in \{0, \dots, n_i - 2\}$ such that $j = m_1 + \cdots + \widehat{m_q} + \cdots + m_r$. For instance, when $j = 1$ there are exactly $r - 1$ possibilities; $1 = j = m_i$ for $i = 1, \dots, \widehat{q}, \dots, r$.

<u>*Claim* 1:</u> $\mathrm{null}(A) \subseteq \mathrm{null}(B)$.

Suppose that $Av = 0$ for some non-zero vector v. Then

$$-(a_1X_1 + \cdots + a_rX_r)v = c(X_1, \ldots, X_r)v \in J^2v, \quad \text{and} \tag{2}$$

$$Bv = w(X_1, \ldots, X_r)v \in J^sv. \tag{3}$$

Hence, showing $J^sv = 0$ is sufficient to prove *Claim* 1. We will use $a_q \neq 0$ together with the generic elements $[\cdots]^{M-j}$ and induction on l to show $J^{\tau-l}v = 0$ for $l \in \{-1, 0, \ldots, n_1 - 2\}$. Recall that $s = \tau - n_1 + 2$. For $0 \leq l \leq n_1 - 2$, we have $s \leq \tau - l \leq \tau$. The case $l = n_1 - 2$ gives the desired $J^sv = 0$. The first step of the induction, namely $J^{\tau+1} = 0$, is true by equation (1). Suppose that $J^{\tau-t}v = 0$ for some $t \in \{-1, 0, 1, \ldots, n_1 - 3\}$.

<u>*Claim* 2:</u> $J^{\tau-(t+1)}v = 0$.

Notice that for $X_1^{t_1} \ldots X_r^{t_r} \in J^{\tau-t-1}$, if $t_i < n_i - t - 2$ for some $i \in \{1, \ldots, r\}$, then

$$\tau - t - 1 \leq t_1 + \cdots + t_r < (n_1 - 1) + \cdots + (n_r - 1) - t - 1 = \tau - t - 1,$$

which is a contradiction. Hence $t_i \geq n_i - t - 2 \geq n_1 - t - 2$ for all $i \in \{1, \ldots, r\}$. In particular, $X_1^{n_1-t-2} \cdots X_r^{n_r-t-2}$ is a factor of every element of $J^{\tau-t-1}$. In addition, the induction hypothesis $t \leq n_1 - 3$ implies $t_i \geq 1$.

Let $\mathcal{B}$ be a basis for $J^{\tau-t-1}$ consisting of elements of the form $X_1^{t_1} \ldots X_r^{t_r}$ such that $\mathcal{C} \subset \mathcal{B}$ is a basis for $J^{\tau-t}$. Due to the induction hypothesis that $J^{\tau-t}v = 0$, the proof of *Claim* 2 is reduced to showing $\beta v = 0$ for every β in $\mathcal{B} \setminus \mathcal{C}$. Let $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_{t+1}$ be the subsets of $\mathcal{B}$ forming a partition of $\mathcal{B} \setminus \mathcal{C}$ such that the general form of an element β_j of $\mathcal{B}_j$ is given by $\beta_j := X_q^{m+j}[\cdots]^{M-j} \in J^{\tau-t-1}$ where $m = n_q - t - 2$ is the lower bound for n_q and $[\cdots]^{M-j}$ is a generic element as described above. It should be noted that $m \geq 1$ and $m_i \in \{0, 1, \ldots, t+1\}$ due to the induction hypothesis $t \leq n_1 - 3$.

The proof of *Claim* 2 is now reduced to another induction, namely, to show that $\mathcal{B}_l$ annihilates v for $l = 0, \ldots, t+1$. Note that $\mathcal{B}_0 = \{\beta_0\}$ where $\beta_0 = X_q^m[\cdots]^M \in J^{\tau-t-1}$. Multiplying the equation (2) by the unique element $X_q^{m-1}[\cdots]^M \in J^{\tau-t-2}$ gives that

$$-a_qX_q^m[\cdots]^Mv = -a_q\beta_0v = X_q^{m-1}[\cdots]^Mc(X_1, \ldots, X_r) \in J^{\tau-t}v = 0.$$

Since $a_q \neq 0$, we obtain the desired $\beta_0v = 0$. Assume that $\mathcal{B}_{k-1}$ annihilates v for some $k \in \{1, \ldots, t+1\}$.

<u>*Claim* 3:</u> $\mathcal{B}_k$ annihilates v.

Note that multiplying a fixed generic element $X_q^{m+k-1}[\cdots]^{M-k} \in J^{\tau-t-2}$ by X_i for $i \in \{1, \ldots, \widehat{q}, \ldots, r\}$ gives a generic element $X_q^{m+k-1}[\cdots]^{M-k-1}$ belonging to $\mathcal{B}_{k-1}$. Thus the left hand side of the equation (2), after multiplication by a generic element $X_q^{m+k-1}[\cdots]^{M-k}$, becomes

$$\begin{aligned} &-\Big(a_qX_q^{m+k}[\cdots]^{M-k} + k\text{-linear combinations of elements of } \mathcal{B}_{k-1}\Big)v \\ = &-a_qX_q^{m+k}[\cdots]^{M-k}v, \end{aligned}$$

where the equality is due to the induction hypothesis that $\mathcal{B}_{k-1}$ annihilates v. Meanwhile the right hand side of the equation (2) after multiplication by a generic element $X_q^{m+k-1}[\cdots]^{M-k} \in J^{\tau-t-2}$ belongs to $J^{\tau-t}v$, hence it is 0 by the previous induction hypothesis that $J^{\tau-t}v = 0$. Since $a_q \neq 0$, we obtain the desired $X_q^{m+k}[\cdots]^{M-k}v = 0$. Therefore $\mathcal{B}_k$ annihilates v which proves *Claim* 3. Thus, all $\mathcal{B}_1, \ldots, \mathcal{B}_{t+1}$ and hence $J^{\tau-t-1}$ annihilates v proving *Claim* 2. Therefore *Claim* 1 and hence Theorem is proved.

4. Corollary

A natural application of Theorem is for modules over group algebras of finite p-groups where the field k is of characteristic p. Let $G = C_{p^{e_1}} \times \cdots \times C_{p^{e_r}}$ be an abelian p-group of rank r with generators g_i of order p^{e_i}, $e_1 \leq \cdots \leq e_r$, $s = p^{e_2} + \cdots + p^{e_r} - r + 2$, and V be a finitely generated $k[G]$-module, let X_i denote the action of the $g_i - 1$ on V. Since $(g_i - 1)^{p^{e_i}} = g_i^{p^{e_i}} - 1^{p^{e_i}} = 0$, X_i is nilpotent with nilpotency at most p^{e_i}. Let I denote the Jacobson radical of $k[G]$. Furthermore, for $a \in I \backslash I^2$, we write $V{\downarrow}_{\langle 1+a \rangle}$ when a $k[G]$-module V is regarded as a $k[\langle 1+a \rangle]$-module. We obtain a straightforward corollary of Theorem.

Corollary. *If V is a $k[G]$-module and a, b are elements of $I \backslash I^2$ with $a \equiv b$ $(\text{mod } I^s)$, then the null spaces of a^i and b^i are the same when considered as nilpotent operators on V for all $i = 1, \ldots, m$. In particular, the Jordan types of $V{\downarrow}_{\langle 1+a \rangle}$ and $V{\downarrow}_{\langle 1+b \rangle}$ are the same.*

The Jordan type of $V{\downarrow}_{\langle 1+a \rangle}$ is well defined modulo I^s by Corollary. The number s is not small, for the special case of $V{\downarrow}_{\langle 1+a \rangle}$ being of maximal Jordan type we introduced a subspace $\mathbb{J}^{(2)}$ in [4] with $I \supseteq \mathbb{J}^{(2)} \supseteq I^s$ and showed that $V{\downarrow}_{\langle 1+a \rangle}$ is of maximal Jordan type if and only if $V{\downarrow}_{\langle 1+b \rangle}$ is of maximal Jordan type for $a \equiv b$ $(\text{mod } \mathbb{J}^{(2)})$, see Theorem 3.5 in [4]. When the group G is an elementary abelian p-group, namely, $G = C_p \times \cdots \times C_p$, the space $\mathbb{J}^{(2)}$ coincides with I^2 where I is the Jacobson radical of $k[G]$.

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