J. Korean Math. Soc. **55** (2018), No. 6, pp. 1435–1458
https://doi.org/10.4134/JKMS.j170731
pISSN: 0304-9914 / eISSN: 2234-3008

MONOTONICITY OF THE FIRST EIGENVALUE OF THE LAPLACE AND THE p-LAPLACE OPERATORS UNDER A FORCED MEAN CURVATURE FLOW

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Abstract. In this paper, we would like to give an answer to **Problem 1** below issued firstly in [17]. In fact, by imposing some conditions on the mean curvature of the initial hypersurface and the coefficient function of the forcing term of a forced mean curvature flow considered here, we can obtain that the first eigenvalues of the Laplace and the p-Laplace operators are monotonic under this flow. Surprisingly, during this process, we get an interesting byproduct, that is, without any complicate constraint, we can give lower bounds for the first nonzero closed eigenvalue of the Laplacian provided additionally the second fundamental form of the initial hypersurface satisfies a pinching condition.

1. Introduction

The mathematical genius, Perelman, in his famous work [19] introduced a functional, which is called $\mathcal{F}$-functional, for a prescribed closed Riemannian manifold (M,g) and a function f on M defined as follows

$$\mathcal{F}(g,f) := \int_M \left(R + |\nabla f|^2\right) e^{-f} d\mu,$$

with R here the scalar curvature and $d\mu$ the volume element of M. Denote by ∇ and Δ the gradient and the Laplace operators of M, respectively. For the following coupled system

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \\ \frac{\partial}{\partial t} f = -\Delta f - R + |\nabla f|^2, \end{cases}$$

with the first equation the famous Ricci-Hamilton flow, he proved that the $\mathcal{F}$-functional is nondecreasing under the Ricci flow, i.e.,

$$\frac{d}{dt}\mathcal{F} = 2\int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu \geq 0.$$

Received November 12, 2017; Accepted February 21, 2018.
2010 *Mathematics Subject Classification.* 58C40, 53C44.
Key words and phrases. Ricci-Hamilton flow, mean curvature flow, Laplace operator, p-Laplace operator.

Define

$$\lambda(g) := \inf \left\{ \mathcal{F}(g,f) \;\middle|\; f \text{ runs over all smooth functions, and satisfies } \int_M e^{-f} d\mu = 1 \right\},$$

and then $\lambda(g)$ is the lowest eigenvalue of the operator $(-4\Delta + R)$. This fact can be obtained easily by making a transformation $u = e^{-f/2}$. Then $\lambda(g)$ can be defined equivalently as follows:

$$\lambda(g) := \inf \left\{ \int_M (4|\nabla u|^2 + Ru^2) d\mu \;\middle|\; u \text{ runs over all smooth functions, and } \int_M u^2 d\mu = 1 \right\},$$

which implies that $\lambda(g) = \lambda_1(-4\Delta + R)$, the first eigenvalue of $(-4\Delta + R)$. Besides, $\lambda(g)$ is nondecreasing since $\mathcal{F}$ is nondecreasing. By using this fact, Perelman has shown that there are no nontrivial steady or expanding breathers on compact manifolds (see Sections 2, 3, and 4 of [19]).

From Perelman's this work, we know that monotonicity of the first eigenvalue of some operator related to the Laplacian under curvature flows, like the Ricci flow, should be worthy to be investigated. Because of this, many mathematicians have made efforts on this direction, and some interesting results have also been obtained after Perelman's pioneering work. For instance, Ma [12] studied the first eigenvalue of the Laplace operator Δ, subject to the Dirichlet boundary condition, on a compact domain, with smooth boundary in a compact or a complete noncompact manifold, under the unnormalized Ricci-Hamilton flow, and obtained the monotonicity of the first eigenvalue of Δ under several assumptions on the scalar curvature of the prescribed manifold therein. Cao [3] showed that, under the Ricci flow, the eigenvalues of the operator $(-\Delta + R/2)$, with R the scalar curvature, are non-decreasing for manifolds with nonnegative curvature operator, and then, by applying this monotonicity of the eigenvalues, he proved that the only steady Ricci breather with nonnegative curvature operator is the trivial one (see Section 4 of [3]). Without assuming the nonnegativity of the curvature operator, Li [11] also proved the nondecreasing property for the eigenvalues of the operator $(-\Delta + R/2)$. Cao [4] proved that, under the unnormalized Ricci flow, the first eigenvalue of $(-\Delta + cR)$, with $c \geq 1/4$ and R the scalar curvature, is nondecreasing, which generalized his previous work [3]. Recently, Cao, Hou, and Ling [5] derived a monotonicity formula for the first eigenvalue of the operator $(-\Delta + aR)$, with $0 < a \leq 1/2$, on closed surfaces with the scalar curvature $R \geq 0$ under the unnormalized Ricci flow.

The mean curvature flow (MCF) also has connections with the Ricci flow which is a powerful tool to solve the 3-dimensional Poincaré conjecture. There are surprising analogies between the Ricci flow and the MCF. Indeed, many results hold in a similar way for both flows, and several ideas have been successfully transferred from one context to the other (see, for instance, [10, Corollary 2.5], where we have used a principle, the maximum principle for tensors, appearing in the Ricci flow, supplied by Hamilton, to prove the convexity-preserving property for the curvature flow considered therein). However, at the moment there is no formal way of transforming one of them into the other.

Because of the deep connection between the MCF and the Ricci flow, it is natural to ask whether or not we could derive monotonicity formulas for the first eigenvalue of some geometric operators related to the Laplacian under the MCF or some other deformations of the MCF, like the volume-preserving MCF, the area-preserving MCF, the forced MCF (MCF with a prescribed forcing term), etc. Recently, under several assumptions on the mean curvature of a given closed Riemannian manifold, Zhao [23] proved that the first eigenvalue of the p-Laplacian on the manifold is nondecreasing along powers of the mth MCF (see, e.g., [2] for the basic information on this flow). This provides us the feasibility of trying to derive the monotonicity of the first eigenvalue of the Laplacian or the p-Laplacian under curvature flows.

Denote by M_0 a compact and strictly convex hypersurface of dimension $n \geq 2$, without boundary, smoothly embedded in the Euclidean space $\mathbb{R}^{n+1}$ and represented locally by a diffeomorphism $X_0 : U \subset \mathbb{R}^n \to X_0(U) \subset M_0 \subset \mathbb{R}^{n+1}$. Consider that M_0 evolves along the forced MCF defined as follows:

$$(1.1)\quad \begin{cases} \frac{\partial}{\partial t} X(x,t) = -H(x,t)\vec{v}(x,t) + \kappa(t)X(x,t), & x \in M_0^n, \quad t > 0, \\ X(\cdot,0) = X_0, \end{cases}$$

with $\vec{v}(x,t)$ the outer unit normal vector of $M_t = X_t(M_0)$ at $X(x,t) = X_t(x)$, H the mean curvature of M_t, and $\kappa(t)$ a continuous function of t. Li, Mao and Wu [10] proved that the convexity is preserving as the case of MCF, and the evolving convex hypersurfaces may shrink to a point in finite time if the forcing term is small, or exist for all time and expand to infinity if it is large enough (see [10, Theorem 1.1] or Theorem 2.1 here for the precise statement). In fact, the forced MCF (1.1) can be obtained by adding a forcing term in direction of the position vector to the classical MCF (only when the ambient space is a Euclidean space), and this type of forced (or forced hyperbolic) mean curvature flows has been studied in [10, 13, 14, 18] with some interesting results on the convergence or the long time existence obtained.

As pointed out in [10], the tangent component of $X(x,t)$ does not affect the behavior of the evolving hypersurface, but usually the normal component of $X(x,t)$ is not a unit normal vector, which leads to the fact that the flow (1.1) differs from the classical MCF. Readers can find that the convergent situation of our flow (1.1) is more complicated than that of the MCF even if the initial hypersurface is a sphere (see Remark 2.2). In fact, it can be seen as an extension of the MCF, since the flow (1.1) degenerates to be the MCF if $\kappa(t) \equiv 0$.

Based on the result concerning the convergence or the long time existence we have obtained in [10], and the fact that Zhao can get a monotonicity formula for the first eigenvalue of the p-Laplacian under powers of the mth MCF in [23], we might consider the following problem.

Problem 1. *For a compact and strictly convex hypersurface M_0 of dimension $n \geq 2$, without boundary, which is embedded smoothly in $\mathbb{R}^{n+1}$ and can be represented locally by a diffeomorphism $X_0 : U \subset \mathbb{R}^n \to X_0(U) \subset M_0 \subset \mathbb{R}^{n+1}$, could we derive a monotonicity formula for the first eigenvalue of the Laplace and the p-Laplace operators on M_t under the forced MCF defined by* (1.1)*?*

Several eigenvalue problems have been studied by the author in [6, 15–17] and some interesting conclusions have been obtained therein. This experience somehow supplies the possibility to answer the above *Problem 1.* In fact, based on the main conclusions for the flow (1.1) in [10], we can give an answer to this problem (see Theorem 5.1 for the details).

As mentioned in the Abstract, during the process of trying to get the monotonicity of the first non-zero closed eigenvalue, we can obtain an interesting byproduct, which somehow reveals the convergence or expansion of the evolving hypersurfaces under the flow (1.1) from the aspect of eigenvalues. As in Section 2, denote by H the mean curvature, h_{ij} and g_{ij} the components of the second fundamental form and the Riemannian metric of the prescribed manifold, respectively. By imposing a pinching condition for the second fundamental form of the initial hypersurface, we can prove the following.

Theorem 1.1. *If, in addition, there exist positive constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that the initial hypersurface M_0 satisfies*

$$h_{ij} = \alpha_i H g_{ij}, \qquad \text{where } \sum_{i=1}^{n} \alpha_i = 1 \text{ and } \left|\alpha_i - \frac{1}{n}\right| \leq \epsilon \tag{1.2}$$

for small enough ϵ only depending on n, then under the flow (1.1) *we have*

$$\lambda_1(t) \geq e^{-2\int_0^t \kappa(\tau)d\tau} \cdot \lambda_1(0)$$

for any $0 \leq t < T_{\mathrm{m}}$, where, of course, $\lambda_1(0)$ and $\lambda_1(t)$ are the first nonzero closed eigenvalues of the Laplace operator on M_0 and M_t respectively, and T_{m} is defined by (3.7)*.*

Remark 1.2. For an n-dimensional compact, connected and oriented Riemannian manifold (M, g) without boundary isometrically immersed in $\mathbb{R}^{n+1}$, it is said to be *almost-umbilical* if there exists $\theta \in (0, 1)$ such that $\|A - cg\|_\infty \leq \epsilon$ for a positive constant c, with ϵ small enough depending on n, c and θ, where A is the second fundamental form of M. So, clearly, if the initial hypersurface M_0 satisfies the pinching condition (1.2), then it is almost-umbilical. A well-known result states that a totally umbilical hypersurface of $\mathbb{R}^{n+1}$ which is not totally geodesic is a round sphere. Clearly a totally umbilical hypersurface of $\mathbb{R}^{n+1}$ must be almost-umbilical with $c = H/n$. However, an almost-umbilical hypersurface of $\mathbb{R}^{n+1}$ may not be totally umbilical. For instance, considering a sphere with ideal elasticity in $\mathbb{R}^3$, and orthogonally and very slightly squashing this sphere at a pair of antipodal points such that the new geometric object (might be an ellipsoid) obtained by this deformation satisfies the almost-umbilical condition. In this case, the deformation of the sphere might be ignored but it do has deformation. Therefore, it is natural to ask if and how the almost-umbilical hypersurfaces are "*close*" to round spheres. In fact, there are many interesting conclusions walking on this direction. For instance, Shiohama and Xu [21, 22] proved that almost-umbilical hypersurfaces of Euclidean space are homeomorphic to the sphere if imposing a condition on Betti numbers. Recently, Roth [20] proved that an n-dimensional compact, connected and oriented *almost-umbilical* Riemannian manifold M without boundary isometrically immersed in $\mathbb{R}^{n+1}$ is diffeomorphic and θ-quasi-isometric to $\mathbb{S}^n(\frac{1}{c})$, i.e., there exists a diffeomorphism F from M into $\mathbb{S}^n(\frac{1}{c})$ such that, for any $x \in M$ and any unitary vector $X \in T_xM$, we have $\big||d_xF(X)|^2 - 1\big| \leq \theta$. Hence, according to these facts, our pinching condition (1.2) is feasible and also reasonable. Especially, for (1.2), when $\alpha_i = 1/n$ for each $1 \leq i \leq n$, then the initial hypersurface M_0 must be a sphere with a prescribed radius, say r_0, and moreover, the evolving hypersurface M_t must be a sphere with radius $r(t)$ given by (2.10) (see Remark 2.2 for details). Correspondingly, $\lambda_1(t) = n/r^2(t)$, which clearly satisfies the conclusion of Theorem 1.1.

The paper is organized as follows. We recall some basic knowledge about the Laplacian and the p-Laplacian in the next section. Besides, we also mention some useful conclusions of the forced MCF (1.1). In Section 3, we give the proofs of Theorems 3.1 and 3.3. In Section 4, by applying Theorem 3.1, we successfully give lower bounds for the first nonzero closed eigenvalue of the Laplace operator provided, in addition, the initial hypersurface satisfies the pinching condition (1.2). Theorem 5.1 will be proved in the last section.

2. Preliminaries

In this section, we would like to give a brief introduction to the eigenvalue problem first and then recall some facts about the forced MCF (1.1).

In fact, due to the related conditions, the eigenvalue problem can be classified into several types, but here we just focus on the closed eigenvalue problem. For the consistency of the symbols, as before, let M_0 be an n-dimensional compact Riemannian manifold without boundary. The so-called closed eigenvalue problem is actually to find all possible real λ such that there exists non-trivial functions u satisfying

$$\Delta u + \lambda u = 0 \quad \text{on } M_0$$

with Δ the Laplacian on M_0, which is given by

$$\Delta u = \operatorname{div}(\nabla u) = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det(g_{ij})} g^{ij} \frac{\partial u}{\partial x_j} \right)$$

in a local coordinate system $\{x_1, x_2, \ldots, x_n\}$ of M_0. Here div and ∇ denote the divergence operator and the gradient operator on M_0, respectively. Moreover, $|\nabla u|^2 = |\nabla u|_g^2 = \sum_{i,j=1}^n g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$, and $(g^{ij}) = (g_{ij})^{-1}$ is the inverse of the metric matrix. It is well-known that Δ only has discrete spectrum in this setting (M_0 is compact without boundary). Each element in the discrete spectrum is called the eigenvalue of the Laplacian Δ. It is easy to find that 0 is an eigenvalue of Δ and whose eigenfunction should be chosen to be a constant function. By Rayleigh's theorem and Max-min principle, together with the fact that eigenfunctions belonging to different eigenvalues are orthogonal, we know that the first non-zero (i.e., the lowest non-zero) closed eigenvalue $\lambda_1(M)$ (λ_1 for short) can be characterized by

$$\lambda_1 = \inf \left\{ \frac{\int_{M_0} |\nabla u|^2 d\mu_0}{\int_{M_0} |u|^2 d\mu_0} \Big| u \neq 0, u \in W^{1,2}(M_0), \text{ and } \int_{M_0} u d\mu_0 = 0 \right\}, \tag{2.1}$$

where $W^{1,2}(M_0)$ is the completion of the set $C^\infty(M_0)$ of the smooth functions on M_0 under the Sobolev norm

$$\|u\|_{1,2} := \left(\int_{M_0} |u|^2 d\mu_0 + \int_{M_0} |\nabla u|^2 d\mu_0 \right)^{1/2},$$

and $d\mu_0$ denotes the volume element of M_0.

Now, we would like to make an agreement. That is, for the convenience, in the sequel we will drop the volume element for each integration appearing below. We also make an agreement on the range of indices as follows

$$1 \leq i, j, \ldots \leq n.$$

The p-Laplacian ($1 < p < \infty$) is a natural generalization of the Laplace operator. In fact, the so-called p-Laplacian eigenvalue problem is to consider the following nonlinear second-order partial differential equation (PDE for short)

$$\Delta_p u + \lambda |u|^{p-2} u = 0 \quad \text{on } M_0,$$

where, in local coordinates $\{x_1, \ldots, x_n\}$ on M_0, Δ_p is defined by

$$\Delta_p u = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\sqrt{\det(g_{ij})} g^{ij} |\nabla u|^{p-2} \frac{\partial u}{\partial x_j} \right).$$

Similar to the case of the linear Laplace operator, Δ_p has discrete spectrum on M_0 when M_0 is compact. However, we do not know whether it only has the discrete spectrum or not. This situation is different from the case of the Laplacian, when the domain considered is bounded. Besides, the first non-zero closed eigenvalue $\lambda_{1,p}(M_0)$ ($\lambda_{1,p}$ for short) of Δ_p can be characterized by

$$\lambda_{1,p} = \inf \left\{ \frac{\int_{M_0} |\nabla u|^p}{\int_{M_0} |u|^p} \Big| u \in W^{1,p}(M_0), u \neq 0, \text{ and } \int_{M_0} |u|^{p-2} u = 0 \right\}, \tag{2.2}$$

with $W^{1,p}(M_0)$ the completion of the set $C^\infty(M_0)$ under the Sobolev norm

$$\|u\|_{1,p} := \left(\int_{M_0} |u|^p + \int_{M_0} |\nabla u|^p\right)^{1/p}.$$

Now, we would like to recall several evolution equations derived in [10], which will be used to prove our main conclusions. In fact, for the unnormalized forced MCF (1.1), we have (cf. [10, Lemma 2.2])

$$\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij} + 2\kappa(t) g_{ij} \tag{2.3}$$

and

$$\begin{aligned} \frac{\partial h_{ij}}{\partial t} &= \Delta h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij} + \kappa(t) h_{ij}, \\ \frac{\partial H}{\partial t} &= \Delta H + |A|^2 H - \kappa(t) H, \end{aligned} \tag{2.4}$$

with g_{ij} the component of the Riemannian metric on M_t, H the mean curvature and h_{ij}, $|A|^2$ the component and the squared norm of the second fundamental form of M_t, respectively. Denote by $T_{\max}$ the maximal existence time of the forced MCF (1.1). In fact, the existence of $T_{\max} > 0$ can be obtained by the fact that the flow (1.1) is a parabolic equation and which can be converted to a second-order strictly parabolic PDE, leading to the existence of the maximal time interval $[0, T_{\max})$ (see, for instance, [13] for a detailed explanation of this kind of trick). In order to know more information about the flow (1.1) as $t \to T_{\max}$, as the case of the classical MCF, we have to make a rescale to this flow. More precisely, for any $t \in [0, T_{\max})$, let $\phi(t)$ be a positive factor such that the hypersurface $\widetilde{M}_t$ defined by $\widetilde{X}(x,t) = \phi(t) X(x,t)$ has total area equal to $|M_0|$ (i.e., the area of M_0). That is to say, $\int_{\widetilde{M}_t} = |M_0|$. Differentiating this equality with respect to t, we have

$$\phi^{-1}\frac{\partial \phi}{\partial t} = \frac{1}{n}\frac{\int_{M_t} H^2}{\int_{M_t}} - \kappa(t) = \frac{1}{n} h - \kappa(t). \tag{2.5}$$

At the same time, choosing a new time variable

$$\widetilde{t}(t) = \int_0^t \phi^2(\tau) d\tau = \int_0^t \phi^2(\tau),$$

then we have

$$\widetilde{g}_{ij} = \phi^2 g_{ij}, \quad \widetilde{h}_{ij} = \phi h_{ij}, \quad \widetilde{H} = \phi^{-1} H, \quad |\widetilde{A}|^2 = \phi^{-2}|A|^2,$$

and the evolution equation (1.1) becomes

$$\begin{cases} \frac{\partial}{\partial \widetilde{t}} \widetilde{X}(x,t) = -\widetilde{H} \cdot \widetilde{\vec{v}} + \frac{1}{n}\widetilde{h}\widetilde{X}, \\ \widetilde{X}(\cdot, 0) = \widetilde{X}_0, \end{cases} \tag{2.6}$$

where $\widetilde{h} = \phi^{-2}h = \int_{\widetilde{M_{\tilde{t}}}} \widetilde{H}^2 / \int_{\widetilde{M_{\tilde{t}}}}$. Clearly, we can obtain the normalized evolution equation for the metric as follows

$$\frac{\partial \widetilde{g}_{ij}}{\partial \tilde{t}} = \frac{\partial t}{\partial \tilde{t}} \frac{\partial(\phi^2 g_{ij})}{\partial t} = \frac{2}{n}\widetilde{h}\widetilde{g}_{ij} - 2\widetilde{H}\widetilde{h}_{ij}. \tag{2.7}$$

By [10], we know there always exists a time sequence $\{T_i\}$ in $[0, T_{\max})$ such that $T_i \to T_{\max}$ as $i \to \infty$, and moreover the limit

$$\lim_{T_i \to T_{\max}} \phi(T_i) = \Xi \tag{2.8}$$

holds (see the end of Section 4 of [10] for the detailed statement). About the forced MCF (1.1) and its normalized flow (2.6), Li, Mao and Wu proved the following conclusion (cf. [10, Theorem 1.1]).

Theorem 2.1. *Let M_0 be an n-dimensional smooth, compact and strictly convex hypersurface immersed in $\mathbb{R}^{n+1}$ with $n \geq 2$. Then for any continuous function $\kappa(t)$, there exists a unique, smooth solution to evolution equation* (1.1) *on a maximal time interval* $[0, T_{\max})$*. If additionally the following limit exists and satisfies*

$$\lim_{t \to T_{\max}} \kappa(t) = \overline{\kappa} \qquad \text{and} \qquad |\overline{\kappa}| < +\infty,$$

then we have

(I) *If $\Xi = \infty$, then $T_{\max} < \infty$ and the flow* (1.1) *converges uniformly to a point as $t \to T_{\max}$. Moreover, the normalized equation* (2.6) *has a solution $\widetilde{X}(x, \widetilde{t})$ for all times $0 \leq \widetilde{t} \leq \infty$, and its hypersurfaces $\widetilde{M}(x, \widetilde{t}) = \widetilde{M_{\tilde{t}}}$ converge to a round sphere of area $|M_0|$ in the C^∞-topology as $\widetilde{t} \to \infty$.*

(II) *If $0 < \Xi < \infty$, then $T_{\max} = \infty$ and the solutions to* (1.1) *converge uniformly to a sphere in the C^∞-topology as $t \to \infty$.*

(III) *If $\Xi = 0$, then $\overline{\kappa} \geq 0$ and $T_{\max} = \infty$. Moreover, if $\overline{\kappa} > 0$, the solutions to* (1.1) *expand uniformly to ∞ as $t \to \infty$, and the limit of the rescaled solutions to* (2.6) *must be a round sphere of total area $|M_0|$ if they converge to a smooth hypersurface.*

Remark 2.2. Here we want to reveal the difference between the flow (1.1) and the MCF by an example, through which readers can find that the flow (1.1) is not a simple and trivial extension of the classical MCF. Now, if the n-dimensional initial hypersurface M_0 is a sphere with radius r_0, clearly, it can be represented by

$$\begin{aligned} & X_0(r_0, \theta_1, \dots, \theta_n) \\ := & \,(r_0 \cos(\theta_1), r_0 \sin(\theta_1)\cos(\theta_2), r_0 \sin(\theta_1)\sin(\theta_2)\cos(\theta_3), \dots, \\ & \quad r_0 \sin(\theta_1)\cdots\sin(\theta_{n-1})\cos(\theta_n), r_0 \sin(\theta_1)\cdots\sin(\theta_{n-1})\sin(\theta_n)), \end{aligned}$$

where $r_0 > 0$ and $(\theta_1, \dots, \theta_{n-1}, \theta_n) \in \mathbb{S}^n$. Then the flow (1.1) becomes

$$\begin{cases} \frac{\partial}{\partial t} r(t) = -\frac{n}{r(t)} + \kappa(t) r(t), \\ r(0) = r_0, \end{cases} \tag{2.9}$$

since in this case the evolving hypersurfaces M_t $(0 < t < T_{\max})$ should be spheres under the flow (1.1) and can be represented by

$$\begin{aligned} &X_t(r_0,\theta_1,\ldots,\theta_n)\\ :=\ &(r(t)\cos(\theta_1), r_t\sin(\theta_1)\cos(\theta_2), r(t)\sin(\theta_1)\sin(\theta_2)\cos(\theta_3),\ldots,\\ &\quad r(t)\sin(\theta_1)\cdots\sin(\theta_{n-1})\cos(\theta_n), r(t)\sin(\theta_1)\cdots\sin(\theta_{n-1})\sin(\theta_n)). \end{aligned}$$

In fact, the assertion that M_t $(0 < t < T_{\max})$ is a sphere can be obtained by the fact that the flow (1.1) can preserve the property of being totally umbilical, i.e., $h_{ij} = Hg_{ij}/n$ (cf. Lemma 4.3). The first equation of (2.9) is a Bernoullie equation, and by direct computation, we can get

$$r(t) = \left(r_0^2 - 2n\int_0^t e^{-2\int_0^\tau \kappa(\xi)d\xi}d\tau\right)^{1/2}\cdot e^{\int_0^t \kappa(\tau)d\tau}. \tag{2.10}$$

Clearly, from (2.10) we know that the contraction or expansion of M_t depends on $\kappa(t)$ and r_0, and we can also get information of $T_{\max}$ by considering the first zero-point (if exists) of the function $r_0^2 - 2n\int_0^t e^{-2\int_0^\tau \kappa(\xi)d\xi}d\tau$. More precisely, if there exists some $t_0 < +\infty$ such that $r_0^2/2n = \int_0^{t_0} e^{-2\int_0^\tau \kappa(\xi)d\xi}d\tau$, then we have $T_{\max} = t_0$, i.e., M_t contracts to a single point at t_0; if there does not exist, then $T_{\max} = +\infty$, i.e., M_t expands to infinity. In order to let readers realize this clearly, we would like to investigate several different $\kappa(t)$ which let the flow (1.1) have different behaviors. For instance, if we choose $\kappa(t) = 1/(t+1)$, then by (2.10) we have

$$r(t) = \left(r_0^2 - 2n + \frac{2n}{t+1}\right)^{1/2}\cdot(t+1).$$

Clearly, if $0 < r_0 < \sqrt{2n}$, then $T_{\max} = r_0^2/(2n - r_0^2) < \infty$, and M_t contracts to a single point as $t \to T_{\max}$; if $\sqrt{2n} \le r_0 < \infty$, then $T_{\max} = +\infty$, and M_t expands uniformly to ∞ as $t \to \infty$. If we choose $\kappa(t) = -1/(t+1)$, then by (2.10) we have

$$r(t) = \left[r_0^2 - 2n\frac{(t+1)^3}{3} + \frac{2n}{3}\right]^{1/2}\cdot\frac{1}{t+1}.$$

Clearly, no matter how much r_0 is, M_t contracts to a single point as $t \to T_{\max}$ and $T_{\max} = \sqrt[3]{1 + \frac{3r_0^2}{2n}} - 1 < +\infty$. From these two examples, we know that different $\kappa(t)$ might let the flow (1.1) have different behaviors (i.e., contraction and expansion are all possible). However, Huisken [8] proved that an n-dimensional smooth, compact and strictly convex hypersurface immersed in $\mathbb{R}^{n+1}$ with $n \ge 2$ evolves under the MCF *would only contract to single point at a finite time*. In fact, if M_0 is a sphere which can be represented as above, then the MCF should become

$$\begin{cases} \frac{\partial}{\partial t}r(t) = -\frac{n}{r(t)},\\ r(0) = r_0. \end{cases}$$

So, $r(t)=\sqrt{r_0^2-2nt}$ and the maximal time is $T_{\max}=\frac{r_0^2}{2n}$. Clearly, even in this special setting (i.e., the initial hypersurface is a sphere), the situation of our flow (1.1) is more complicated than that of the MCF. Hence, the flow (1.1) cannot be seen as a simple extension of the MCF. From the above argument, one can realize that one needs to study the function $\kappa(t)$ and might also (if needed) the diameter (or equivalently, the mean curvature) of the initial hypersurface if he or she wants to investigate behaviors of the evolving hypersurfaces under the flow (1.1), and this difficulty has been solved in [10] by successfully finding a breakthrough, i.e., discussing the limit Ξ determined by (2.8), which in essence has relation with $\kappa(t)$ and the mean curvature of the initial hypersurface. However, in the case of the classical MCF, this problem does not exist. One cannot get Theorem 2.1 *only* by applying Huisken's method (i.e., L^p-estimate) in [8]. In fact, to prove Theorem 2.1, except the L^p-estimate tool, one might also have to use other tools introduced in [1, 9] (see [10] for the details).

However, the above process might only works for this special case (i.e., the initial hypersurface is a sphere) in which we can compute X_t directly. Actually, even in this special case when $\kappa(t)$ is complicated, for instance, choose $\kappa(t)=\sqrt{1+\frac{1}{t+4}\sqrt{\frac{1}{t+3}\sqrt{\frac{1}{t+2}\sqrt{\frac{1}{t+1}}}}}$, then it is not easy to compute directly. Of course, in this case, we might get the numerical value of $T_{\max}=t_0<\infty$ (if exits) by software once r_0 and n are given. Therefore, *it should be interesting to know how M_t behaves and $T_{\max}$ once $\kappa(t)$ is given and the initial hypersurface M_0 is not so special as above.* Theorem 2.1 can supply us this possibility. In fact, if $\kappa(t)$ is given, then the rescaled factor $\phi(t)$ might be solved by (2.5) (if feasible), and then applying Theorem 2.1 the behavior of M_t and the information of $T_{\max}$ can be known.

3. Evolution equations for the first eigenvalues of the Laplace and the *p*-Laplace operators

In this section, based on the evolution equations mentioned in Section 2, we would like to derive evolution equations for the first eigenvalues of the Laplacian and the p-Laplacian as follows.

Theorem 3.1. *Let $\lambda_1(t)$ be the first non-zero closed eigenvalue of the Laplacian on an n-dimensional compact and strictly convex hypersurface M_t which evolves by the forced MCF* (1.1)*, and let u be the normalized eigenfunction corresponding to λ_1, i.e., $-\Delta u=\lambda_1 u$ and $\int_{M_t}u^2=1$. Then we have*

$$\frac{d}{dt}\lambda_1(t)=-2\lambda_1\kappa(t)+2\int_{M_t}Hh^{ij}\nabla_i u\nabla_j u+2\int_{M_t}uH\nabla_i h^{ij}\nabla_j u. \tag{3.1}$$

Similarly, under the normalized flow (2.6)*, we have*

$$\frac{d}{d\tilde{t}}\widetilde{\lambda}_1(\tilde{t})=-\frac{2\widetilde{h}}{n}\cdot\widetilde{\lambda}_1(\tilde{t})+2\int_{\widetilde{M}_{\tilde{t}}}\widetilde{H}\cdot\widetilde{h}^{ij}\nabla_i u\nabla_j u+2\int_{\widetilde{M}_{\tilde{t}}}u\widetilde{H}\nabla_i\widetilde{h}^{ij}\nabla_j u,$$

where $\widetilde{\lambda}_1(\tilde{t})$ is the first non-zero closed eigenvalue of the Laplacian on the rescaled hypersurface $\widetilde{M}_{\tilde{t}}$.

Proof. Let u be an eigenfunction of the first non-zero closed eigenvalue λ_1 of Δ on the evolving compact hypersurface M_t. For simplicity, we normalize the function u, i.e., $\int_{M_t} u^2 = 1$. By (2.1), we know that u also satisfies

$$-\Delta u = \lambda_1 u, \qquad \text{where} \quad \int_{M_t} u = 0.$$

Clearly, we have

$$-\frac{\partial}{\partial t}(\Delta u) = \left(\frac{d}{dt}\lambda_1\right) u + \lambda_1 \frac{\partial u}{\partial t} \tag{3.2}$$

by taking derivatives with respect to t for the above equation. By multiplying u to both sides of (3.2) and then integrating over M_t, we have

$$-\int_{M_t} u \frac{\partial}{\partial t}(\Delta u) = \left(\frac{d}{dt}\lambda_1\right)\int_{M_t} u^2 + \lambda_1 \int_{M_t} u\frac{\partial u}{\partial t}.$$

Therefore, we can obtain

$$\frac{d}{dt}\lambda_1 = -\int_{M_t} u\frac{\partial}{\partial t}(\Delta u) - \lambda_1 \int_{M_t} u \frac{\partial u}{\partial t}. \tag{3.3}$$

Hence, if we want to get the evolution equation of λ_1, we need to derive the evolution equation of Δu under the flow (1.1). First, by (2.3) we have

$$\begin{aligned}
\frac{\partial}{\partial t} g^{ij} &= -g^{im}\left(\frac{\partial}{\partial t} g_{mq}\right) g^{qj} \\
&= 2g^{im}\left[H h_{mq} - \kappa(t) g_{mq}\right] g^{qj} \\
&= 2H g^{im} h_{mq} g^{qj} - 2\kappa(t) g^{ij},
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{\partial}{\partial t}(\Delta u) &= \frac{\partial}{\partial t}\left(g^{ij}\nabla_i\nabla_j u\right) \\
&= \frac{\partial}{\partial t}(g^{ij})\nabla_i\nabla_j u + g^{ij}\frac{\partial}{\partial t}(\nabla_i\nabla_j u) \\
&= 2\left[H g^{im} h_{mq} g^{qj} - \kappa(t) g^{ij}\right]\nabla_i\nabla_j u + g^{ij}\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x_i \partial x_j} - \Gamma^m_{ij}\frac{\partial u}{\partial x_m}\right) \\
&= 2H g^{im} h_{mq} g^{qj}\nabla_i\nabla_j u - 2\kappa(t)\Delta u + \Delta\frac{\partial u}{\partial t} - g^{ij}\frac{\partial}{\partial t}\left(\Gamma^m_{ij}\right)\frac{\partial u}{\partial x_m}. \qquad (3.4)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
g^{ij}\frac{\partial}{\partial t}\left(\Gamma^m_{ij}\right) &= \frac{1}{2} g^{ij} g^{ml}\left(\nabla_i \frac{\partial g_{jl}}{\partial t} + \nabla_j \frac{\partial g_{il}}{\partial t} - \nabla_l \frac{\partial g_{ij}}{\partial t}\right) \\
&= \frac{1}{2} g^{ij} g^{ml}\Big\{\nabla_i\left[-2H h_{jl} + 2\kappa(t) g_{jl}\right] + \nabla_j\left[-2H h_{il} + 2\kappa(t) g_{il}\right]
\end{aligned}$$

$$
\begin{aligned}
&\quad - \nabla_l \left[-2Hh_{ij} + 2\kappa(t)g_{ij}\right] \Big\} \\
&= -2\nabla_i H \cdot g^{ij} g^{ml} h_{jl}.
\end{aligned}
$$

Substituting the above equality into (3.4) results in

$$
\begin{aligned}
\frac{\partial}{\partial t}(\Delta u) = {}& 2Hg^{im}h_{mq}g^{qj}\nabla_i\nabla_j u - 2\kappa(t)\Delta u + \Delta\frac{\partial u}{\partial t} \\
&+ 2\nabla_i H \cdot g^{ij} g^{ml} h_{jl} \frac{\partial u}{\partial x_m}.
\end{aligned}
\tag{3.5}
$$

By substituting (3.5) into (3.3), and then integrating by parts, we have

$$
\begin{aligned}
\frac{d}{dt}\lambda_1 = {}& -\int_{M_t} u\Big[2Hg^{im}h_{mq}g^{qj}\nabla_i\nabla_j u - 2\kappa(t)\Delta u + \Delta\frac{\partial u}{\partial t} + 2\nabla_i H \cdot g^{ij}g^{ml}h_{jl}\nabla_m u\Big] \\
&- \lambda_1 \int_{M_t} u\frac{\partial u}{\partial t} \\
= {}& 2\int_{M_t} Hh^{ij}\nabla_i u\nabla_j u - 2\lambda_1\kappa(t) - \int_{M_t} u\left(\Delta\frac{\partial u}{\partial t}\right) - \lambda_1\int_{M_t} u\frac{\partial u}{\partial t} \\
= {}& -2\lambda_1\kappa(t) + 2\int_{M_t} Hh^{ij}\nabla_i u\nabla_j u + 2\int_{M_t} uH\nabla_i h^{ij}\nabla_j u,
\end{aligned}
\tag{3.6}
$$

where $h^{ij} = g^{im}h_{mq}g^{qj}$. Here the last equality in (3.6) holds since

$$
\int_{M_t} u\left(\Delta\frac{\partial u}{\partial t}\right) = \int_{M_t} \Delta u\frac{\partial u}{\partial t} = -\lambda_1\int_{M_t} u\frac{\partial u}{\partial t}.
$$

This completes the proof of (3.1).

Similarly, under the normalized flow (2.6), we can obtain

$$
\frac{d}{d\tilde{t}}\lambda_1(\tilde{t}) = -\frac{2\widetilde{h}}{n}\cdot\lambda_1(\tilde{t}) + 2\int_{\widetilde{M}_{\tilde{t}}} \widetilde{H}\cdot\widetilde{h}^{ij}\nabla_i u\nabla_j u + 2\int_{\widetilde{M}_{\tilde{t}}} u\widetilde{H}\nabla_i\widetilde{h}^{ij}\nabla_j u,
$$

since the evolution equations (2.3) and (2.7) *almost* have the same form except the function $\kappa(t)$ replaced by $\widetilde{h}/n$ with $\widetilde{h} = \phi^{-2}h = \int_{\widetilde{M}_{\tilde{t}}} \widetilde{H}^2 / \int_{\widetilde{M}_{\tilde{t}}}$. □

Remark 3.2. Here we want to emphasize one thing, that is, we need to require that M_t should be compact on a prescribed time interval, since the compactness of M_t can assure the existence of the eigenvalues of the Laplace and the p-Laplace operators. This implies that it cannot be avoided investigating the evolving behavior of the forced flow (1.1). In fact, by Theorem 2.1, we know that it is feasible to consider the evolution equation (3.1) of the first nonzero closed eigenvalue of the Laplace operator on $[0, T_{\mathrm{m}})$ with T_{m} defined by

$$
T_{\mathrm{m}} = \begin{cases} T_{\max}, & \text{if } 0 < \Xi \leq \infty, \\ T < T_{\max}, & \text{if } \Xi = 0, \end{cases}
\tag{3.7}
$$

where Ξ is the limit given by (2.8) and $[0, T_{\max})$ corresponds to the maximal time interval of the flow (1.1). Clearly, on $[0, T_{\mathrm{m}})$, the evolving hypersurface M_t is compact.

In the case of the p-Laplace operator, since we do not know whether the first nonzero closed eigenvalue $\lambda_{1,p}(t)$ of Δ_p is differentiable under the forced flow (1.1) or not, it seems like that we cannot use a similar method to that of the proof of Theorem 3.1. However, in fact, we can use a similar method to the one in [3,4] to avoid discussing the differentiation of $\lambda_{1,p}(t)$ under the flow (1.1). More precisely, on the time interval $[0, T_{\rm m})$ where the flow (1.1) exists and M_t is compact, we can define a smooth function $\lambda_{1,p}(u,t)$ as follows

$$\lambda_{1,p}(u,t) := -\int_{M_t} \Delta_p u(x,t) \cdot u(x,t) dv_t = \int_{M_t} |\nabla u|^p dv_t, \tag{3.8}$$

where $u(x,t)$ is an arbitrary smooth function satisfying

$$\int_{M_t} |u(x,t)|^p = 1 \qquad \text{and} \qquad \int_{M_t} |u(x,t)|^{p-2} u(x,t) = 0. \tag{3.9}$$

Clearly, for any $t \in [0, T_{\rm m})$, if, furthermore, $u(x,t)$ is the eigenfunction of the first eigenvalue $\lambda_{1,p}(t)$, then, by (3.9), we have

$$\lambda_{1,p}(u,t) = -\int_{M_t} \Delta_p u(x,t) \cdot u(x,t) = \lambda_{1,p}(t) \int_{M_t} |u(x,t)|^p = \lambda_{1,p}(t).$$

Now, by using the function $\lambda_{1,p}(u,t)$ defined by (3.8), we can prove the following result.

Theorem 3.3. *Let $\lambda_{1,p}(t)$ be the first non-zero closed eigenvalue of the p-Laplacian $(1 < p < \infty)$ on an n-dimensional compact and strictly convex hypersurface M_t which evolves by the forced MCF* (1.1)*, and let u be the eigenfunction of $\lambda_{1,p}(t)$ at time $t \in [0, T_{\rm m})$ satisfying $\int_{M_t} u^p = 1$, where $T_{\rm m}$ is defined by* (3.7)*. Let $\lambda_{1,p}(u,t)$ be the smooth function defined by* (3.8)*. Then at time t we have*

$$\begin{aligned} \frac{d}{dt}\lambda_{1,p}(u,t) = & -p\kappa(t)\lambda_{1,p}(t) + p\int_{M_t} |\nabla u|^{p-2} H h^{ij} \nabla_i u \cdot \nabla_j u \\ & + 2\int_{M_t} |\nabla u|^{p-2} u H \nabla_i h^{ij} \nabla_j u. \end{aligned} \tag{3.10}$$

Similarly, under the normalized flow (2.6)*, we have*

$$\begin{aligned} \frac{d}{d\tilde{t}}\widetilde{\lambda}_{1,p}(u,\tilde{t}) = & -\frac{p\widetilde{h}}{n} \cdot \widetilde{\lambda}_{1,p}(\tilde{t}) + p\int_{\widetilde{M}_{\tilde{t}}} |\nabla u|^{p-2} \widetilde{H} \cdot \widetilde{h}^{ij} \nabla_i u \cdot \nabla_j u \\ & + 2\int_{\widetilde{M}_{\tilde{t}}} |\nabla u|^{p-2} u \widetilde{H} \nabla_i \widetilde{h}^{ij} \nabla_j u \end{aligned}$$

at time $\tilde{t} \in [0, \widetilde{T}_{\rm m})$. Here $\widetilde{T}_{\rm m} := \int_0^{T_{\rm m}} \phi^2(s) ds$ with $\phi(t)$ the rescaled factor determined by (2.5)*. Moreover, $\widetilde{\lambda}_{1,p}(\tilde{t})$ is the first nonzero closed eigenvalue of the p-Laplacian on the rescaled hypersurface $\widetilde{M}_{\tilde{t}}$, and $\widetilde{\lambda}_{1,p}(u,\tilde{t})$ is a smooth* *function defined by*

$$\widetilde{\lambda}_{1,p}(u,\tilde{t}) := -\int_{\widetilde{M}_{\tilde{t}}} \Delta_p u(x,\tilde{t}) \cdot u(x,\tilde{t}),$$

where $u(x,\tilde{t})$ *is an arbitrary smooth function satisfying*

$$\int_{\widetilde{M}_{\tilde{t}}} |u(x,\tilde{t})|^p = 1 \qquad \text{and} \qquad \int_{\widetilde{M}_{\tilde{t}}} |u(x,\tilde{t})|^{p-2} u(x,\tilde{t}) = 0.$$

Proof. Taking derivatives with respect to t on both sides of (3.8), we have

$$-\frac{d}{dt}\lambda_{1,p}(u,t) = \frac{d}{dt}\int_{M_t} u\Delta_p u dv_t. \tag{3.11}$$

For convenience in the computation below, set $B = |\nabla u|^{p-2}$, and then $\Delta_p u = \mathrm{div}[B(\nabla u)]$. Furthermore, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{M_t} u\Delta_p u dv_t \\
&= \frac{\partial}{\partial t}\int_{M_t} g^{ij}\nabla_i[B(\nabla_j u)] u dv_t \\
&= \int_{M_t} \frac{\partial}{\partial t}\left[g^{ij}\nabla_i B\nabla_j u + B\Delta u\right] u dv_t + \int_{M_t} g^{ij}\nabla_i[B(\nabla_j u)](u_t dv_t + u(dv_t)_t) \\
&= \int_{M_t}\left[\left(\frac{\partial}{\partial t}g^{ij}\right)\nabla_i B\nabla_j u + g^{ij}\nabla_i B_t\nabla_j u + g^{ij}\nabla_i B\nabla_j u_t + B_t\Delta u + B\frac{\partial}{\partial t}(\Delta u)\right] u dv_t \\
&\quad + \int_{M_t} g^{ij}\nabla_i[B(\nabla_j u)](u_t dv_t + u(dv_t)_t),
\end{aligned}$$

where, except dv_t, the subscript $(\cdot)_t$ means taking derivative with respect to t for the prescribed function. Substituting the corresponding evolution equations of g^{ij}, Δu under the flow (1.1) derived in the proof of Theorem 3.1 into the above equality results in

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{M_t} u\Delta_p u dv_t \\
&\quad = \int_{M_t} u\Big[\left(2Hh^{ij} - 2\kappa(t)g^{ij}\right)\nabla_i B\nabla_j u + g^{ij}\nabla_i B_t\nabla_j u + g^{ij}\nabla_i B\nabla_j u_t \\
&\qquad + B_t\Delta u + B\left(2Hh^{ij}\nabla_i\nabla_j u - 2\kappa(t)\Delta u + \Delta\frac{\partial u}{\partial t} + 2\nabla_i H\cdot h^{im}\nabla_m u\right)\Big] dv_t \\
&\qquad + \int_{M_t} g^{ij}\nabla_i[B(\nabla_j u)](u_t dv_t + u(dv_t)_t) \\
&\quad = \int_{M_t} u\left(2Hh^{ij} - 2\kappa(t)g^{ij}\right)\nabla_i(B\nabla_j u) dv_t + \int_{M_t} g^{ij}\nabla_i(B_t\nabla_j u) u dv_t \\
&\qquad + \int_{M_t} g^{ij}\nabla_i(B\nabla_j u_t) u dv_t + 2\int_{M_t} Bu\nabla_i H\cdot h^{im}\nabla_m u dv_t
\end{aligned} \tag{3.12}$$

$$
\begin{aligned}
&+\int_{M_t} g^{ij}\nabla_i[B(\nabla_j u)](u_t dv_t+u(dv_t)_t)\\
=&\int_{M_t} u\left(2Hh^{ij}-2\kappa(t)g^{ij}\right)\nabla_i(B\nabla_j u)dv_t-\int_{M_t} g^{ij}B_t\nabla_i u\cdot\nabla_j u dv_t\\
&-\int_{M_t} g^{ij}B\nabla_i u\cdot\nabla_j u_t dv_t+2\int_{M_t} Bu\nabla_i H\cdot h^{im}\nabla_m u dv_t\\
&+\int_{M_t} g^{ij}\nabla_i[B(\nabla_j u)](u_t dv_t+u(dv_t)_t).
\end{aligned}
$$

Since

$$
\begin{aligned}
B_t&=\frac{\partial B}{\partial t}=\frac{\partial}{\partial t}|\nabla u|^{p-2}\\
&=\frac{\partial}{\partial t}\left(|\nabla u|^2\right)^{\frac{p-2}{2}}\\
&=\frac{\partial}{\partial t}\left(g^{ij}\nabla_i u\nabla_j u\right)^{\frac{p-2}{2}}\\
&=(p-2)|\nabla u|^{p-4}\left[Hh^{ij}-\kappa(t)g^{ij}\right]\nabla_i u\nabla_j u+(p-2)|\nabla u|^{p-4}g^{ij}\nabla_i u_t\cdot\nabla_j u,
\end{aligned}
$$

then substituting the above equality into (3.12) yields

$$
\begin{aligned}
(3.13)\quad &\frac{\partial}{\partial t}\int_{M_t} u\Delta_p u dv_t\\
=&\,2\int_{M_t} u\left[Hh^{ij}-\kappa(t)g^{ij}\right]\nabla_i(B\nabla_j u)dv_t-(p-2)\int_{M_t}|\nabla u|^{p-2}\\
&\cdot\left[Hh^{ij}-\kappa(t)g^{ij}\right]\nabla_i u\nabla_j u dv_t-(p-1)\int_{M_t}|\nabla u|^{p-2}g^{ij}\nabla_i u_t\cdot\nabla_j u dv_t\\
&+2\int_{M_t} Bu\nabla_i H\cdot h^{im}\nabla_m u dv_t+\int_{M_t} g^{ij}\nabla_i[B(\nabla_j u)](u_t dv_t+u(dv_t)_t)\\
=&-p\int_{M_t} B\left[Hh^{ij}-\kappa(t)g^{ij}\right]\nabla_i u\nabla_j u dv_t-2\int_{M_t} BuH\nabla_i h^{ij}\nabla_j u dv_t\\
&-(p-1)\int_{M_t} Bg^{ij}\nabla_i u_t\nabla_j u dv_t+\int_{M_t} g^{ij}\nabla_i[B(\nabla_j u)](u_t dv_t+u(dv_t)_t).
\end{aligned}
$$

By divergence theorem, we have

$$
-(p-1)\int_{M_t} Bg^{ij}\nabla_i u_t\nabla_j u dv_t=(p-1)\int_{M_t} g^{ij}\nabla_i[B(\nabla_j u)]u_t dv_t.
$$

Substituting the above equality into (3.13), we have

$$
\begin{aligned}
(3.14)\qquad \frac{\partial}{\partial t}\int_{M_t} u\Delta_p u dv_t=&-p\int_{M_t} B\left[Hh^{ij}-\kappa(t)g^{ij}\right]\nabla_i u\nabla_j u dv_t\\
&-2\int_{M_t} BuH\nabla_i h^{ij}\nabla_j u dv_t
\end{aligned}
$$

$$+\int_{M_t} g^{ij}\nabla_i[B(\nabla_j u)](pu_t dv_t + u(dv_t)_t).$$

If now u is the eigenfunction of the first non-zero closed eigenfunction $\lambda_{1,p}(t)$, then, as pointed out before, we have

$$\lambda_{1,p}(u,t)=\lambda_{1,p}(t) \quad \text{and} \quad \Delta_p u = -\lambda_{1,p}(t)|u|^{p-2}u.$$

By applying this fact and (3.9), we can obtain

$$\begin{aligned}\frac{d}{dt}\int_{M_t}|u(x,t)|^p dv_t &= \frac{d}{dt}\int_{M_t} B\cdot\left(g^{ij}\nabla_i u\nabla_j u\right)dv_t\\ &= \int_{M_t} Bu(pu_t dv_t + u(dv_t)_t)\\ &= -(\lambda_{1,p})^{-1}\int_{M_t} g^{ij}\nabla_i[B(\nabla_j u)](pu_t dv_t + u(dv_t)_t) = 0.\end{aligned}$$

Together the above equality with (3.14), we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_{M_t} u\Delta_p u dv_t = &-p\int_{M_t} B\left[Hh^{ij}-\kappa(t)g^{ij}\right]\nabla_i u\nabla_j u dv_t\\ &-2\int_{M_t} BuH\nabla_i h^{ij}\nabla_j u dv_t.\end{aligned}\tag{3.15}$$

By substituting (3.8) and (3.11) into (3.15), we have

$$\begin{aligned}&\frac{d}{dt}\lambda_{1,p}(u,t)\\ &= p\int_{M_t} B\left[Hh^{ij}-\kappa(t)g^{ij}\right]\nabla_i u\nabla_j u dv_t + 2\int_{M_t} BuH\nabla_i h^{ij}\nabla_j u dv_t\\ &= p\int_{M_t} BHh^{ij}\nabla_i u\cdot\nabla_j u dv_t + 2\int_{M_t} BuH\nabla_i h^{ij}\nabla_j u dv_t - p\kappa(t)\lambda_{1,p}(t),\end{aligned}$$

which completes the proof of (3.10).

Similarly, under the normalized flow (2.5), we can obtain

$$\begin{aligned}\frac{d}{d\tilde{t}}\lambda_{1,p}(u,\tilde{t}) = p&\int_{\widetilde{M}_{\tilde{t}}}|\nabla u|^{p-2}\widetilde{H}\cdot\widetilde{h}^{ij}\nabla_i u\cdot\nabla_j u dv_{\tilde{t}}\\ &+2\int_{\widetilde{M}_{\tilde{t}}}|\nabla u|^{p-2}u\widetilde{H}\nabla_i\widetilde{h}^{ij}\nabla_j u dv_{\tilde{t}} - \frac{p\widetilde{h}}{n}\cdot\lambda_{1,p}(\tilde{t}),\end{aligned}$$

which completes the second claim of Theorem 3.3. □

Remark 3.4. Since (3.10) does not depend on the particular evolution of u, we have $d\lambda_{1,p}(u,t)/dt = d\lambda_{1,p}(t)/dt$ at some time t. Clearly, at some time $t\in[0,T_{\mathrm{m}})$, (3.1) can be directly obtained by choosing $p=2$ in (3.10), which gives an explanation to the fact that the nonlinear Laplacian Δ_p is an extension of the linear Laplacian Δ from the viewpoint of the evolution equation. *Because of this, one may ask that maybe it is not necessary to derive* (3.1) *independently.* However, readers can find that the way for proving (3.1) cannot be used to derive (3.10) directly because of indeterminacy of the differentiability of $\lambda_{1,p}(t)$, and we have to construct a smooth function $\lambda_{1,p}(u,t)$ defined by (3.8) to overcome this problem. This is the reason why we separately give evolution equations of the first eigenvalues of the Laplace and the p-Laplace operators.

4. Lower bounds of the first eigenvalue of the Laplacian

In this section, we would like to give lower bounds for the first nonzero closed eigenvalue of the Laplace operator if additionally the initial hypersurface M_0 satisfies the pinching condition (1.2). However, first, we want to show that this pinching condition (1.2) is preserved under the forced MCF (1.1), i.e., the evolving hypersurface M_t also satisfies (1.2) for any $t \in [0, T_{\max})$. To prove this, we need to use Hamilton's maximum principle for tensors on manifolds (cf. [7, Theorem 9.1]). For convenience, we prefer to list its details here.

Theorem 4.1 (Hamilton)**.** *Suppose that on $0 \le t < T$ the evolution equation*

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + u^k \nabla_k M_{ij} + N_{ij}$$

holds, where $N_{ij} = p(M_{ij}, g_{ij})$, a polynomial in M_{ij} formed by contracting products of M_{ij} with itself using the metric, satisfies the null-eigenvector condition below. If $M_{ij} \ge 0$ at $t = 0$, then it remains so on $0 \le t < T$.

Remark 4.2. Here we would like to make an explanation to the so-called *null-eigenvector condition.* In fact, $N_{ij} = p(M_{ij}, g_{ij})$ satisfies the null-eigenvector condition implies that for any null-eigenvector X of M_{ij}, we have $N_{ij} X^i X^j \ge 0$.

By applying Theorem 4.1, we can prove the following result.

Lemma 4.3. *If, in addition, the initial hypersurface M_0 satisfies the pinching condition* (1.2)*, then the evolving hypersurface M_t remains so under the flow* (1.1) *for any $0 \le t < T_{\max}$.*

Proof. By (1.2), we have

$$h_{ij} = \alpha_i H g_{ij} \quad \text{on } M_0,$$

that is,

$$\alpha_i H g_{ij} \le h_{ij} \le \alpha_i H g_{ij} \quad \text{on } M_0.$$

On the other hand, by (2.3) and (2.4), we have

$$\begin{aligned} \frac{\partial}{\partial t}\left(h_{ij} - \alpha_i H g_{ij}\right) = {} & \Delta\left(h_{ij} - \alpha_i H g_{ij}\right) + |A|^2 \left(h_{ij} - \alpha_i H g_{ij}\right) \\ & + \kappa(t)\left(h_{ij} - \alpha_i H g_{ij}\right) - 2H\left(h_{il} g^{lm} h_{mj} - \alpha_i H h_{ij}\right). \end{aligned}$$

Now, we use Theorem 4.1 to prove Lemma 4.3. In fact, we can choose

$$M_{ij} = h_{ij} - \alpha_i H g_{ij}$$

and

$$N_{ij} = |A|^2 \left(h_{ij} - \alpha_i H g_{ij}\right) + \kappa(t)\left(h_{ij} - \alpha_i H g_{ij}\right) - 2H\left(h_{il} g^{lm} h_{mj} - \alpha_i H h_{ij}\right).$$

Clearly, $M_{ij} \geq 0$ at $t = 0$. It only needs to check that N_{ij} is nonnegative on the null-eigenvectors of M_{ij}. Assume that, for some vector $X = \{X^i\}$, we have

$$h_{ij} X^j = \alpha_i H X_i.$$

So, we can obtain

$$\begin{aligned} N_{ij} X^i X^j =& \left[|A|^2 + \kappa(t)\right]\left(\alpha_i H X_i X^i \alpha_i H g_{ij} X^i X^j\right) \\ &- 2H\left(h_{il} g^{lm} \alpha_m H X_m X^i - \alpha_i^2 H^2 X_i X^i\right) \\ =& \left[|A|^2 + \kappa(t)\right]\left(\alpha_i H X_i X^i - \alpha_i H g_{ij} X^i X^j\right) \\ &- 2H\left(\alpha_m \alpha_l H^2 g^{lm} X_m X_l - \alpha_i^2 H^2 X_i X^i\right) = 0. \end{aligned}$$

Hence, $M_{ij} \geq 0$ on M_t for any $0 \leq t < T_{\max}$, i.e., $h_{ij} \geq \alpha_i H g_{ij}$ for any $t \in [0, T_{\max})$. Similarly, one can easily get $h_{ij} \leq \alpha_i H g_{ij}$ for any $0 \leq t < T_{\max}$. So, we have

$$h_{ij} = \alpha_i H g_{ij} \quad \text{on} \quad M_t \quad \text{for } 0 \leq t < T_{\max},$$

which implies our conclusion. □

By applying Theorem 3.1 and Lemma 4.3, we can prove Theorem 1.1 as follows.

Proof of Theorem 1.1. By applying Theorem 3.1 and Lemma 4.3 directly, we can obtain

$$\begin{aligned} &\frac{d}{dt}\lambda_1(t) \\ =& -2\lambda_1\kappa(t) + 2\int_{M_t} H g^{im} h_{mq} g^{qj} \nabla_i u \nabla_j u + 2\int_{M_t} u H \nabla_i\left(g^{im} h_{mq} g^{qj}\right)\nabla_j u \\ =& -2\lambda_1\kappa(t) + 2\left[\int_{M_t} H g^{im} \alpha_m H g_{mq} g^{qj} \nabla_i u \nabla_j u + \int_{M_t} u H \nabla_i\left(g^{im}\alpha_m H g_{mq} g^{qj}\right)\nabla_j u\right] \\ \geq& -2\lambda_1\kappa(t) + 2\left(\frac{1}{n} - \epsilon\right)\int_{M_t} H^2 |\nabla u|^2 + 2\int_{M_t} u H \alpha_j \nabla_i H \cdot g^{ij} \cdot \nabla_j u. \end{aligned} \tag{4.1}$$

On the other hand, by integrating by parts to the last term of the right hand side of (4.1), the pinching condition (1.2) and the fact that the first nonzero closed eigenvalue $\lambda_1(t)$ is always positive, we have

$$\begin{aligned} &\int_{M_t} u H \alpha_j \nabla_i H \cdot g^{ij} \cdot \nabla_j u \\ =& -\frac{1}{2}\left(\int_{M_t} H^2 \alpha_j \nabla_i u \cdot g^{ij} \cdot \nabla_j u + \int_{M_t} u H^2 \alpha_j \cdot g^{ij} \nabla_i \nabla_j u\right) \\ \geq& -\frac{\left(\frac{1}{n} + \epsilon\right)}{2}\int_{M_t} H^2 |\nabla u|^2 + \frac{\lambda_1(t)}{2n}\int_{M_t} u^2 H^2 \end{aligned} \tag{4.2}$$

$$\begin{aligned}&-\frac{1}{2}\int_{M_t} uH^2\left(\frac{1}{n}-\alpha_j\right)\cdot g^{ij}\nabla_i\nabla_j u\\ \geq\ & -\frac{\left(\frac{1}{n}+\epsilon\right)}{2}\int_{M_t} H^2|\nabla u|^2+\frac{\left(\frac{1}{n}-2\epsilon\right)\lambda_1(t)}{2}\int_{M_t} u^2H^2.\end{aligned}$$

The last inequality holds since, on one hand, for $0 < t < T_0 < T_{\max}$, we know that M_t is strictly convex (cf. [10, Corollary 2.5]) and bounded, and H is continuous. Then H has positive maximum and minimum on M_t, which are finite. Define $H_{\max}(t) = \max_{x\in M_t} H(x,t)$ and $H_{\min}(t) = \min_{x\in M_t} H(x,t)$, so

$$\begin{aligned}\min_{1\leq j\leq n}\left|\frac{1}{n}-\alpha_j\right|\cdot H^2_{\min}(t)\lambda_1(t) &\leq \left|\int_{M_t}\left(\frac{1}{n}-\alpha_j\right)uH^2g^{ij}\nabla_i\nabla_j u\right|\\ &\leq \epsilon H^2_{\max}(t)\lambda_1(t).\end{aligned}$$

Therefore, by suitably choose ϵ, the equality

$$-\int_{M_t}\left(\frac{1}{n}-\alpha_j\right)uH^2g^{ij}\nabla_i\nabla_j u \geq -2\epsilon\lambda_1(t)\int_{M_t} u^2H^2$$

always holds. On the other hand, by Theorem 2.1, we know that $H_{\min}(t)/H_{\max}(t) \to 1$ as $t \to T_{\max}$ (this is because M_t converges *spherically* as $t \to T_{\max}$). So, for sufficiently small $\epsilon > 0$, there exists some $\delta > 0$ such that $|H_{\min}(t)/H_{\max}(t) - 1| \leq \epsilon$ for $T_{\max} - \delta \leq t < T_{\max}$. This implies that $|\int_{M_t} u^2H^2/H^2_{\max}(t)-1|$ must be small enough for $T_{\max}-\delta \leq t < T_{\max}$. Hence, by suitably choose ϵ, we can also get the above inequality. Now, substituting (4.2) into (4.1) results in

$$\frac{d}{dt}\lambda_1(t) \geq -2\lambda_1\kappa(t)+\left(\frac{1}{n}-3\epsilon\right)\int_{M_t} H^2|\nabla u|^2+\left(\frac{1}{n}-2\epsilon\right)\lambda_1(t)\int_{M_t} u^2H^2.$$

Since ϵ is small enough, without loss of generality, choose $\epsilon \ll \frac{1}{3n}$, then we have

$$\frac{d}{dt}\lambda_1(t) \geq -2\lambda_1\kappa(t). \tag{4.3}$$

Dividing both sides of (4.3) by λ_1 and then integrating from 0 to t ($0 < t < T_{\mathrm{m}}$), we have

$$\log\lambda_1(t)-\log\lambda_1(0) \geq -2\int_0^t \kappa(\tau)d\tau,$$

which implies the assertion of Theorem 1.1. □

Of course, under the assumption of Lemma 4.3, we can also give a lower bound for the first eigenvalue of the Laplace operator under the normalized flow (2.6) by repeating almost the same process as above, since from Theorem 3.1, we know that there is no essential difference between the evolution equation of the first eigenvalue under the unnormalized flow and the corresponding one under the normalized flow. In fact, we can easily get

$$\widetilde{\lambda}_1(\tilde{t}) \geq e^{-2\int_0^{\tilde{t}}\frac{\tilde{h}}{n}d\tau}\cdot\widetilde{\lambda}_1(0)$$

for $0 \leq \tilde{t} < \widetilde{T}_{\mathrm{m}}$.

However, we cannot just repeat the above process to try to get a similar conclusion for the p-Laplace operator when $p \neq 2$, since, as mentioned in Section 3, we do not know whether $\lambda_{1,p}(t)$ is differentiable or not.

5. Monotonicity of the first eigenvalues of the Laplacian and the p-Laplacian

By applying Theorems 3.1 and 3.3, we can easily obtain the following monotonicity for the first eigenvalue.

Theorem 5.1. *Let M_t, $\lambda_1(t)$, $\widetilde{\lambda}_1(\tilde{t})$, $\lambda_{1,p}(t)$, and $\widetilde{\lambda}_{1,p}(\tilde{t})$ be defined as in Theorems 3.1 and 3.3. Let T_{m} be defined by* (3.7)*, and let $\widetilde{T}_{\mathrm{m}}$ be defined as in Theorem 3.3. Denote by $H_{\max}(0)$ and $H_{\min}(0)$ the maximal and the minimal values of the mean curvature on the initial hypersurface M_0, respectively. Assume that M_0 satisfies the pinching condition* (1.2)*. Then we have*

(I) *If*

$$e^{-2\int_0^t \kappa(\tau)d\tau}\left[\frac{H_{\max}^{-1}(0) - 2H_{\max}(0)\int_0^t e^{-2\int_0^\tau \kappa(s)ds}d\tau}{H_{\max}(0)}\right]^{-1} \leq n\kappa(t)$$

for $0 \leq t < T_{\mathrm{m}}$, then $\lambda_1(t)$ is non-increasing for $0 \leq t < T_{\mathrm{m}}$ under the flow (1.1)*, and $\lambda_{1,p}(t)$ is non-increasing and differentiable almost everywhere for $0 \leq t < T_{\mathrm{m}}$ under the flow* (1.1)*. If*

$$e^{-2\int_0^t \kappa(\tau)d\tau}\left[\frac{H_{\min}^{-1}(0) - 2H_{\min}(0)\int_0^t e^{-2\int_0^\tau \kappa(s)ds}d\tau}{nH_{\min}(0)}\right]^{-1} \geq n\kappa(t)$$

for $0 \leq t < T_{\mathrm{m}}$, then $\lambda_1(t)$ is nondecreasing for $0 \leq t < T_{\mathrm{m}}$ under the flow (1.1)*, and $\lambda_{1,p}(t)$ is nondecreasing and differentiable almost everywhere for $0 \leq t < T_{\mathrm{m}}$ under the flow* (1.1)*.*

(II) *If*

$$e^{-2\int_0^{\tilde{t}} \frac{\tilde{h}}{n}d\tau}\left[\frac{H_{\max}^{-1}(0) - 2H_{\max}(0)\int_0^{\tilde{t}} e^{-2\int_0^\tau \frac{\tilde{h}}{n}ds}d\tau}{H_{\max}(0)}\right]^{-1} \leq \widetilde{h}$$

for $0 \leq \tilde{t} < \widetilde{T}_{\mathrm{m}}$, then $\widetilde{\lambda}_1(\tilde{t})$ is non-increasing under the normalized flow (2.6)*, and $\widetilde{\lambda}_{1,p}(\tilde{t})$ is non-increasing and differentiable almost everywhere under the normalized flow* (2.6)*. If*

$$e^{-2\int_0^{\tilde{t}} \frac{\tilde{h}}{n}d\tau}\left[\frac{H_{\min}^{-1}(0) - 2H_{\min}(0)\int_0^{\tilde{t}} e^{-2\int_0^\tau \frac{\tilde{h}}{n}ds}d\tau}{nH_{\min}(0)}\right]^{-1} \geq \widetilde{h}$$

for $0 \leq \tilde{t} < \widetilde{T}_{\mathrm{m}}$, then $\widetilde{\lambda}_1(\tilde{t})$ is nondecreasing under the normalized flow (2.6)*, and $\widetilde{\lambda}_{1,p}(\tilde{t})$ is nondecreasing and differentiable almost everywhere under the normalized flow* (2.6)*.*

Proof. By (2.4) and the fact that the convexity is preserved under the forced MCF (1.1), that is, M_t is convex (cf. [10, Corollary 2.5]), we have

$$\begin{aligned}\frac{\partial H}{\partial t} &= \Delta H + |A|^2 H - \kappa(t) H \\ &\leq \Delta H + H^3 - \kappa(t) H.\end{aligned}$$

Let $\rho(t)$ be the solution of the initial value problem

$$\begin{cases} \frac{d}{dt}\rho(t) = \rho^3(t) - \kappa(t)\rho(t), \\ \rho(0) = H_{\max}(0) := \max\limits_{x\in M_0} H(x,0). \end{cases}$$

By applying the maximum principle to the function $H(x,t) - \rho(t)$, we can obtain

$$H(x,t) \leq \rho(t) = e^{-\int_0^t \kappa(\tau)d\tau} \left[\frac{H_{\max}^{-1}(0) - 2H_{\max}(0)\int_0^t e^{-2\int_0^\tau \kappa(s)ds} d\tau}{H_{\max}(0)}\right]^{-1/2}.$$

Similarly, by (2.4) we have

$$\begin{aligned}\frac{\partial H}{\partial t} &= \Delta H + |A|^2 H - \kappa(t) H \\ &\geq \Delta H + \frac{H^3}{n} - \kappa(t) H.\end{aligned}$$

Let $\sigma(t)$ be the solution of the initial value problem

$$\begin{cases} \frac{d}{dt}\sigma(t) = \sigma^3(t) - \kappa(t)\sigma(t), \\ \sigma(0) = H_{\min}(0) := \min\limits_{x\in M_0} H(x,0). \end{cases}$$

By applying the maximum principle to the function $H(x,t) - \sigma(t)$, we can obtain

$$H(x,t) \geq \sigma(t) = e^{-\int_0^t \kappa(\tau)d\tau} \left[\frac{H_{\min}^{-1}(0) - 2H_{\min}(0)\int_0^t e^{-2\int_0^\tau \kappa(s)ds} d\tau}{nH_{\min}(0)}\right]^{-1/2}.$$

From the proof of Theorem 1.1, we know that once the initial hypersurface M_0 satisfies the pinching condition (1.2), M_t remains so and

$$\frac{d}{dt}\lambda_1(t) \geq -2\lambda_1\kappa(t) + \left(\frac{1}{n} - 3\epsilon\right)\int_{M_t} H^2|\nabla u|^2 + \left(\frac{1}{n} - 2\epsilon\right)\lambda_1(t)\int_{M_t} u^2 H^2.$$

Hence, we have

$$\begin{aligned}\frac{d}{dt}\lambda_1(t) \geq &\ -2\lambda_1\kappa(t) + \sigma^2\left[\left(\frac{1}{n} - 3\epsilon\right)\int_{M_t}|\nabla u|^2 + \left(\frac{1}{n} - 2\epsilon\right)\lambda_1(t)\int_{M_t} u^2\right] \\ = &\ 2\lambda_1 \cdot \left[-\kappa(t) + \left(\frac{1}{n} - \frac{5}{2}\epsilon\right)\sigma^2\right],\end{aligned}$$

which implies that $\lambda_1(t)$ is non-decreasing under the flow (1.1) provided $\sigma^2 \geq n\kappa(t)$.

On the other hand, similar to the proof of Theorem 1.1, one can easily get

$$\frac{d}{dt}\lambda_1(t) \leq -2\lambda_1\kappa(t) + 2\left(\frac{1}{n}+\epsilon\right)\int_{M_t} H^2|\nabla u|^2 + 2\int_{M_t} uH\alpha_j\nabla_i H\cdot g^{ij}\cdot\nabla_j u$$

and

$$\int_{M_t} uH\alpha_j\nabla_i H\cdot g^{ij}\cdot\nabla_j u \leq -\frac{\left(\frac{1}{n}-\epsilon\right)}{2}\int_{M_t} H^2|\nabla u|^2 + \frac{\left(\frac{1}{n}+2\epsilon\right)\lambda_1(t)}{2}\int_{M_t} u^2H^2$$

by applying Theorem 3.1 and Lemma 4.3, and suitably choosing ϵ. Combining the above two inequalities yields

$$\begin{aligned}\frac{d}{dt}\lambda_1(t) &\leq -2\lambda_1\kappa(t) + \rho^2\left[\left(\frac{1}{n}+3\epsilon\right)\int_{M_t}|\nabla u|^2 + \left(\frac{1}{n}+2\epsilon\right)\lambda_1(t)\int_{M_t} u^2\right]\\ &= 2\lambda_1\cdot\left[-\kappa(t) + \left(\frac{1}{n}+\frac{5}{2}\epsilon\right)\rho^2\right],\end{aligned}$$

which implies that $\lambda_1(t)$ is non-increasing under the flow (1.1) provided $\rho^2 \leq n\kappa(t)$.

Now, for the case of the p-Laplacian, by Lemma 4.3, if the evolving hypersurface M_t satisfies (1.2), M_t remains so. Then, together with Theorem 3.3 and similar to the proof of Theorem 1.1, at some time $t_0 \in [0, T_{\mathrm{m}})$ we have

$$\begin{aligned}&\frac{d}{dt}\lambda_{1,p}(u,t)\\ &\geq -p\lambda_{1,p}(t)\kappa(t) + \sigma^2\left\{\left[\frac{p-1}{n}-(p+1)\epsilon\right]\int_{M_t}|\nabla u|^p + \left(\frac{1}{n}-2\epsilon\right)\lambda_{1,p}(t)\int_{M_t} u^p\right\}\\ &= p\cdot\lambda_{1,p}(t)\cdot\left[-\kappa(t) + \left(\frac{1}{n}-\frac{p+3}{p}\epsilon\right)\sigma^2\right]\end{aligned}$$

at the time t_0, which implies that

$$\frac{d}{dt}\lambda_{1,p}(u,t)\Big|_{t=t_0} \geq 0$$

provided $\sigma^2 \geq n\kappa(t)$. Since $\lambda_{1,p}(u,t)$ defined by (3.8) is a smooth function with respect to t, then, for any sufficiently small number $\xi > 0$, we have

$$\frac{d}{dt}\lambda_{1,p}(u,t) \geq 0$$

on the interval $[t_0-\xi, t_0]$. Integrating the above inequality on $[t_0-\xi, t_0]$, we can obtain

$$\lambda_{1,p}(u(\cdot,t_0-\xi),t_0-\xi) \leq \lambda_{1,p}(u(\cdot,t_0),t_0). \tag{5.1}$$

By the definition (3.8) of $\lambda_{1,p}(u,t)$, we know that $\lambda_{1,p}(u(\cdot,t_0),t_0) = \lambda_{1,p}(t_0)$ and $\lambda_{1,p}(u(\cdot,t_0-\xi),t_0-\xi) \geq \lambda_{1,p}(t_0-\xi)$ at time t_0. Together this fact with (5.1), we have

$$\lambda_{1,p}(t_0-\xi) \leq \lambda_{1,p}(t_0)$$

for sufficiently small $\xi > 0$. It follows that $\lambda_{1,p}(t)$ is monotone non-decreasing under the flow (1.1), since t_0 can be chosen arbitrarily. The fact that $\lambda_{1,p}(t)$ is differentiable everywhere on $[0, T_{\mathrm{m}})$ can be derived by applying the classical Lebesgue's theorem. Similarly, if $\rho^2 \leq n\kappa(t)$, then $\lambda_{1,p}(t)$ is monotone non-increasing and differentiable everywhere under the flow (1.1). The second assertion (II) of Theorem 5.1 for the normalized flow can be obtained by almost the same process. $\square$

Remark 5.2. It is surprising that $\lambda_1(t)$ and $\lambda_{1,p}(t)$ have the same monotonicity under the same assumptions, and one may think that it is not necessary to derive the monotonicity of $\lambda_1(t)$ independently, since $\lambda_1(t)$ is only a special case of $\lambda_{1,p}(t)$, i.e., $\lambda_{1,p}(t) = \lambda_1(t)$ when $p = 2$. However, readers can find that one cannot use the way for proving the monotonicity of $\lambda_1(t)$ to get the monotonicity of $\lambda_{1,p}(t)$ directly (see the proof of Theorem 5.1 in Section 5). Besides, by applying Theorem 2.1, we can know more about $\widetilde{T}_{\mathrm{m}}$. More precisely, we can obtain: if $\Xi = \infty$ with Ξ defined by (2.8), then $T_{\mathrm{m}} < \infty$, and $\phi(t) \to \infty$ as $t \to T_{\max}$, which implies $\widetilde{T}_{\mathrm{m}} = \int_0^{T_{\max}} \phi^2(s)ds = \infty$; if $0 < \Xi < \infty$, then $T_{\max} = \infty$, which implies $\widetilde{T}_{\mathrm{m}} = \int_0^{\infty} \phi^2(t)dt$; if $\Xi = 0$, then $T_{\max} = \infty$, while $\widetilde{T}_{\mathrm{m}} = \int_0^{T_{\mathrm{m}}} \phi^2(t)dt = \int_0^{T} \phi^2(t)dt < \infty$.

Acknowledgments. This paper was started when the author visited the Beijing International Center for Mathematical Research (BICMR), Peking University from March 2013 to April 2013, and the author here is grateful to Prof. Yu-Guang Shi for the hospitality during his visit to BICMR. The last revision of this paper was carried out when the author visited the Chern Institute of Mathematics (CIM), Nankai University in December 2013, and the author here is grateful to Prof. Shao-Qiang Deng for the hospitality during his visit to CIM. The author would like to thank CIM for supplying the financial support during his visit through the Visiting Scholar Program. This work was partially supported by the NSF of China (Grant No. 11401131), the Fok Ying-Tong Education Foundation (China), and Hubei Key Laboratory of Applied Mathematics (Hubei University).

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