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UNIQUENESS OF FAMILIES OF MINIMAL SURFACES IN $\mathbb{R}^3$

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Abstract. We show that an umbilic-free minimal surface in $\mathbb{R}^3$ belongs to the associate family of the catenoid if and only if the geodesic curvatures of its lines of curvature have a constant ratio. As a corollary, the helicoid is shown to be the unique umbilic-free minimal surface whose lines of curvature have the same geodesic curvature. A similar characterization of the deformation family of minimal surfaces with planar lines of curvature is also given.

1. Introduction

In differential geometry, Liouville's equation in $\mathbb{R}^3$ is a nonlinear partial differential equation satisfied by the conformal factor of a metric

$$\mathrm{d}s^2 = e^{2w}(\mathrm{d}u^2 + \mathrm{d}v^2)$$

on a surface of constant Gaussian curvature K:

$$\triangle w = -Ke^{-2w}, \tag{1.1}$$

where $\triangle = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$.

It is, in fact, a consequence of the Gauss equation in isothermal coordinates. In [1], it was shown that, when $K \equiv -1$, the entire solution of (1.1) determines a unique global meromorphic function g such that

$$e^{w(u,v)} = \frac{1 + |g(z)|^2}{2|g'(z)|}, \quad z = u + iv \in \mathbb{C} \tag{1.2}$$

up to the Möbius transformations $\frac{ag - \bar{b}}{bg + \bar{a}}$, $|a|^2 + |b|^2 > 0$.

The function g is closely related to the Weierstrass representation formula for a minimal surface:

$$X(u,v) = \mathrm{Re} \int \left(\frac{1}{2}(1 - g^2), \frac{i}{2}(1 + g^2), g \right) \eta,$$

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with a meromorphic function g and a homomorphic one form η with $g^2\eta$ being holomorphic over a simply connected domain in $\mathbb{C}$. Such $\{\eta, g\}$ is called a Weierstrass pair. It is now clear that this representation formula relates the minimal surface theory to the complex analysis. In particular, a minimal surface can be determined solely by g when the coordinate curves are the lines of curvature.

The relation between Liouville's equation and minimal surfaces is well described in [2], where some solutions of (1.2) and the corresponding minimal surfaces are listed. In addition, in [3] it was shown how Liouville's equation can be used in the classification of minimal surfaces with planar lines of curvature in $\mathbb{R}^3$. In fact, Nitsche gave a complete classification of minimal surfaces with planar lines of curvature in $\mathbb{R}^3$ by analyzing the orthogonal families of circles [7]. His method was generalized by Leite [6] to give a full classification of maximal surfaces with planar lines of curvature in Lorentz–Minkowski space $\mathbb{R}^{2,1}$.

On a separate note, by using the notion of a Chebyshev net, Riveros and Corro [8] characterized the catenoid as the only non-planar minimal surface in $\mathbb{R}^3$ whose asymptotic lines have the same geodesic curvature. The fact that the associate minimal surfaces share the same first fundamental form enabled them to conclude that a set of coordinate curves of the minimal surface associated to the catenoid have the same geodesic curvature. Moreover, in [9], the authors classified GICM-surfaces, defined by the class of minimal surfaces in $\mathbb{R}^3$ such that one family of coordinate curves among its isothermal coordinates have zero geodesic curvature.

In this article, we show that in the line of curvature coordinates, the geodesic curvatures of a pair of coordinate curves can be expressed by the conformal factor of the metric. This observation leads us to the characterization theorems on the helicoid, the catenoid, and the minimal surfaces associated to them. More specifically, the minimal surfaces associated to the catenoid are shown to be the only minimal surfaces such that geodesic curvatures of their lines of curvature have a constant ratio. As a corollary, the helicoid is shown to be the unique umbilic-free minimal surface whose lines of curvature have the same geodesic curvature. In addition, by applying this result to the conjugate surfaces of the helicoid, we obtain the same characterization result obtained in [8] mentioned above.

It is noteworthy that according to our result, the ratio of geodesic curvatures of lines of curvature indicates a specific surface among the surfaces associated to the catenoid. In other words, each value of the ratio of the geodesic curvatures corresponds to one specific surface in the associate family of the catenoid. As for the method of proofs, our setting avoids solving complicated partial differential equations. Finally, a different characterization of the deformation family of minimal surfaces introduced in [3] that includes Enneper surface and Bonnet surfaces will also be given.

2. Preliminaries

It is well known that the lines of curvature can be introduced as parameter curves in a neighborhood of any point, except umbilical or planar points, provided that the surface has parametrization with continuous derivatives of the third order. See [10] for details. Because we consider umbilic-free minimal surfaces, lines of curvature are allowed to be used as parameter curves.

Conversely, given a minimal surface without umbilic points in a simply connected region, one can always find the parameters such that the coordinate lines are lines of curvature and the conformal factor in the metric corresponds to a solution of the Liouville equation (1.1). The following is a collection of well-known facts on the relation between Liouville's equation and minimal surfaces that was investigated in [2] and [3].

Proposition 1. *Let $X : \Sigma \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a conformal minimal immersion of a simply connected domain Σ. Let the metric be given by $\mathrm{d}s^2 = E(\mathrm{d}u^2 + \mathrm{d}v^2)$ and let the second fundamental form be represented by $l\,\mathrm{d}u^2 + 2m\,\mathrm{d}u\,\mathrm{d}v + n\,\mathrm{d}v^2$. Let $z = u + iv$ be a complex coordinate in the neighborhood of a point in Σ. Then, there always exists a change of variables $z \to \tilde{z}$ such that the Hopf differential $\mathcal{H} := \left(\frac{l-n}{2} - im\right)\mathrm{d}z^2$ becomes $\mathcal{H} = -\mathrm{d}\tilde{z}^2$. $\tilde{z}$ is called a Liouville parameter. Moreover, in the Liouville parameter, the following hold.*

(i) *$E = e^{2w}$ with the Liouville equation $\triangle w = e^{-2w}$.*

(ii) *Coordinate lines are lines of curvatures.*

(iii) *The Gauss–Weingarten equations become*

$$\begin{cases} X_{uu} = w_u\,X_u - w_v\,X_v - \vec{N}, \\ X_{uv} = w_v\,X_u + w_u\,X_v, \\ X_{vv} = -w_u\,X_u + w_v\,X_v + \vec{N}. \end{cases}$$

(iv) *The Weierstrass pair is given by $\{\frac{\mathrm{d}z}{g'}, g\}$, where g is the stereographic projection of the oriented normal of X, globally meromorphic with $g'(z) \neq 0$ at all regular points and admitting only simple poles.*

(v) *An entire solution of the Liouville equation (1.1) is expressed by*

$$e^w(u,v) = \frac{1 + |g(z)|^2}{2|g'(z)|}, \quad z = u + iv \in \mathbb{C}. \tag{2.1}$$

Moreover, g and its transformations $\frac{ag-\bar{b}}{bg-\bar{a}}$, $|a|^2 + |b|^2 > 0$, give all the possibilities for (2.1) to hold.

Proof. Fact (iii) is easily derived from the general Gauss–Weingarten equations by using the fact that the coordinates are conformal with $E = e^{2w}$, $l = -1 = -n$, and $m = 0$. The other facts follow from the series of propositions in section 2 of [2]. □

Note that every minimal surface has one parameter family of minimal surfaces that share the same Weierstrass data. More specifically, if a minimal surface $X \subset \mathbb{R}^3$ has the following Weierstrass representation,

$$X(u,v) = \mathrm{Re} \int \left(\frac{1}{2} f(1-g^2), \frac{i}{2} f(1+g^2), fg \right) \mathrm{d}z,$$

with Weierstrass pair $\{\eta := f(z)\mathrm{d}z,\ g\}$, then a family of isometric minimal surfaces associated to X is defined by

$$X^\theta(u,v) = \mathrm{Re} \left\{ e^{i\theta} \int \left(\frac{1}{2} f(1-g^2), \frac{i}{2} f(1+g^2), fg \right) \mathrm{d}z \right\}.$$

The collection of such surfaces $(X^\theta)_{0 \leq \theta < 2\pi}$ is often called the associate family of X. Intuitively, any minimal surface in the family can be deformed to another one in the same family without tearing or stretching. See [4] for more details. The most famous example of this is the associate family of the catenoid. Let us denote it by Cat^θ. Recall that the catenoid can be given by taking $g(z) = -e^z$ and $f(z) = e^{-z}$. Therefore, it is easy to see that an explicit parametrization of Cat^θ is

$$\begin{aligned} Cat^\theta(u,v) = (&\cos\theta \cosh u \cos v + \sin\theta \sinh u \sin v, \\ &\cos\theta \cosh u \sin v - \sin\theta \sinh u \cos v, u\cos\theta + v \sin\theta). \end{aligned}$$

Note that as θ varies from 0 to $\pi/2$, the corresponding surface transforms from a catenoid to a helicoid.

3. Main results

Let Σ be a non-umbilic minimal surface in the lines of curvature coordinates. Then, owing to Proposition 1, the first and second fundamental forms of Σ become $I = e^{2w}(\mathrm{d}u^2 + \mathrm{d}v^2)$ and $II = -\mathrm{d}u^2 + \mathrm{d}v^2$, respectively.

Lemma 1. *Let $(\kappa_g)_{l_1}$ and $(\kappa_g)_{l_2}$ be the geodesic curvatures of a pair of the lines of curvature l_1 (u-parameter curve) and l_2 (v-parameter curve) at a point $p \in \Sigma$. Then, $(\kappa_g)_{l_1} = \frac{-w_v}{e^w}$ and $(\kappa_g)_{l_2} = \frac{w_u}{e^w}$.*

Proof. Let $C(t) = X(t\cos\phi, t\sin\phi)$ be a curve on Σ parametrized by $X(u,v)$ in the neighborhood of a non-umbilic point $p \in \Sigma$, where ϕ is the angle between $C'(t)$ and the u-parameter curve $X_u(u, v_0)$ at p, independent of t. Then, an elementary calculation together with Proposition 1(iii) shows that

$$\begin{aligned} \frac{\mathrm{d}^2 C}{\mathrm{d}s^2} = e^{-2w} \Big\{ &(-\cos\phi\, w_u - \sin\phi\, w_v)(\cos\phi\, X_u + \sin\phi\, X_v) \\ &+ \cos^2\phi \left(w_u X_u - w_v X_v - \vec{N} \right) + 2\cos\phi \sin\phi\, (w_v X_u + w_u X_v) \\ &+ \sin^2\phi (-w_u X_u + w_v X_v + \vec{N}) \Big\}. \end{aligned}$$

The tangential component of $\frac{\mathrm{d}^2 C}{\mathrm{d}s^2}$ becomes

$$\left. \frac{\mathrm{d}^2 C}{\mathrm{d}s^2} \right|_{T\Sigma} = e^{-w} (\sin\phi\, w_u - \cos\phi\, w_v) \cdot e^{-w} (\cos\phi\, X_v - \sin\phi\, X_u).$$

Therefore, $\kappa_g = e^{-w}(\sin\phi\, w_u - \cos\phi\, w_v)$ and, in particular,

$$(\kappa_g)_{l_1} = \kappa_g \bigg|_{\phi=0} = \frac{-w_v}{e^w} \text{ and } (\kappa_g)_{l_2} = \kappa_g \bigg|_{\phi=\pi/2} = \frac{w_u}{e^w}. \qquad \square$$

In the previous lemma, we saw that the geodesic curvatures of lines of curvature of a minimal surface can be expressed in the partial derivatives of w with respect to u and v separately. Hence, conditions on the geodesic curvatures of lines of curvature can be converted into partial differential equations of w. Note that $w(u, v)$ also satisfies Liouville's equation (1.1), because (u, v) is the line of curvature coordinates. Along with Liouville's equation, we thus obtain a system of partial differential equations. Solving these equations enables us to obtain characterizations of certain families of minimal surfaces.

First, let us provide a characterization of the associate family of the catenoid. It was denoted by Cat^θ and an explicit parametrization of this family was given in Section 2.

Theorem 1. *The ratio λ of the geodesic curvatures of the lines of curvatures on a minimal surface Σ is constant if and only if Σ is $Cat^{2\theta}$, one of the associate family of the catenoid to the helicoid, with $\theta = \arctan\lambda$. Here λ varies from* 0 *to* 1 *as Σ changes from a catenoid to a helicoid.*

The following lemma is essential in proving our theorem. Recall that a conformal parameter $z = u + iv \in \mathbb{C}$ is called a Liouville parameter if the Hopf differential $\mathcal{H} = \left(\frac{l-n}{2} - im\right) \mathrm{d}z^2$ has the local coefficient -1.

Lemma 2. *Let D be a simply connected domain in $\mathbb{C}$. If $z = u+iv \in D \subset \mathbb{C}$ is a Liouville parameter for a minimal immersion $X : D \to \mathbb{R}^3$, then $\tilde{z} := e^{\frac{i\theta}{2}} z = \left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)(u + iv)$ is a Liouville parameter for the isometric immersion X^θ associated to X.*

Proof. Let $\{f(z)\,\mathrm{d}z, g(z)\}$ be a Weierstrass pair for X. Then, the Weierstrass pair for X^θ is $\left(e^{i\theta} f(z)\,\mathrm{d}z, g(z)\right)$. Because z is a Liouville parameter for X and, therefore, coordinate lines are lines of curvature by Proposition 1, we have $f(z)g'(z) = 1$. Again from Proposition 1, for $\tilde{z}$ to be a Liouville parameter, $\tilde{z}$ should satisfy $\mathcal{H}(\tilde{z}) = -\mathrm{d}\tilde{z}^2$. This is equivalent to

$$e^{i\theta} f(z) g'(z) \left(\frac{\mathrm{d}z}{\mathrm{d}\tilde{z}}\right)^2 = 1.$$

Hence, $\mathrm{d}\tilde{z} = \sqrt{e^{i\theta}}\,\mathrm{d}z$, i.e., $\tilde{z} = e^{i\theta/2} z$ is a Liouville parameter for X^θ. $\square$

Proof of Theorem 1. First, assume that the ratio λ of the geodesic curvatures of the lines of curvatures on Σ is constant. Then, without loss of generality, the constant ratio condition can be written as $(\kappa_g)_{l_1} = \lambda(\kappa_g)_{l_2}$. Therefore, the following holds:

$$(\kappa_g)_{l_1} = \lambda(\kappa_g)_{l_2}$$

$$\Leftrightarrow \frac{1}{\sqrt{1+\lambda^2}}(\kappa_g)_{l_1} - \frac{\lambda}{\sqrt{1+\lambda^2}}(\kappa_g)_{l_2} = 0$$

$$\Leftrightarrow \cos\theta\,\Big(-\frac{w_v}{e^w}\Big) - \sin\theta\,\frac{w_u}{e^w} = 0, \text{ where } \theta = \arctan\lambda$$

$$\Leftrightarrow \cos\theta\, w_v + \sin\theta\, w_u = 0. \tag{3.1}$$

Put $s := \cos\theta\, u - \sin\theta\, v$, $t := \sin\theta\, u + \cos\theta\, v$. Then,

$$\begin{cases} w_u = w_s s_u + w_t t_u = \cos\theta\, w_s + \sin\theta\, w_t, \\ w_v = w_s s_v + w_t t_v = -\sin\theta\, w_s + \cos\theta\, w_t. \end{cases}$$

Plugging these terms into (3.1), we obtain $w_t = 0$. Because $\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}$, we have

$$\begin{cases} w_t = 0, & \text{(3.2a)} \\ \triangle\, w = e^{-2w}, & \text{(3.2b)} \end{cases}$$

where $\triangle = \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2}$. Observe that (3.2a) implies that $w(s,t) = w(s)$. Hence, (3.2b) is equivalent to

$$w''(s) = e^{-2w(s)}.$$

Multiplying $2w'$ on both sides, we obtain

$$(w')^2 = c - e^{-2w} \quad \text{for some constant } c.$$

As w'' would be negative if $w' = -\sqrt{c - e^{-2w}}$, we have $w' = \sqrt{c - e^{-2w}}$. Thus,

$$\begin{aligned} s &= \int \frac{\mathrm{d}w}{\sqrt{c - e^{-2w}}} = \int \frac{e^w}{\sqrt{ce^{2w} - 1}}\,\mathrm{d}w \\ &= \frac{1}{\sqrt{c}} \int \frac{\sinh x}{\sqrt{\cosh^2 x - 1}}\,\mathrm{d}x \quad \text{by putting } \sqrt{c}\, e^w = \cosh x \\ &= \frac{1}{\sqrt{c}} x + d = \frac{1}{\sqrt{c}} \cosh^{-1}(\sqrt{c} e^w) + d \quad \text{for some constant } d. \end{aligned}$$

Therefore, we have $w = \log \frac{1}{\sqrt{c}} \cosh(\sqrt{c}(s - d))$. Observe that c and d can be seen as a homothety factor and a translation factor, respectively. Hence, by setting $c = 1$ and $d = 0$, we have

$$w = \log\cosh s = \log\cosh(\cos\theta\, u - \sin\theta\, v).$$

We claim that $w = \log\cosh(\cos\theta\, u - \sin\theta\, v)$ is the solution of the Liouville equation for $Cat^{2\theta}$. To verify this, first observe that $g(z) = e^z$ globally represents the catenoid. This can be seen directly by calculating the coordinates explicitly through the Weierstrass representation formula:

$$\begin{cases} x_1 = \mathrm{Re} \int \frac{1-g^2}{2g'} = \mathrm{Re} \int \frac{1-e^{2z}}{2e^z} = -\mathrm{Re} \int \sinh z = -\cosh u \cos v, \\ x_2 = \mathrm{Re} \int \frac{1+g^2}{2g'} = \mathrm{Re} \int i \frac{1+e^{2z}}{2e^z} = \mathrm{Re} \int i \cosh z = \cosh u \sin v, \\ x_3 = \mathrm{Re} \int \frac{g}{g'} = \mathrm{Re} \int 1 = \mathrm{Re} z = u, \end{cases}$$

up to constants. Therefore, by Lemma 2, it is clear that

$$g(e^{i\theta z}) = e^{(\cos\theta + i\sin\theta)(u+iv)}$$

represents $Cat^{2\theta}$. Substituting it in (2.1) in Proposition 1, we conclude that

$$w = \log \frac{1+|g(z)|^2}{2|g'(z)|} = \log\cosh(\cos\theta\, u - \sin\theta\, v)$$

is the solution for Liouville's equation for $Cat^{2\theta}$.

Conversely, from the claim above, it is clear that each $Cat^{2\theta}$ is represented by $w = \log\cosh(\cos\theta\, u - \sin\theta\, v)$. Plugging this w into the equations $(\kappa_g)_{l_1} = \frac{-w_v}{e^w}$ and $(\kappa_g)_{l_2} = \frac{w_u}{e^w}$ in Lemma 1, it is straightforward that

$$(\kappa_g)_{l_1} = \frac{\sin\theta\, \tanh(\cos\theta\, u - \sin\theta\, v)}{e^w},\ (\kappa_g)_{l_2} = \frac{\cos\theta\, \tanh(\cos\theta\, u - \sin\theta\, v)}{e^w}.$$

This is equivalent to $(\kappa_g)_{l_1} = \lambda(\kappa_g)_{l_2}$ where $\lambda = \tan\theta$. □

This theorem leads us to the following characterization of the helicoid and the catenoid, which coincide with the result obtained in [8].

Corollary 1. (i) *The helicoid is the only minimal surface at every point of which two lines of curvature have the same geodesic curvature.*

(ii) *The catenoid is the only minimal surface at every point of which two asymptotic lines have the same geodesic curvature.*

Proof. (i) We have $(\kappa_g)_{l_1} = (\kappa_g)_{l_2} \Leftrightarrow \lambda = 1 \Leftrightarrow \Sigma = Cat^{2\cdot\frac{\pi}{4}} = Cat^{\frac{\pi}{2}}$. Because $Cat^{\frac{\pi}{2}}$ is the helicoid, we obtain the result.

(ii) In general, for a curve parametrized as $X(u(t), v(t))$ on a surface Σ, the geodesic curvature is given by

$$\kappa_g = \sqrt{EG - F^2}\ \left[\Gamma_{11}^2(u')^3 - \Gamma_{22}^1(v')^3 + \left(2\Gamma_{12}^2 - \Gamma_{11}^1\right)(u')^2 v' \right.$$
$$\left. - \left(2\Gamma_{12}^1 - \Gamma_{22}^2\right) u'(v')^2 - u''v' + v''u'\right] \left(E(u')^2 + 2Fu'v' + G(v')^2\right)^{-3/2},$$

where E, F, and G are coefficients of the first fundamental form of Σ and Γ_{ij}^k are the Christoffel symbols of the second kind. (See [5, pp. 544–545] for a proof.) This formula implies that the geodesic curvature depends only on the first fundamental form of the surface. On the other hand, it is well known that asymptotic lines of Σ become lines of curvature of $\Sigma^{\frac{\pi}{2}}$ as Σ transforms to $\Sigma^{\frac{\pi}{2}}$.

From these two properties, it is clear that if we assume the two asymptotic lines at each point of Σ have the same geodesic curvature, then the two lines of curvature at each point of $\Sigma^{\frac{\pi}{2}}$ have the same geodesic curvature. From Corollary 1(i), $\Sigma^{\frac{\pi}{2}}$ should be the helicoid and, therefore, Σ should be the catenoid. Likewise, the converse holds true. □

The notion of GICM-surfaces was introduced in [9] as the class of minimal surfaces in $\mathbb{R}^3$ having isothermal coordinates such that one of its coordinate curves are geodesic. The authors then proved that if a minimal surface with isothermal coordinates such that the geodesic curvatures κ_g^1 and κ_g^2 of coordinate curves satisfy $\alpha\kappa_g^1 + \beta\kappa_g^2 = 0$ for some $\alpha, \beta \in \mathbb{R}$, then it should be a GICM-surface. It is clear that, by definition, at least one of κ_g^1 and κ_g^2 is zero for a GICM-surface, therefore the converse holds. The following corollary shows the relation between their result and Theorem 1. Note that $(\alpha, \beta) \neq (0,0)$ for the expression $\alpha\kappa_g^1 + \beta\kappa_g^2 = 0$ to be non-trivial.

Corollary 2. *Let Σ be a minimal surface such that geodesic curvatures of its coordinate curves l_1, l_2 satisfy $\alpha\left(\kappa_g\right)_{l_1} + \beta\left(\kappa_g\right)_{l_2} = 0$, $\alpha, \beta \in \mathbb{R}$, $(\alpha, \beta) \neq (0,0)$. If l_1 and l_2 are lines of curvature, then Σ should be $Cat^{2\theta}$ where $\theta =$ $\arcsin \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$.*

Proof. Without loss of generality, assume $\alpha \neq 0$. Then, it is clear that $\left(\kappa_g\right)_{l_1} = \lambda\left(\kappa_g\right)_{l_2}$ with $\lambda = -\beta/\alpha$. Thus, if l_1 and l_2 are lines of curvature, by Theorem 1, $\Sigma = Cat^{2\theta}$ with $\theta = \arctan\lambda$ or, equivalently, $\theta = \arcsin \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$. □

Put another way, if Σ is a minimal surface with $\alpha\left(\kappa_g\right)_{l_1} + \beta\left(\kappa_g\right)_{l_2} = 0$, $\alpha, \beta \in \mathbb{R}$, $(\alpha, \beta) \neq (0,0)$, it should be a GICM-surface. If, in addition, the coordinate curves are lines of curvature, then, among all GICM-surfaces, Σ should be $Cat^{2\theta}$ with $\theta = \arcsin \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$.

Next, we turn our attention to the minimal surfaces with planar lines of curvature. Nitsche [7, §175 on p. 165] showed that these surfaces must be the plane, the catenoid, the Enneper surface, or one of the Bonnet surfaces. Using Lemma 1, let us calculate geodesic curvatures of lines of curvature of such surfaces.

(i) **Catenoid**. The catenoid is Cat^0 in the previous theorem, which implies $w = \log\cosh u$. Therefore, the geodesic curvatures are $\left(\kappa_g\right)_{l_1} = 0$ and $\left(\kappa_g\right)_{l_2} = \frac{\sinh u}{\cosh^2 u} = \frac{\sinh u}{e^{2w}}$.

(ii) **Enneper surface**. It is well known that the Enneper surface can be parametrized by $X(u,v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$ with the Weierstrass data expressed by $f(z) = 1$ and $g(z) = z$, $z = u + iv \in \mathbb{C}$. As $Im\{fg'\} = 0$, the coordinate curves are lines of curvature and $w = \log \frac{1+|g(z)|^2}{2|g'(z)|} = \log \frac{1+u^2+v^2}{2}$. Therefore, $\left(\kappa_g\right)_{l_1} = -\frac{v}{e^{2w}}$ and $\left(\kappa_g\right)_{l_2} = \frac{u}{e^{2w}}$.

(iii) **Bonnet surfaces**. For each $t > 0$, a parametrization of the Bonnet surface can be given by $\mathcal{B}_t(u,v) = (\cosh u\cos v + tu, \cosh u\sin v + tv, u) - \frac{te^{-u}}{2}(t\cos v, -t\sin v, 2\cos v)$. The corresponding Weierstrass data are $f(z) = e^{-z}$ and $g(z) = e^z + t$ with $t > 0$. Again, $Im\{fg'\} = 0$ implies $w = \log \frac{1+|g(z)|^2}{2|g'(z)|} = \log \frac{1+(e^u\cos v+t)^2+(e^u\sin v)^2}{2e^u} = \log \frac{e^u+(1+t^2)e^{-u}+2t\cos v}{2}$. Therefore, $\left(\kappa_g\right)_{l_1} = \frac{t\sin v}{e^{2w}}$ and $\left(\kappa_g\right)_{l_2} = \frac{e^u-(1+t^2)e^{-u}}{2e^{2w}}$.

Observe that the catenoid, the Enneper surface, and the Bonnet surfaces have geodesic curvatures of their lines of curvature in the form $\left(\kappa_g\right)_{l_1} = \zeta(v)/e^{2w}$

and $(\kappa_g)_{l_2} = \eta(u)/e^{2w}$ for some single-variable functions $\zeta(v)$ and $\eta(u)$. In fact, the following theorem shows that the converse is true.

Let E be the conformal factor of the first fundamental form of a minimal surface Σ and let $(\kappa_g)_{l_1}$ and $(\kappa_g)_{l_2}$ be the geodesic curvatures of a pair of lines of curvature l_1 and l_2, as before.

Theorem 2. *If* $(\kappa_g)_{l_1} = \zeta(v)/E$ *and* $(\kappa_g)_{l_2} = \eta(u)/E$ *for some single-variable* C^2*-functions* $\zeta(v)$ *and* $\eta(u)$*, then* Σ *must be one of the following*:

- *plane*;
- *catenoid*;
- *Enneper's surface*;
- *one of the Bonnet family.*

Proof. We have

$$\begin{cases} (\kappa_g)_{l_1} = \zeta(v)/E, \\ (\kappa_g)_{l_2} = \eta(u)/E \end{cases}$$

is equivalent to

$$\begin{cases} e^w w_u = \zeta(v), \\ e^w w_v = \eta(u). \end{cases}$$

This is equivalent to $(e^w w_u)_v = 0 = (e^w w_v)_u$, i.e., $e^w(w_u w_v + w_{uv}) = 0$. We claim that $(w_u w_v + w_{uv}) = 0$ if and only if the lines of curvature of Σ are plane curves. To see this, we follow the computation of Lemma 2.1 in [3]: Let $X(u,v)$ be a parametrization of Σ in the line of curvature coordinates in the neighborhood of a given point. By Proposition 1(iii), it is a straightforward computation that $\langle X_{uuu}, X_v\rangle = -2(w_v w_v + w_{uv})$ and $\langle X_{uuu}, \vec{N}\rangle = -w_u$. Thus,

$$\begin{aligned} \det(X_u, X_{uu}, X_{uuu}) &= \det(X_u, w_u X_u - w_v X_v - \vec{N}, X_{uuu}) \\ &= \det(X_u, -w_v X_v, \langle X_{uuu}, \vec{N}\rangle \vec{N}) \\ &\quad + \det(X_u, -\vec{N}, \langle X_{uuu}, X_v\rangle X_v) \\ &= (-w_v\langle X_{uuu}, \vec{N}\rangle + \langle X_{uuu}, X_v\rangle) \det(X_u, X_v, \vec{N}) \\ &= -(w_u w_v + w_{uv})|X_u||X_v| \\ &= -(w_u w_v + w_{uv})e^{2w} = 0. \end{aligned}$$

Hence, the u-parameter curves are torsion-free, and, in the same way, so are the v-parameter curves. In other words, the lines of curvature of Σ are plane curves and the claim holds true. Therefore, by Nitsche's classification theorem on minimal surfaces with plane lines of curvature, Σ should be one of the surfaces listed in the theorem. □

Note that the principal curvatures for a point in Σ in our setting are e^{-2w} and $-e^{-2w}$. This allows us to restate the theorem as follows.

Theorem 3 (Restated)**.** *Let* κ *be the maximum of principal curvature at a point of a minimal surface* Σ*. If* $(\kappa_g)_{l_1} = \kappa\,\zeta(v)$ *and* $(\kappa_g)_{l_2} = \kappa\,\eta(u)$ *for* *some single-variable C^2-functions $\zeta(v)$ and $\eta(u)$, then Σ must be the plane, the catenoid, the Enneper surface, or one of the Bonnet family.*

Finally, in [3], Cho and Ogata obtained a one-parameter family of these minimal surfaces by proving the existence of axial directions, which then enabled them to recover the Weierstrass data. Therefore, Theorems 2 and 3 are characterizations of families of minimal surfaces with planar lines of curvature in terms of geodesic curvatures.

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References

[1] F. Brito and M. L. Leite, *Uniqueness and globality of the Liouville formula for entire solutions of* $\frac{\partial^2 \log \lambda}{\partial z \partial \bar{z}} + \frac{\lambda}{2} = 0$, Arch. Math. (Basel) **80** (2003), no. 5, 501–506.

[2] F. Brito, M. L. Leite, and V. de Souza Neto, *Liouville's formula under the viewpoint of minimal surfaces*, Commun. Pure Appl. Anal. **3** (2004), no. 1, 41–51.

[3] J. Cho and Y. Ogata, *Deformation of minimal surfaces with planar curvature lines*, J. Geom. **108** (2017), no. 2, 463–479.

[4] Y. Fang, *Lectures on minimal surfaces in* $\mathbb{R}^3$, Proceedings of the Centre for Mathematics and its Applications, Australian National University, **35**, Australian National University, Centre for Mathematics and its Applications, Canberra, 1996.

[5] A. Gray, *Modern Geometry of Curves and Surfaces with Mathematica*, 3rd ed. Chapman and Hall/CRC Press, Boca Raton, FL, 2006.

[6] M. L. Leite, *Surfaces with planar lines of curvature and orthogonal systems of cycles*, J. Math. Anal. Appl. **421** (2015), no. 2, 1254–1273.

[7] J. C. C. Nitsche, *Lectures on Minimal Surfaces. Vol. 1*, translated from the German by Jerry M. Feinberg, Cambridge University Press, Cambridge, 1989.

[8] C. M. C. Riveros and A. M. V. Corro, *A characterization of the catenoid and helicoid*, Internat. J. Math. **24** (2013), no. 6, 1350045, 11 pp.

[9] ———, *Geodesics in minimal surfaces*, Math. Notes **101** (2017), no. 3-4, 497–514.

[10] J. J. Stoker, *Differential Geometry*, Pure and Applied Mathematics, Vol. XX, Interscience Publishers John Wiley & Sons, New York, 1969.

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