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EQUIDIMENSIONAL LOCAL RINGS WITH FINITE COUSIN COHOMOLOGY MODULES

Alireza Vahidi

Abstract. It is shown that any equidimensional local ring which has finite Cousin cohomology modules with respect to the dimension filtration has a uniform local cohomological annihilator and is universally catenary.

1. Introduction

Throughout R will denote a commutative Noetherian ring with non-zero identity and X an arbitrary R-module which is not necessarily finite (i.e., finitely generated). We will say that X is equidimensional if $\dim_R(X) = \dim(R/\mathfrak{p})$ for each $\mathfrak{p}$ of $\mathrm{Min}_R(X)$ (i.e., the set of minimal elements of $\mathrm{Supp}_R(X)$). For basic results, notations, and terminology not given in this paper, the reader is referred to [1, 2, 10].

The notion of Cousin complex for an R-module was introduced by Sharp [11] as an analogue of Hartshorne [7]. Sharp in [13] generalized this concept to the Cousin complex for an R-module X with respect to a filtration $\mathcal{F}$ of $\mathrm{Spec}(R)$ and denoted this complex by $\mathrm{C}_R(\mathcal{F}, X)$. The Cousin cohomology modules (i.e., the cohomology modules of $\mathrm{C}_R(\mathcal{F}, X)$) have been studied by several authors. In [12, 13], Sharp studied the vanishing of Cousin cohomology modules with respect to the height and dimension filtrations. Dibaei, Tousi, Jafari, and Kawasaki, in [3–6, 8], worked on the finiteness of Cousin cohomology modules with respect to the height filtration and, in [9, Proposition 9.3.5], Lipman, Nayak, and Sastry generalized their results to complexes on formal schemes.

In this paper, we study the finiteness of Cousin cohomology modules with respect to the dimension filtration and show that any equidimensional local ring which has finite Cousin cohomology modules with respect to the dimension filtration has a uniform local cohomological annihilator and is universally catenary. Recall that an element r of $R \setminus \bigcup_{\mathfrak{p} \in \mathrm{Min}_R(X)} \mathfrak{p}$ is called a uniform local cohomological annihilator of X, if $r\,\mathrm{H}^i_{\mathfrak{m}}(X) = 0$ for each maximal ideal $\mathfrak{m}$ of R

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and for all $i < \dim_{R_\mathfrak{m}}(X_\mathfrak{m})$, where $\mathrm{H}^i_\mathfrak{m}(X)$ is the ith local cohomology module of X with respect to $\mathfrak{m}$.

2. Main results

Let R be a local ring with maximal ideal $\mathfrak{m}$ and X an arbitrary R-module. Let $\mathcal{D} = \{D_i\}_{i\geq 0}$ be the family of subsets of $\mathrm{Spec}(R)$ with

$$D_i = \{\mathfrak{p} \in \mathrm{Spec}(R) : \dim(R) - \dim(R/\mathfrak{p}) \geq i\}.$$

Then, for all $i \geq 0$, we have

- $\mathrm{Supp}_R(X) \subseteq D_0$,
- $D_i \supseteq D_{i+1}$, and
- $D_i \setminus D_{i+1}$ is low with respect to D_i (i.e., each member of $D_i \setminus D_{i+1}$ is a minimal member of D_i with respect to inclusion).

The family $\mathcal{D}$ is called the dimension filtration of R. The Cousin complex for X with respect to the dimension filtration is of the form

$$C_R(\mathcal{D}, X) : 0 \xrightarrow{d^{-2}} X \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{i-2}} X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} \cdots$$

where, for all $i \geq 0$,

- $X^i = \bigoplus_{\mathfrak{p}\in D_i\setminus D_{i+1}} (\mathrm{Coker}\, d^{i-2})_\mathfrak{p}$ and
- $d^{i-1}(x) = \{\frac{x+\mathrm{Im}\, d^{i-2}}{1}\}_{\mathfrak{p}\in D_i\setminus D_{i+1}}$ for every element x of X^{i-1};

and satisfies

- $\mathrm{Supp}_R(X^i) \subseteq D_i$,
- $\mathrm{Supp}_R(\mathrm{Coker}\, d^{i-2}) \subseteq D_i$, and
- $\mathrm{Supp}_R(H^{i-1}(C_R(\mathcal{D}, X))) \subseteq D_{i+1}$

(see [11, 13] for details). For all $i \geq 0$, we will use the notations $C^{i-2} := \mathrm{Coker}\, d^{i-2}$ and $H^{i-1} := H^{i-1}(C_R(\mathcal{D}, X))$.

The following lemma is needed in this paper.

Lemma 2.1. *Assume that*

$$X' \longrightarrow X \longrightarrow X''$$

is an exact sequence of R-modules. Then

$$(0 :_R X')(0 :_R X'') \subseteq (0 :_R X).$$

Proof. This is easy and left to the reader. □

Lemma 2.2. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and X an arbitrary R-module. Then $\mathrm{H}^i_\mathfrak{m}(X^j) = 0$ for all $i \geq 0$ and for all $j < \dim(R)$.*

Proof. Let i and j be integers such that $i \geq 0$ and $j < \dim(R)$. It is enough to show that $\mathrm{H}^i_\mathfrak{m}((\mathrm{Coker}\, d^{j-2})_\mathfrak{p}) = 0$ for all $\mathfrak{p} \in D_j \setminus D_{j+1}$ because $X^j = \bigoplus_{\mathfrak{p}\in D_j\setminus D_{j+1}} (\mathrm{Coker}\, d^{j-2})_\mathfrak{p}$. Let $\mathfrak{p}$ be a prime ideal in $D_j \setminus D_{j+1}$. Thus $\mathfrak{p} \subsetneqq \mathfrak{m}$ and so there is an element $r \in \mathfrak{m} \setminus \mathfrak{p}$. Since the multiplication map

$$(\mathrm{Coker}\, d^{j-2})_\mathfrak{p} \xrightarrow{r} (\mathrm{Coker}\, d^{j-2})_\mathfrak{p}$$

is an R-isomorphism, the multiplication map

$$\mathrm{H}^i_{\mathfrak{m}}((\operatorname{Coker} d^{j-2})_{\mathfrak{p}}) \xrightarrow{r} \mathrm{H}^i_{\mathfrak{m}}((\operatorname{Coker} d^{j-2})_{\mathfrak{p}})$$

is also an R-isomorphism and so $\mathrm{H}^i_{\mathfrak{m}}((\operatorname{Coker} d^{j-2})_{\mathfrak{p}}) = 0$. □

Lemma 2.3. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and X an arbitrary R-module. Then*

$$\prod_{i=0}^{j}(0 :_R \mathrm{H}^{j-i}_{\mathfrak{m}}(H^{i-1})) \subseteq (0 :_R \mathrm{H}^j_{\mathfrak{m}}(X))$$

for all $j < \dim(R)$.

Proof. Let j be an integer such that $j < \dim(R)$ and let i be an integer such that $0 \leq i \leq j$. By considering the short exact sequences

$$0 \longrightarrow \frac{C^{i-2}}{H^{i-1}} \longrightarrow X^i \longrightarrow C^{i-1} \longrightarrow 0$$

and

$$0 \longrightarrow H^{i-1} \longrightarrow C^{i-2} \longrightarrow \frac{C^{i-2}}{H^{i-1}} \longrightarrow 0,$$

we have the long exact sequences

$$\begin{array}{llllll}
0 & \longrightarrow & \Gamma_{\mathfrak{m}}(\frac{C^{i-2}}{H^{i-1}}) & \longrightarrow & \Gamma_{\mathfrak{m}}(X^i) & \longrightarrow & \Gamma_{\mathfrak{m}}(C^{i-1}) \\
 & \longrightarrow & \mathrm{H}^1_{\mathfrak{m}}(\frac{C^{i-2}}{H^{i-1}}) & \longrightarrow & \mathrm{H}^1_{\mathfrak{m}}(X^i) & \longrightarrow & \mathrm{H}^1_{\mathfrak{m}}(C^{i-1}) \\
 & \longrightarrow & \cdots & & & & \\
 & \longrightarrow & \mathrm{H}^{j-i-1}_{\mathfrak{m}}(\frac{C^{i-2}}{H^{i-1}}) & \longrightarrow & \mathrm{H}^{j-i-1}_{\mathfrak{m}}(X^i) & \longrightarrow & \mathrm{H}^{j-i-1}_{\mathfrak{m}}(C^{i-1}) \\
 & \longrightarrow & \mathrm{H}^{j-i}_{\mathfrak{m}}(\frac{C^{i-2}}{H^{i-1}}) & \longrightarrow & \mathrm{H}^{j-i}_{\mathfrak{m}}(X^i) & \longrightarrow & \mathrm{H}^{j-i}_{\mathfrak{m}}(C^{i-1}) \\
 & \longrightarrow & \cdots & & & &
\end{array}$$

and

$$\begin{array}{llllll}
0 & \longrightarrow & \Gamma_{\mathfrak{m}}(H^{i-1}) & \longrightarrow & \Gamma_{\mathfrak{m}}(C^{i-2}) & \longrightarrow & \Gamma_{\mathfrak{m}}(\frac{C^{i-2}}{H^{i-1}}) \\
 & \longrightarrow & \mathrm{H}^1_{\mathfrak{m}}(H^{i-1}) & \longrightarrow & \mathrm{H}^1_{\mathfrak{m}}(C^{i-2}) & \longrightarrow & \mathrm{H}^1_{\mathfrak{m}}(\frac{C^{i-2}}{H^{i-1}}) \\
 & \longrightarrow & \cdots & & & & \\
 & \longrightarrow & \mathrm{H}^{j-i}_{\mathfrak{m}}(H^{i-1}) & \longrightarrow & \mathrm{H}^{j-i}_{\mathfrak{m}}(C^{i-2}) & \longrightarrow & \mathrm{H}^{j-i}_{\mathfrak{m}}(\frac{C^{i-2}}{H^{i-1}}) \\
 & \longrightarrow & \cdots . & & & &
\end{array}$$

We have $\mathrm{H}^{j-i-1}_{\mathfrak{m}}(C^{i-1}) \cong \mathrm{H}^{j-i}_{\mathfrak{m}}(\frac{C^{i-2}}{H^{i-1}})$ because $\mathrm{H}^{j-i-1}_{\mathfrak{m}}(X^i) = 0 = \mathrm{H}^{j-i}_{\mathfrak{m}}(X^i)$ by Lemma 2.2. Thus

$$(0 :_R \mathrm{H}^{j-i}_{\mathfrak{m}}(H^{i-1}))(0 :_R \mathrm{H}^{j-i-1}_{\mathfrak{m}}(C^{i-1})) \subseteq (0 :_R \mathrm{H}^{j-i}_{\mathfrak{m}}(C^{i-2}))$$

from Lemma 2.1. Therefore we get

$$\begin{aligned}
\prod_{i=0}^{j}(0 :_R \mathrm{H}^{j-i}_{\mathfrak{m}}(H^{i-1})) &\subseteq (\prod_{i=0}^{j-1}(0 :_R \mathrm{H}^{j-i}_{\mathfrak{m}}(H^{i-1})))(0 :_R \mathrm{H}^0_{\mathfrak{m}}(C^{j-2})) \\
&\subseteq (\prod_{i=0}^{j-2}(0 :_R \mathrm{H}^{j-i}_{\mathfrak{m}}(H^{i-1})))(0 :_R \mathrm{H}^1_{\mathfrak{m}}(C^{j-3}))
\end{aligned}$$

$$\begin{aligned}&\subseteq \cdots\\ &\subseteq (0 :_R \mathrm{H}^j_{\mathfrak{m}}(H^{-1}))(0 :_R \mathrm{H}^{j-1}_{\mathfrak{m}}(C^{-1}))\\ &\subseteq (0 :_R \mathrm{H}^j_{\mathfrak{m}}(C^{-2}))\\ &= (0 :_R \mathrm{H}^j_{\mathfrak{m}}(X))\end{aligned}$$

as desired. □

Lemma 2.4. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and X an arbitrary R-module. Then*

$$\prod_{i=0}^{j}(0 :_R H^{i-1}) \subseteq \bigcap_{i=0}^{j}(0 :_R \mathrm{H}^i_{\mathfrak{m}}(X))$$

for all $j < \dim(R)$.

Proof. This follows from Lemma 2.3. □

In the following theorem, which is the main result of this paper, we show that any equidimensional local ring which has finite Cousin cohomology modules with respect to the dimension filtration has a uniform local cohomological annihilator.

Theorem 2.5 (see [4, Theorem 2.7])**.** *Suppose that R is an equidimensional local ring with maximal ideal $\mathfrak{m}$ such that $H^i(C_R(\mathcal{D}, R))$ is finite for all i. Then R has a uniform local cohomological annihilator.*

Proof. For all $i \geq -1$, we use the notation $H^i := H^i(C_R(\mathcal{D}, R))$. Since R is equidimensional, $D_1 \cap \mathrm{Min}_R(R) = \emptyset$. Also, we have

$$\begin{aligned}\{\mathfrak{p} \in \mathrm{Spec}(R) : (0 :_R H^i) \subseteq \mathfrak{p}\} &= \mathrm{Supp}_R(H^i)\\ &\subseteq D_{i+2}\\ &\subseteq D_1\end{aligned}$$

for all $i \geq -1$ because H^i is finite for all $i \geq -1$. Hence

$$\prod_{i=-1}^{\dim(R)-2}(0 :_R H^i) \nsubseteq \bigcup_{\mathfrak{p}\in \mathrm{Min}_R(R)} \mathfrak{p}$$

and so there exists an element

$$r \in \prod_{i=-1}^{\dim(R)-2}(0 :_R H^i) \setminus \bigcup_{\mathfrak{p}\in \mathrm{Min}_R(R)} \mathfrak{p}.$$

On the other hand, we have

$$\prod_{i=-1}^{\dim(R)-2}(0 :_R H^i) \subseteq \bigcap_{i<\dim(R)}(0 :_R \mathrm{H}^i_{\mathfrak{m}}(R))$$

by Lemma 2.4. Thus, for all $i < \dim(R)$, $r\,\mathrm{H}^i_{\mathfrak{m}}(R) = 0$. □

We end this paper by showing that any equidimensional local ring which has finite Cousin cohomology modules with respect to the dimension filtration is universally catenary.

Corollary 2.6. *Suppose that R is an equidimensional local ring such that $H^i(C_R(\mathcal{D}, R))$ is finite for all i. Then R is universally catenary.*

Proof. This follows from Theorem 2.5 and [14, Theorem 2.1(ii)]. □

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Alireza Vahidi
Department of Mathematics
Payame Noor University (PNU)
P.O.BOX, 19395-4697, Tehran, Iran
Email address: vahidi.ar@pnu.ac.ir