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SQUAREFREE ZERO-DIVISOR GRAPHS OF STANLEY-REISNER RINGS

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Abstract. Let Δ be a simplicial complex, I_Δ its Stanley-Reisner ideal and $K[\Delta]$ its Stanley-Reisner ring over a field K. Assume that $\Gamma(R)$ denotes the zero-divisor graph of a commutative ring R. Here, first we present a condition on two reduced Noetherian rings R and R', equivalent to $\Gamma(R) \cong \Gamma(R')$. In particular, we show that $\Gamma(K[\Delta]) \cong \Gamma(K'[\Delta'])$ if and only if $|\mathrm{Ass}(I_\Delta)| = |\mathrm{Ass}(I_{\Delta'})|$ and either $|K|, |K'| \leq \aleph_0$ or $|K| = |K'|$. This shows that $\Gamma(K[\Delta])$ contains little information about $K[\Delta]$. Then, we define the squarefree zero-divisor graph of $K[\Delta]$, denoted by $\Gamma_{\mathrm{sf}}(K[\Delta])$, and prove that $\Gamma_{\mathrm{sf}}(K[\Delta]) \cong \Gamma_{\mathrm{sf}}(K[\Delta'])$ if and only if $K[\Delta] \cong K[\Delta']$. Moreover, we show how to find $\dim K[\Delta]$ and $|\mathrm{Ass}(K[\Delta])|$ from $\Gamma_{\mathrm{sf}}(K[\Delta])$.

1. Introduction

In this paper all rings are commutative with identity and K is a field. Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates over K. By a squarefree monomial ideal of S we mean an ideal generated by a set of squarefree monomials of S. In the last few decades, the study of squarefree monomial ideals has got a large attention (for example, see [3, 8, 10]). This is because if we know algebraic properties of squarefree monomial ideals well, then we can understand many algebraic properties of much larger classes of ideals such as monomial and graded ideals (see [7]).

Squarefree monomial ideals have a combinatorial nature and many have tried to derive algebraic properties of a squarefree monomial ideal I, from different combinatorial structures associated to I, such as graphs, hypergraphs, clutters, simplicial complexes, posets, etc (see, for instance, [6, 9, 11] and Part III of [7]). Here we use the concept of simplicial complexes. Recall that a *simplicial complex* Δ on $[n] = \{1, \ldots, n\}$ is a family of subsets of $[n]$, called faces of Δ, with the following properties:

(i) if $A \in \Delta$ and $B \subseteq A$, then $B \in \Delta$;

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(ii) $\{i\} \in \Delta$ for all $i \in [n]$.

For every subset $F \subseteq [n]$ we set $x_F = \prod_{i\in F} x_i$. Then the ideal $I_\Delta = \langle X_F \mid F \subseteq [n], F \notin \Delta\rangle$ of S is called the *Stanley-Reisner ideal* of Δ and the ring $K[\Delta] = S/I_\Delta$ is called the *Stanley-Reisner ring of* Δ *over* K. For more on (squarefree) monomial ideals, simplicial complexes and their interrelations, the reader is referred to [7].

Now assume that R is a commutative ring with identity. Recently many graphs have been associated to R and many researchers have studied the relation between graph theoretic properties of these graphs and algebraic properties of R. One of the first and most studied graphs associated to R is the zero-divisor graph of R defined in [2]. Suppose that $\mathrm{Z}(R)$ denotes the set of zero-divisors of R and $\mathrm{Z}^*(R) = \mathrm{Z}(R) \setminus \{0\}$. The *zero-divisor graph* of R is the graph $\Gamma(R)$ on the vertex set $\mathrm{V}(\Gamma(R)) = \mathrm{Z}^*(R)$, in which, vertices x and y are adjacent if and only if $xy = 0$.

It is natural to ask how much the theory of zero-divisor graphs can help us to study squarefree monomial ideals, that is, what information about $R = K[\Delta]$ can be derived from $\Gamma(R)$, where Δ is a simplicial complex. As every Stanley-Reisner ring is a *reduced* ring (a ring without nonzero nilpotent elements), here in Section 2, we first study the structure of $\Gamma(R)$ when R is a reduced ring and present an algebraic condition on Noetherian reduced rings R and S equivalent to $\Gamma(R) \cong \Gamma(S)$. As a corollary, it is shown that $\Gamma(K[\Delta]) \cong \Gamma(K'[\Delta'])$ if and only if $|\mathrm{Ass}(K[\Delta])| = |\mathrm{Ass}(K'[\Delta'])|$ and either $|K| = |K'|$ or $|K|, |K'| \le \aleph_0$, where $\aleph_0$ is the smallest infinite cardinal.

Thus $\Gamma(R)$ provides us with little information about R, when R is a Stanley-Reisner ring. To remedy this weakness, in Section 3, we define squarefree zero-divisor graph of a Stanley-Reisner ring R, denoted $\Gamma_{\mathrm{sf}}(R)$, and prove that $\Gamma_{\mathrm{sf}}(R) \cong \Gamma_{\mathrm{sf}}(S)$ if and only if $S \cong R$, when R and S are Stanley-Reisner rings over a fixed field K. This shows that all algebraic properties of a Stanley-Reisner ring which do not depend on the base field can be recovered from its squarefree zero-divisor graph. In particular, we show how Krull dimension of $R = K[\Delta]$ and $|\mathrm{Ass}(R)|$ can be found from $\Gamma_{\mathrm{sf}}(R)$. Moreover, we study connectedness and the diameter of $\Gamma_{\mathrm{sf}}(R)$.

2. Zero-divisor graphs of Stanley-Reisner rings

To see what kind of information can be derived from the zero-divisor graph of a Stanley-Reisner ring and since every Stanley-Reisner ring is reduced, we first study the structure of $\Gamma(R)$ for a reduced ring R. Recall that an *independent set* of a graph is a set of vertices which are mutually nonadjacent.

Proposition 2.1. *Suppose that R is a reduced ring and* $\mathrm{Min}(R) = \{P_\lambda \mid \lambda \in \Lambda\}$. *For each $\emptyset \neq I \subseteq \Lambda$ set*

$$P_I = \bigcap_{\lambda\in I} P_\lambda \setminus \bigcup_{\lambda\in(\Lambda\setminus I)} P_\lambda.$$

Then $\{P_I \mid \emptyset \neq I \neq \Lambda, P_I \neq \emptyset\}$ *is a partition of* $\mathrm{V}(\Gamma(R))$*, each such* P_I *is an independent set of* $\Gamma(R)$ *and for each* $x \in P_I$ *and* $y \in P_J$*,* x *and* y *are adjacent if and only if* $I \cup J = \Lambda$*.*

Proof. Suppose that $x \in \mathrm{V}(\Gamma(R)) = \mathrm{Z}^*(R)$ and $0 \neq y \in \mathrm{Ann}(x)$. If x is not contained in any of the P_λ's, then as $0 = xy \in P_\lambda$, we get $y \in \cap_{\lambda\in\Lambda} P_\lambda =$ the nilradical of R =0, a contradiction. Thus for some $\emptyset \neq I \subseteq \Lambda$, we have $x \in P_I$. As $x \neq 0$, $I \neq \Lambda$. It is clear that $P_I \cap P_J = \emptyset$ for $I \neq J \subseteq \Lambda$ and hence $\{P_I \mid \emptyset \neq I \neq \Lambda, P_I \neq \emptyset\}$ is a partition of $\mathrm{V}(\Gamma(R))$.

Now note that as $0 = xy \in P_\lambda$, y should lie in all P_λ with $\lambda \notin I$. Conversely if $z \in \cap_{\lambda\in\Lambda\setminus I} P_\lambda$, then $xz \in P_\Lambda = 0$. Thus if $z \in P_J$, then x and z are adjacent if and only if $\Lambda \setminus I \subseteq J$, if and only if $I \cup J = \Lambda$. In particular, if $x, z \in P_I$, then x and z are not adjacent, which concludes the proof. ☐

In what follows, if x is a vertex of a (fixed) graph, we denote the neighborhood of x in that graph by $\mathrm{N}(x)$.

Theorem 2.2. *Suppose that* R *is a reduced ring with finitely many minimal primes* $P_1, \ldots, P_n$ *(for example, if* R *is a Noetherian reduced ring). Also assume that* S *is a reduced ring. Then* $\Gamma(R) \cong \Gamma(S)$ *if and only if* $\mathrm{Min}(S) = \{Q_1, \ldots, Q_n\}$ *and for a permutation* σ *on* $[n]$ *and every nonempty proper subset* I *of* $[n]$*, we have*

$$|P_I| = |Q_{\sigma(I)}|,$$

where $P_I = \cap_{i\in I} P_i \setminus \cup_{i\in[n]\setminus I} P_i$ *and* $Q_J = \cap_{i\in I} Q_i \setminus \cup_{i\in[n]\setminus I} Q_i$*.*

Proof. ($\Leftarrow$): A direct consequence of Proposition 2.1.

($\Rightarrow$): Assume that $\phi : \Gamma(R) \to \Gamma(S)$ is a graph isomorphism. Also let $\mathrm{Min}(S) = \{Q_\lambda \mid \lambda \in \Lambda\}$ and Q_J be as above (with $[n]$ replaced by Λ). First note that if $\emptyset \neq I \subsetneq [n]$, then by the prime avoidance theorem $P_I \neq \emptyset$. If $x \in P_I$ and $\phi(x) \in Q_J$, then it follows from Proposition 2.1 that $P_I = \{y \in \mathrm{Z}^*(R) \mid \mathrm{N}(y) = \mathrm{N}(x)\}$ and $Q_J = \{y \in \mathrm{Z}^*(S) \mid \mathrm{N}(y) = \mathrm{N}(\phi(x))\}$. Since ϕ is an isomorphism of graphs, we deduce that $\phi(P_I) = Q_J$. Similarly for each nonempty Q_J, there is an $I \subsetneq [n]$ such that $\phi^{-1}(Q_J) = P_I$. Therefore the set $\widetilde{Q} = \{Q_J \mid \emptyset \neq J \subsetneq \Lambda, Q_J \neq \emptyset\}$ is finite. But for each $\lambda \in \Lambda$, we have $Q_\lambda = \cup_{\lambda\in J} Q_J$, that is, each minimal prime of S is a union of sets in $\widetilde{Q}$. Since $|\widetilde{Q}| < \infty$, it follows that the number of minimal primes of S is finite and because of the one-to-one correspondence ϕ should be n, so we assume $\Lambda = [n]$.

Consider the graph with vertices P_I, in which P_I is adjacent to $P_{I'}$ when $I \cup I' = [n]$ and consider the similar graph on Q_J's. Since for each I, $\phi(P_I) = Q_J$ for some J, ϕ can be considered as an isomorphism between these two graphs on P_I's and Q_J's. In the rest of the proof we work in these two graphs. It can be readily checked that if $\phi(P_I) = Q_J$, then $2^{|I|} - 1 = |\mathrm{N}(P_I)| = |\mathrm{N}(Q_J)| = 2^{|J|} - 1$, hence $|I| = |J|$. In particular, it follows that for a permutation σ on $[n]$, $\phi(P_{\{i\}}) = Q_{\{\sigma(i)\}}$. We show that this σ satisfies the required condition. For simplicity we assume $\sigma(i) = i$, so that we must show $|P_I| = |Q_I|$ for each nonempty $I \subsetneq [n]$. For this, we prove by induction on $|I|$ that $\phi(P_I) = Q_I$ and $\phi(P_{[n]\setminus I}) = Q_{[n]\setminus I}$ for each nonempty $I \subsetneq [n]$.

If $|I| = 1$, then by assumption $\phi(P_I) = Q_I$. Also the only vertex adjacent to P_I is $P_{[n]\setminus I}$, therefore, $\phi(P_{[n]\setminus I})$ should be the only vertex adjacent to $\phi(P_I) = Q_I$ which is $Q_{[n]\setminus I}$. This sets the basis of the induction. Now assume $|I| > 1$ and let $\phi(P_I) = Q_J$. Note that for each $i \in I$, $P_{[n]\setminus(I\setminus\{i\})} \in \mathrm{N}(P_I)$ and by the induction hypothesis, $\phi(P_{[n]\setminus(I\setminus\{i\})}) = Q_{[n]\setminus(I\setminus\{i\})}$. Hence $Q_{[n]\setminus(I\setminus\{i\})} \in \mathrm{N}(Q_J)$, that is $[n] \setminus J \subseteq [n] \setminus (I \setminus \{i\})$ or equivalently, $I \setminus \{i\} \subseteq J$. Since $i \in I$ is arbitrary and $|I| > 1$, we see that $I \subseteq J$. Because, as proved above, $|I| = |J|$, we must have $I = J$. Now $\phi(P_{[n]\setminus I}) = Q_{[n]\setminus I}$ follows from the facts that $P_{[n]\setminus I}$ (resp. $Q_{[n]\setminus I}$) is the only vertex with index of size $n - |I|$ adjacent to P_I (resp. Q_I) and that ϕ preserves adjacency and also the size of the index sets. □

Theorem 2.2 generalizes the following result which was first proved in [1].

Corollary 2.3 ([1, Theorem 1.4])**.** *Assume that R and S are reduced finite rings. Then $\Gamma(R) \cong \Gamma(S)$ if and only if $R \cong S$.*

Proof. Note that R and S are finite direct products of finite fields, say $R \cong \prod_{i=1}^n K_i$ and $S \cong \prod_{i=1}^m F_i$. So minimal primes of R and S are $P_j = \prod_{i\neq j=1}^n K_i$ and $Q_j = \prod_{i\neq j=1}^m F_i$, respectively. Now the result follows from Theorem 2.2. □

As another corollary of Theorem 2.2, we can now answer the question which was the starting point of this research: "when zero-divisor graphs of two Stanley-Reisner rings are isomorphic?" Recall that the maximal faces of a simplicial complex are called its *facets*. By [7, Lemma 1.5.4], the number of facets of a simplicial complex Δ equals $|\mathrm{Ass}(K[\Delta])| = |\mathrm{Min}(K[\Delta])|$.

Corollary 2.4. *Suppose that $R = K[\Delta]$ and $S = K'[\Delta']$ where K, K' are fields and Δ, Δ' are simplicial complexes. Then $\Gamma(R) \cong \Gamma(S)$ if and only if the following two conditions hold*:

(i) $|\mathrm{Ass}(R)| = |\mathrm{Ass}(S)|$ (*or equivalently, Δ and Δ' have the same number of facets*);

(ii) *either $|K| = |K'|$ or $|K|, |K'| \leq \aleph_0$, where $\aleph_0$ denotes the smallest infinite cardinal.*

Proof. Let $\{P_i\}_{i=1}^n$ be the minimal primes of I_Δ in $K[x_1, \dots, x_t]$ and P_I be as defined in Theorem 2.2. According to [7, Lemma 1.5.4], each P_i is generated by a set of variables and hence for each $\emptyset \neq I \subsetneq [n]$, $q_I = \cap_{i\in I} P_i$ is a squarefree monomial ideal. By [7, Proposition 2.2.5(a)], the set of monomials in $q_I \setminus I_\Delta$ is a K-basis for $\bar{q}_I$, where $\bar{\ }$ denotes image in R. Since we have a countable number of monomials, we see that $|\bar{P}_I| \leq |\bar{q}_I| \leq |K|^{(\aleph_0)}$, where by $|K|^{(\aleph_0)}$ we mean the cardinality of a K-vector space with dimension $\aleph_0$. On the other hand, if $u = x_1^{\alpha_1} \cdots x_t^{\alpha_t} \in P_I$ with for example $\alpha_1 \neq 0$, then $\bar{x}_1^j \bar{u} \in \bar{P}_I$ for each $j \in \mathbb{N}$. Indeed since each P_i is a monomial ideal, every K-linear combination of $\{\bar{x}_1^j \bar{u} \mid j \in \mathbb{N}\}$ is in $\bar{P}_I$. Therefore, $|\bar{P}_I| = |K|^{(\aleph_0)}$. Noting that $|K|^{(\aleph_0)} = \aleph_0$ for $|K| \leq \aleph_0$ and $|K|^{(\aleph_0)} = |K|$ for $|K| \geq \aleph_0$, the result follows from Theorem 2.2. $\square$

In particular, in the case that $K = K'$, the above result asserts that $\Gamma(K[\Delta]) \cong \Gamma(K[\Delta'])$ if and only if Δ and Δ' have the same number of facets. In other words, the only information which can be read off the zero-divisor graph of $K[\Delta]$ is the number of facets of Δ. Thus this infinite graph contains little information about Δ and its Stanley-Reisner ring. In the next section, we present a finite graph, based on the zero-divisor structure of $K[\Delta]$, from which we can reconstruct Δ completely.

3. Squarefree zero-divisor graphs of Stanley-Reisner rings

Since Stanley-Reisner rings are defined by squarefree monomial ideals, to get the "most important" information about such rings, we should look at the structure of squarefree monomials in these rings. This is our idea in defining the squarefree zero-divisor graph of Stanley-Reisner rings.

Definition 3.1. Suppose that Δ is a simplicial complex on $[n]$ and $S = K[x_1, \ldots, x_n]$. Let V be the set of image in $R = K[\Delta]$ of all squarefree monomials of S which are not in I_Δ. By the squarefree zero-divisor graph of R (or Δ), we mean the graph on vertex set V, in which, two vertices u and v are adjacent if and only if $uv = 0$. We denote this graph by $\Gamma_{\mathrm{sf}}(R)$ or $\Gamma_{\mathrm{sf}}(\Delta)$.

The squarefree zero-divisor graph of Δ could directly be described using Δ: its vertices correspond to nonempty faces of Δ and two faces $A, B \in \Delta$ are adjacent if and only if $A \cup B \notin \Delta$. Recall that if $F_1, \ldots, F_t \subseteq [n]$, then $\Delta = \langle F_1, \ldots, F_t \rangle$ means that Δ is the simplicial complex with facets $F_1, \ldots, F_t$.

Example 3.2. Suppose that $n = 3$ and $\Delta = \langle \{1,2\}, \{1,3\}, \{2,3\} \rangle$. Then $R = K[\Delta] = K[x_1, x_2, x_3]/\langle x_1x_2x_3 \rangle$ and $\Gamma(R)$ is the graph in Figure 1(a) (note that, for simplicity, we denote the image of x_i in R again by x_i). If $\Delta = \langle \{1,3\}, \{2,3\} \rangle$, then $K[\Delta] = K[x_1, x_2, x_3]/\langle x_1x_2 \rangle$ and $\Gamma_{\mathrm{sf}}(\Delta)$ is Figure 1(b). Note that in this case $\Gamma_{\mathrm{sf}}(R)$ has an isolated vertex which corresponds to the non-zero-divisor vertex x_1.

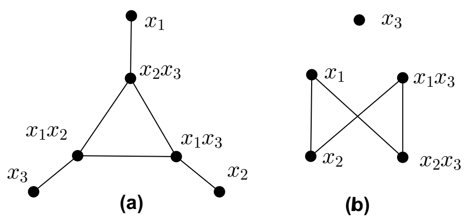

FIGURE 1. $\Gamma_{\mathrm{sf}}(R)$ for (a) $R = \frac{K[x_1,x_2,x_3]}{\langle x_1x_2x_3 \rangle}$, (b) $R = \frac{K[x_1,x_2,x_3]}{\langle x_1x_2 \rangle}$

Clearly, $\Gamma_{\mathrm{sf}}(K[\Delta])$ is independent of K and has no information about K. But we are going to show that we can reconstruct Δ form this graph. For this we need the following lemma.

Lemma 3.3. *Suppose that* $\Delta = \langle F_1, \ldots, F_t\rangle$ *and* $\Delta' = \langle F'_1, \ldots, F'_t\rangle$ *are two simplicial complexes. If there exists a permutation* σ *on* $[t]$ *such that for each* $\emptyset \neq I \subseteq [t]$ *we have* $|\cap_{i\in I} F_i| = |\cap_{i\in I} F_{\sigma(i)}|$, *then* $\Delta \cong \Delta'$.

Proof. Assume that Δ is on $[n]$ and Δ' is on $[n']$. For each $I \subseteq [t]$, let $F_I = \cap_{i\in I} F_i \setminus \cup_{i\in[t]\setminus I} F_i$ and similarly define F'_I. Then we can compute $|F_I|$ (resp. $|F'_I|$) from the cardinalities of $\cap_{j\in J} F_j$ (resp. $\cap_{j\in J} F'_j$) for different J's, using the inclusion-exclusion principle. It follows that $|F_I| = |F_{\sigma(I)}|$ for each $I \subseteq [t]$. Suppose that $\phi_I : F_I \to F'_{\sigma(I)}$ is a bijection. Since $[n] = \cup_{i=1}^t F_i = \cup_{I\subseteq[t]} F_I$ and $[n'] = \cup_{I\subseteq[t]} F'_I$, we see that $n = n'$. Because the sets $\{F_I\}_{I\subseteq[t]}$ are pairwise disjoint, we can patch together the bijections ϕ_I to get a permutation $\phi = \cup_{I\subseteq[t]}\phi_I$ of $[n]$. Now using the fact that $F_i = \cup_{i\in I\subseteq[t]} F_I$, it is straightforward to check that $\phi(F_i) = F'_{\sigma(i)}$ and hence the result is established. □

Recall that a *clique* of graph is a set of mutually adjacent vertices of that graph.

Theorem 3.4. *Suppose that* Δ *and* Δ' *are simplicial complexes. The following are equivalent.*

(i) $\Gamma_{\mathrm{sf}}(K[\Delta]) \cong \Gamma_{\mathrm{sf}}(K[\Delta'])$.
(ii) $K[\Delta] \cong K[\Delta']$.
(iii) $\Delta \cong \Delta'$.

Proof. (ii) $\Leftrightarrow$ (iii) is proved in [4] and (iii) $\Rightarrow$ (i) is trivial. According to Lemma 3.3, to show (i) $\Rightarrow$ (iii), we just need to prove that if $\Delta = \langle F_1, \ldots, F_t\rangle$, then the numbers $|\cap_{i\in I} F_i|$ are determined uniquely by $\Gamma_{\mathrm{sf}}(K[\Delta])$. In this proof, we consider vertices of $\Gamma_{\mathrm{sf}}(\Delta)$ to be faces of Δ, as mentioned after Definition 3.1.

Suppose that $\mathcal{C} = \{A_1, \ldots, A_{t'}\}$ is a largest clique in $\Gamma_{\mathrm{sf}}(\Delta)$. Since for $i \neq j$, A_i and A_j are adjacent, we have $A_i \cup A_j \notin \Delta$, that is, there is no facet of Δ containing both A_i and A_j. If $A_1 \subseteq F_i$ and $A_1 \subseteq F_j$ for $i \neq j$, then $\mathcal{C} \setminus \{A_1\} \cup \{F_i, F_j\}$ is a clique larger than $\mathcal{C}$, against the choice of $\mathcal{C}$. So each A_i is contained in exactly one facet of Δ. Also if some facet does not contain any of the A_j's, then by adding that facet to $\mathcal{C}$, we get a larger clique. Therefore, it follows that $t = t'$, and after possibly reordering A_i's, we can assume that $A_1 \subseteq F_1, \ldots, A_t \subseteq F_t$ and $A_i \nsubseteq F_j$ for $j \neq i$. Therefore,

$$B \in \mathrm{N}(A_i) \Leftrightarrow B \cup A_i \notin \Delta \Leftrightarrow B \nsubseteq F_i \Leftrightarrow B \in \mathrm{N}(F_i),$$

hence $\mathrm{N}(A_i) = \mathrm{N}(F_i)$. Set $\Omega_i = \mathrm{V}(\Gamma_{\mathrm{sf}}(\Delta)) \setminus \mathrm{N}(A_i)$. Thus $\Omega_i = \{\emptyset \neq B \in \Delta \,|\, B \subseteq F_i\} = 2^{F_i} \setminus \{\emptyset\}$. If $\emptyset \neq I \subseteq [t]$, then

$$\bigcap_{i\in I} \Omega_i = \bigcap_{i\in I} 2^{F_i} \setminus \{\emptyset\} = 2^{\cap_{i\in I} F_i} \setminus \{\emptyset\}.$$

Consequently, $|\cap_{i\in I} F_i| = \log_2(|\cap_{i\in I} \Omega_i| + 1)$, as required. □

The following is a corollary of the proof of the above theorem. In this corollary by the *clique number* $\omega(G)$ of a graph G, we mean the largest size of a clique in G.

Corollary 3.5. *For a Stanley-Reisner ring R, $\omega\left(\Gamma_{\mathrm{sf}}(R)\right) = |\mathrm{Ass}(R)|$.*

Theorem 3.4, shows that all of the algebraic properties of a Stanley-Reisner ring R, could be read off $\Gamma_{\mathrm{sf}}(R)$ together with the base filed K of R. For example, the following shows how we can find the Krull dimension of R from $\Gamma_{\mathrm{sf}}(R)$.

Corollary 3.6. *Let R be a Stanley-Reisner ring and consider the family of all independent sets of $\Gamma_{\mathrm{sf}}(R)$ which meet a largest clique. If α is the largest size of such an independent set, then* $\dim R = \log_2(\alpha + 1)$.

Proof. Let $\Delta = \langle F_1, \dots, F_t \rangle$ be the simplicial complex corresponding to R. Assume that $\mathcal{I}$ is a largest independent set of $\Gamma_{\mathrm{sf}}(R)$ which meets a largest clique, say in the vertex A. Then $\mathcal{I} \subseteq \mathrm{V}(\Gamma_{\mathrm{sf}}(R)) \setminus \mathrm{N}(A)$. But in the proof of Theorem 3.4, we saw that for some $1 \leq i \leq t$, $\mathrm{V}(\Gamma_{\mathrm{sf}}(R)) \setminus \mathrm{N}(A) = \Omega_i$, where $\Omega_i = 2^{F_i} \setminus \{\emptyset\}$. Since Ω_i is an independent set of $\Gamma_{\mathrm{sf}}(R)$, we deduce that $\mathcal{I} = \Omega_i$ and $\log_2(\alpha+1) = |F_i|$. Since $\mathcal{I}$ is one of the largest such independent sets, F_i is one of the largest facets of Δ. Now the result follows [5, Theorem 5.1.4]. $\square$

Note that not all maximum size independent sets need to meet a largest clique. For example, if G is the graph in Figure 1(a), then $\{x_1, x_2, x_3\}$ is a largest independent but does not meet any largest clique. Note that in this graph the largest size of an independent set meeting a largest clique equals $\alpha(G) = 3$, where $\alpha(G)$ denotes the largest size of an independent set in G. For example, $\{x_1, x_2, x_1x_2\}$ is a such a set. So we end this article with the following question.

Question 3.7. *Is the number α in Corollary* 3.6*, the same as* $\alpha(\Gamma_{\mathrm{sf}}(R))$?

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