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SOLVABILITY OF NONLINEAR ELLIPTIC TYPE EQUATION WITH TWO UNRELATED NON STANDARD GROWTHS

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Abstract. In this paper, we study the solvability of the nonlinear Dirichlet problem with sum of the operators of independent non standard growths

$$-div\left(|\nabla u|^{p_1(x)-2}\nabla u\right)-\sum_{i=1}^{n}D_i\left(|u|^{p_0(x)-2}D_iu\right)+c(x,u)=h(x),\ x\in\Omega$$

in a bounded domain $\Omega\subset\mathbb{R}^n$. Here, one of the operators in the sum is monotone and the other is weakly compact. We obtain sufficient conditions and show the existence of weak solutions of the considered problem by using monotonicity and compactness methods together.

1. Introduction

In this work, we are concerned with the Dirichlet problem for the nonlinear elliptic equation with variable nonlinearity

$$\text{(1.1)}\quad \begin{cases} -div\left(|\nabla u|^{p_1(x)-2}\nabla u\right)-\sum\limits_{i=1}^{n}D_i\left(|u|^{p_0(x)-2}D_iu\right)+c(x,u)=h(x), \\ u\,|_{\partial\Omega}=0, \end{cases}$$

where $x\in\Omega\subset\mathbb{R}^n$ $(n\geq 3)$ is a bounded domain which has sufficiently smooth boundary (at least Lipschitz boundary), $D_i\equiv\partial/\partial x_i$, p_0, p_1 are nonnegative measurable functions defined on Ω, h is a generalized function and $c:\Omega\times\mathbb{R}\to\mathbb{R}$, $c(x,\tau)$ is a function with variable nonlinearity in τ (for example, $c(x,u)=c_0(x)|u|^{\alpha(x)-2}u+c_1(x)$, see Section 2).

We denote the operators A and B with

$$A(u):=-div\left(|\nabla u|^{p_1(x)-2}\nabla u\right),\ B(u):=-\sum_{i=1}^{n}D_i\left(|u|^{p_0(x)-2}D_iu\right)+c(x,u).$$

There has been recently a considerable interest in the study of equations and variational problems with variable exponents of nonlinearities due to their applications. Nonlinear equations including the operator $A(u)$ is a rather common

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nonlinear problem with variable exponent which is known as $p_1(\cdot)$-Laplacian equation. This kind of problems have been studied in various contexts by many authors [1, 3, 4, 6, 28] and have wide range of application areas in the mathematical modeling of non-Newtonian fluids [8, 18, 19], theory of elasticity and hydrodynamics [26], thermistor problem [27] and in image restoration [7] etc. (Indeed, all application areas mentioned above are also valid for problem (1.1)). On the other hand, the equations of the type $B(u) = h$ are rarely researched. For instance in [5], a similar type of problem for $B(u) = h$ was studied by Antontsev and Shmarev who investigated the regularized problem to show the existence of weak solution. Such equations may appear in the mathematical modeling of the process of nonstable filtration of an ideal barotropic gas in a nonhomogeneous porous medium [4]. We also refer [14, 17, 19] for the several of the most important applications of (1.1) and nonlinear partial differential equations with variable exponent arise from mathematical modeling of suitable processes in mechanics, mathematical physics, image processing etc.

To the best of our knowledge, by now there have not been any studies on the existence of solutions for the elliptic equations of the type (1.1) with variable exponents of nonlinearity. However, we note that a similar problem to (1.1) with constant exponents was investigated in [21]. Essentially, in [21] the question of the solvability of an operator equation when the case the operator is in the form of sum of a weakly compact and pseudo-monotone operators was answered. In the present paper, we study the similar type of operator equation in a model problem when the operators in the sum with variable nonlinearity, and obtain the sufficient conditions for solvability. The our goal of studying the model problem is to provide a more understandable and explicit way for the established results in this article.

The main feature of the equation $A(u) + B(u) = h$ is that the exponents $p_0(x)$ and $p_1(x)$ are independent of each other. Thus, neither A nor B is the main part of this equation. It is needed to specify that if A is the main part of the equation, i.e., the exponents are dependent each other, the results for the theory of pseudo-monotone operators can be used to investigate the problem. However, in the case we consider, any methods which is merely related to monotonicity can not be used.

We use the general solvability theorem [21], (Theorem 2.5) to prove the existence of weak solution of the problem (1.1). In order to apply this theorem to existence theorem (Theorem 2.4) for problem (1.1), we obtain sufficient conditions and prove the monotonicity of the operator A and weak compactness of B on proper spaces under these conditions, and then we get the solvability of (1.1) by using monotonicity and compactness simultaneously.

This paper is organized as follows: In the next section, we present the assumptions, definition of the weak solution and description of the main result. For this purpose, we also define some function classes which are required to study the posed problem. In Section 3, firstly, we establish some integral inequalities to investigate the function classes (pseudo-norm spaces) defined in previous section and then verify some necessary lemmas and theorems which indicate the relation of these spaces with variable exponent Lebesgue and Sobolev spaces and show the continuous and compact embeddings of these function spaces etc. In Section 4, we give the proof of the main theorem (Theorem 2.4) of this paper with the help of the embedding results achieved in Section 3.

2. Statement of the problem and the main result

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. We study the problem (1.1)

$$\begin{cases} -div\left(|\nabla u|^{p_1(x)-2}\nabla u\right) - \sum\limits_{i=1}^{n} D_i\left(|u|^{p_0(x)-2} D_i u\right) + c\left(x,u\right) = h\left(x\right), \; x \in \Omega, \\ u \mid_{\partial\Omega} = 0 \end{cases}$$

under the following conditions:

(U1) $2 \leq p_0^- \leq p_0(x) \leq p_0^+ < \infty$, $1 < p_1^- \leq p_1(x) \leq p_1^+ < \infty$ *and* $p_0 \in C^1\left(\bar{\Omega}\right), p_1 \in C^0\left(\bar{\Omega}\right)$.

(U2) *There exists a measurable function* $\alpha : \Omega \longrightarrow [1,\infty)\,, 1 \leq \alpha^- \leq \alpha(x) \leq \alpha^+ < \infty$ *such that the following inequalities hold*

$$|c(x,\tau)| \leq c_0(x)\,|\tau|^{\alpha(x)-1} + c_1(x)$$

and

$$c(x,\tau)\,\tau \geq c_2(x)\,|\tau|^{\alpha(x)},$$

a.e. $(x,\tau) \in \Omega \times \mathbb{R}$.

Here $c(x,\tau)$ *is a Carathédory function and* c_i, $i = 0,1,2$ *are nonnegative measurable functions defined on* Ω, *besides for* $\varepsilon > 0$, $\alpha(x) \geq p_0(x) + \varepsilon$ *and* $c_2(x) \geq \tilde{C} > 0$, $x \in \Omega$ *is satisfied.*

In order to give the definition of weak solution of the problem (1.1), we introduce the required spaces. For this reason, firstly we remind some basic facts about generalized Lebesgue and Sobolev spaces [2,9,11,12,15].

Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$ such that $|\Omega| > 0$ (Throughout the paper, we denote by $|\Omega|$ the Lebesgue measure of Ω). $M(\Omega)$ denotes the family of all measurable functions $p : \Omega \longrightarrow [1,\infty]$ and by $M_0(\Omega)$,

$$M_0(\Omega) := \left\{p \in M(\Omega) : \; 1 \leq p^- \leq p(x) \leq p^+ < \infty, \text{ a.e. } x \in \Omega\right\},$$

where $p^- := \operatorname*{ess\,inf}_{\Omega} |p(x)|\,,\; p^+ := \operatorname*{ess\,sup}_{\Omega} |p(x)|$.

For $p \in M(\Omega)\,, \Omega_\infty^p \equiv \Omega_\infty \equiv \{x \in \Omega |\; p(x) = \infty\}$. On the set of all functions on Ω, define the functional σ_p and $\|\cdot\|_p$ by

$$\sigma_p(u) \equiv \int_{\Omega\setminus\Omega_\infty} |u|^{p(x)}\,dx + \operatorname*{ess\,sup}_{\Omega_\infty} |u(x)|$$

and

$$\|u\|_{L^{p(x)}(\Omega)} \equiv \inf\left\{\lambda > 0 : \sigma_p\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Clearly, if $p \in L^{\infty}(\Omega)$, then $p \in M_0(\Omega)$, $\sigma_p(u) \equiv \int_\Omega |u|^{p(x)}\,dx$ and the generalized Lebesgue space is defined as follows:

$L^{p(x)}(\Omega) := \{u : u$ is a measurable real-valued function such that $\sigma_p(u) < \infty\}$.

If $p^- > 1$, then the space $L^{p(x)}(\Omega)$ becomes a reflexive and separable Banach space under the norm $\|\cdot\|_{L^{p(x)}(\Omega)}$ which is so-called Luxemburg norm.

If $0 < |\Omega| < \infty$, and $p_1,\ p_2 \in M(\Omega)$, then the continuous embedding $L^{p_1(x)}(\Omega) \subset L^{p_2(x)}(\Omega)$ exists $\Longleftrightarrow$ $p_2(x) \le p_1(x)$ for a.e. $x \in \Omega$.

For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ where $p,\ q \in M_0(\Omega)$ and $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ the following inequalities hold:

$$\int_\Omega |uv|\,dx \le 2\,\|u\|_{L^{p(x)}(\Omega)}\,\|v\|_{L^{q(x)}(\Omega)}, \text{ (generalized Hölder inequality)} \tag{2.1}$$

and

$$\min\{\|u\|^{p^-}_{L^{p(x)}(\Omega)}, \|u\|^{p^+}_{L^{p(x)}(\Omega)}\} \le \sigma_p(u) \le \max\{\|u\|^{p^-}_{L^{p(x)}(\Omega)}, \|u\|^{p^+}_{L^{p(x)}(\Omega)}\}. \tag{2.2}$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p \in L^{\infty}(\Omega)$ then generalized Sobolev space is defined as follows:

$$W^{1,\ p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) : \ \ |\nabla u| \in L^{p(x)}(\Omega)\right\}$$

and this space is a separable Banach space under the norm

$$\|u\|_{W^{1,\ p(x)}(\Omega)} \equiv \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

$W_0^{1,\ p(x)}(\Omega)$ defines as the closure of $C_0^{\infty}(\Omega)$ in $W^{1,\ p(x)}(\Omega)$. If $p^- > 1$, then $W^{1,\ p(x)}(\Omega)$ and $W_0^{1,\ p(x)}(\Omega)$ are reflexive and separable Banach spaces. If $\partial\Omega$ is Lipschitz and $p \in C^0(\bar{\Omega})$, then equivalent norm in $W_0^{1,\ p(x)}(\Omega)$ is given by;

$$\|u\|_{W_0^{1,\ p(x)}(\Omega)} \equiv \|\nabla u\|_{L^{p(x)}(\Omega)} \equiv \sum_{i=1}^{n} \|D_i u\|_{L^{p(x)}(\Omega)}.$$

Let $p,\ q \in C(\bar{\Omega}) \cap M_0(\Omega)$ and $p(x) < n$, $q(x) < \frac{np(x)}{n-p(x)} \equiv p^*(x)$ is hold for all $x \in \Omega$, then there is a continuous and compact embedding $W^{1,\ p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

A function $p \in M_0(\Omega)$ is called log-Hölder continuous if there is a constant L such that the inequality

$$-|p(x) - p(y)| \log|x-y| \le L, \qquad \forall x, y \in \Omega$$

holds. (Here, we remark that it can be deduced from mean value theorem and feature of logarithm function, if $p \in C^1(\bar{\Omega})$, then p is log-Hölder continuous.) If p is log-Hölder continuous and $q \in M_0(\Omega)$, then we have the continuous embedding $W^{1,\ p(x)}(\Omega) \subset L^{q(x)}(\Omega)$ for all $q \le p^*$. For more details and embedding results for these spaces see [2, 9–13, 15].

We now define some function classes which are required to study the problem (1.1). These classes are nonlinear spaces which are generalization of the nonlinear spaces with constant exponent studied in [21–24] (see also references therein). We also specify that the necessary properties of these spaces are presented in Section 3.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n\,(n \geq 2)$ be a bounded domain with Lipschitz boundary and $\gamma,\, \beta \in M_0\,(\Omega)$. We introduce $S_{1,\gamma(x),\beta(x)}\,(\Omega)$, the class of functions $u : \Omega \to \mathbb{R}$ and the functional $[\cdot]_{S_{\gamma,\beta}} : S_{1,\gamma(x),\beta(x)}\,(\Omega) \longrightarrow \mathbb{R}_+$ as follows:

$$S_{1,\gamma(x),\beta(x)}\,(\Omega) := \left\{ u \in L^1\,(\Omega) : \int_\Omega |u|^{\gamma(x)+\beta(x)}\,dx + \sum_{i=1}^n \int_\Omega |u|^{\gamma(x)}\,|D_i u|^{\beta(x)}\,dx < \infty \right\},$$

$$[u]_{S_{\gamma,\beta}} := \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{\gamma(x)+\beta(x)} dx + \sum_{i=1}^n \left(\int_\Omega \left| \frac{|u|^{\frac{\gamma(x)}{\beta(x)}}\,D_i u}{\lambda^{\frac{\gamma(x)}{\beta(x)}+1}} \right|^{\beta(x)} \right) dx \leq 1 \right\}.$$

$[\cdot]_{S_{\gamma,\beta}}$ defines a pseudo-norm on $S_{1,\gamma(x),\beta(x)}\,(\Omega)$, actually it can be clearly seen that $[\cdot]_{S_{\gamma,\beta}}$ fulfills all conditions of pseudo-norm (pn) see [24], i.e., $[u]_{S_{\gamma,\beta}} \geq 0$, $u = 0 \Rightarrow [u]_{S_{\gamma,\beta}} = 0$, $[u]_{S_{\gamma,\beta}} \neq [v]_{S_{\gamma,\beta}} \Rightarrow u \neq v$ and $[u]_{S_{\gamma,\beta}} = 0 \Rightarrow u = 0$.

Let $S_{1,\gamma(x),\beta(x)}\,(\Omega)$ be the space given in Definition 2.1 and $\theta\,(x) \in M_0\,(\Omega)$, we denote $S_{1,\gamma(x),\beta(x),\theta(x)}\,(\Omega)$, the class of functions $u : \Omega \to \mathbb{R}$ by the following intersection:

$$S_{1,\gamma(x),\beta(x),\theta(x)}\,(\Omega) := S_{1,\gamma(x),\beta(x)}\,(\Omega) \cap L^{\theta(x)}\,(\Omega)\,, \tag{2.3}$$

with the pseudo-norm

$$[u]_{S_{\gamma,\beta,\theta}} := [u]_{S_{\gamma,\beta}} + \|u\|_{L^{\theta(x)}(\Omega)}\,, \quad \forall u \in S_{1,\gamma(x),\beta(x),\theta(x)}\,(\Omega)\,.$$

We state a proposition which can be easily proved by the embedding results for the variable exponent Lebesgue spaces.

Proposition 2.2. *If $\gamma,\, \beta,\, \theta \in M_0\,(\Omega)$ and $\theta\,(x) \geq \gamma\,(x) + \beta\,(x) + \varepsilon_0$ a.e. $x \in \Omega$ for some $\varepsilon_0 > 0$, then we have the following equivalence*;

$$S_{1,\gamma(x),\beta(x),\theta(x)}\,(\Omega) \equiv \left\{ u \in L^1\,(\Omega) : \Re^{\gamma,\beta,\theta}\,(u) < \infty \right\},$$

where $\Re^{\gamma,\beta,\theta}\,(u) := \int_\Omega |u|^{\theta(x)}\,dx + \sum_{i=1}^n \int_\Omega |u|^{\gamma(x)}\,|D_i u|^{\beta(x)}\,dx$, and the pseudo-norm on this space is

$$[u]_{S_{\gamma,\beta,\theta}} \equiv \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u}{\lambda}\right|^{\theta(x)} dx + \sum_{i=1}^n \left(\int_\Omega \left| \frac{|u|^{\frac{\gamma(x)}{\beta(x)}}\,D_i u}{\lambda^{\frac{\gamma(x)}{\beta(x)}+1}} \right|^{\beta(x)} \right) dx \leq 1 \right\}.$$

We also denote the dual spaces,

$$W^{-1,\,q_0(x)}\,(\Omega) := \left(W_0^{1,\,p_0(x)}\,(\Omega)\right)^*,$$
$$W^{-1,\,q_1(x)}\,(\Omega) := \left(W_0^{1,\,p_1(x)}\,(\Omega)\right)^* \text{ and}$$
$$L^{\alpha'(x)}\,(\Omega) := \left(L^{\alpha(x)}\,(\Omega)\right)^*,$$

here $q_0\,(x) := \frac{p_0(x)}{p_0(x)-1}$, $q_1\,(x) := \frac{p_1(x)}{p_1(x)-1}$ and $\alpha'\,(x) := \frac{\alpha(x)}{\alpha(x)-1}$.

We investigate (1.1) for the functions $h \in W^{-1,\ q_0(x)}(\Omega) + L^{\alpha'(x)}(\Omega) + W^{-1,\ q_1(x)}(\Omega)$. Let us denote $Q(\Omega)$ by

$$Q(\Omega) := \mathring{S}_{1,q_0(x)(p_0(x)-2),q_0(x),\alpha(x)}(\Omega) \cap W_0^{1,\ p_1(x)}(\Omega),$$

where

$$\mathring{S}_{1,q_0(x)(p_0(x)-2),q_0(x),\alpha(x)}(\Omega) := \left\{u \in S_{1,q_0(x)(p_0(x)-2),q_0(x),\alpha(x)}(\Omega) : u\mid_{\partial\Omega} = 0\right\}.$$

Notice that the condition on $\alpha(x)$ in (U2) indicates Proposition 2.2 is valid for $\mathring{S}_{1,q_0(x)(p_0(x)-2),q_0(x),\alpha(x)}(\Omega)$.

We are ready to give the definition of the weak solution of problem (1.1).

Definition 2.3. If the function $u \in Q(\Omega)$ satisfies the following equality

$$\int_\Omega \left(|\nabla u|^{p_1(x)-2}\nabla u\right)\cdot\nabla v dx + \sum_{i=1}^n \int_\Omega \left(|u|^{p_0(x)-2} D_i u\right) D_i v dx + \int_\Omega c(x,u)\, v dx = \int_\Omega h v dx \tag{2.4}$$

for all $v \in W_0^{1,\ p_0(x)}(\Omega) \cap W_0^{1,\ p_1(x)}(\Omega) \cap L^{\alpha(x)}(\Omega)$, then the function u is called the weak solution of the problem (1.1).

Note that it is clear under the conditions (U1) and (U2), all the integrals in (2.4) make sense.

Now, we state the main theorem of this article that is the solvability theorem for problem (1.1).

Theorem 2.4 (Existence Theorem)**.** *Let the conditions* (U1)-(U2) *fulfill and* $c_0, c_2 \in L^\infty(\Omega)$, $c_1 \in L^{\alpha'(x)}(\Omega)$. *Then for all* $h \in W^{-1,\ q_0(x)}(\Omega) + L^{\alpha'(x)}(\Omega) + W^{-1,\ q_1(x)}(\Omega)$, *problem* (1.1) *has a weak solution in the space* $Q(\Omega)$.

We will use the following solvability theorem [21] to prove Theorem 2.4. Let X and Y_0 be reflexive Banach spaces, Y is an arbitrary Banach space and S_{gY_0} is pn-space (pseudo-norm space) [24]. Let $A : X \longrightarrow X^*$ and $B : S_{gY_0} \longrightarrow Y$ be nonlinear operators.

Assume that the following conditions are satisfied:

1) $A : X \longrightarrow X^*$ is a pseudo-monotone operator, i.e.,
   (i) A is bounded and
   (ii) the conditions $u_m \xrightarrow{X} u_0$ and limsup$\langle A(u_m),\ u_m - u\rangle \leq 0$ imply

   $$\text{liminf}\,\langle A(u_m),\ u_m - v\rangle \geq \langle A(u),\ u - v\rangle, \quad \forall v \in X.$$

2) $B : S_{gY_0} \longrightarrow Y$ is weakly compact. Furthermore, there exists a mapping $B_0 : X_0 \cap S_{gY_0} \longrightarrow Y_2 \subseteq Y$ such that B_0 is weakly compact from $X_0 \cap S_{gY_0}$ to Y_2 where X_0 is a separable topological vector space which

is dense in X, Y^* and S_{gY_0} and there exists a continuous nondecreasing function $\varphi : \mathbb{R}^1_+ \longrightarrow \mathbb{R}^1_+$, such that $\varphi \in C^0$ and

$$\langle B(u),\ u\rangle = \varphi\left(\|B_0(u)\|_{Y_2}\right), \quad \forall u \in S_{gY_0}.$$

3) The operator $T := A + B$ is coercive in the generalized sense on X_0, i.e., for each $u \in X_0$ with $\|u\|_X$, $[u]_{S_{gY_0}} \geq M$ we have

$$\langle T(u),\ u\rangle = \langle A(u) + B(u),\ u\rangle \geq \lambda_0\left(\|u\|_X\right)\|u\|_X + \lambda_1\left([u]_{S_{gY_0}}\right)[u]_{S_{gY_0}},$$

where $\lambda_0, \lambda_1 \in C^0$ such that as $\tau \nearrow \infty$, $\lambda_0(\tau), \lambda_1(\tau) \nearrow \infty$ and $M > 0$ is some number.

Theorem 2.5 ([21])**.** *Let conditions* 1)-3) *be satisfied. Then the equation*

$$T(u) = A(u) + B(u) = y, \quad y \in X^* + Y,$$

is solvable in $X \cap S_{gY_0}$ *for any* $y \in X^* + Y$ *satisfying*

$$\sup\left\{\frac{\langle y,\ u\rangle}{\|u\|_X + [u]_{S_{gY_0}}} :\ u \in X_0\right\} < \infty.$$

3. Preliminary results

In this section, we work on the function classes which are defined in Section 2 that is actually required to examine the problem (1.1). First, we establish some integral inequalities to understand the structure of these spaces. Afterwards, we show that these spaces are complete metric spaces. Finally, we prove some lemmas and theorems on continuous and compact embeddings of these spaces and also indicate their relation between the variable exponent Lebesgue and Sobolev spaces.

3.1. Some integral inequalities

In this subsection, we derive some integral inequalities. Here, the proofs of the lemmas can be attained readily by using Young's, Hölder inequalities and by calculations (and also see Lemma 4.1 in [20]), hence we skip the proofs for the sake of brevity. Throughout this section, we assume that $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with Lipschitz boundary.

Lemma 3.1. *Let* $\zeta,\ \xi \in M_0(\Omega)$ *and* $\zeta(x) \geq \xi(x)$ *a.e.* $x \in \Omega$. *Then the inequality*

$$\int_\Omega |u|^{\xi(x)}\,dx \leq \int_\Omega |u|^{\zeta(x)}\,dx + |\Omega|, \quad \forall u \in L^{\zeta(x)}(\Omega) \tag{3.1}$$

holds.

Lemma 3.2. *Assume that* $\zeta \in M_0(\Omega)$ *and the numbers* η *and* ϵ *satisfy* $\eta \geq 1$, $\epsilon > 0$. *Then for every* $u \in L^{\zeta(x)+\epsilon}(\Omega)$

$$\int_\Omega |u|^{\zeta(x)}\left|\ln|u|\right|^\eta\,dx \leq N_1\int_\Omega |u|^{\zeta(x)+\epsilon}\,dx + N_2 \tag{3.2}$$

is satisfied. Here $N_1 \equiv N_1(\epsilon, \eta) > 0$ *and* $N_2 \equiv N_2(\epsilon, \eta, |\Omega|) > 0$ *are constants.*

Corollary 3.3. *Let* ζ, $\eta \in M_0(\Omega)$. *Then for* $\epsilon > 0$, *the inequality*

$$\int_\Omega |u|^{\zeta(x)} |\ln |u||^{\eta(x)} dx \leq N_3 \int_\Omega |u|^{\zeta(x)+\epsilon} dx + N_4, \quad \forall u \in L^{\zeta(x)+\epsilon}(\Omega) \tag{3.3}$$

holds. Here $N_3 \equiv N_3(\epsilon,\ \eta^+) > 0$ *and* $N_4 \equiv N_4(\epsilon, \eta^+, |\Omega|) > 0$ *are constants.*

3.2. Generalized nonlinear spaces and embedding theorems

In this section, we examine the properties of the spaces $S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ and their connection with the variable exponent Lebesgue and Sobolev spaces. Investigating most of boundary value problems on its own space leads to achieve better results. Therefore problem (1.1) is studied here on its own space (i.e., $S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$). Unlike linear boundary value problems, the sets generated by nonlinear problems are subsets of linear spaces, but not have the linear structure [21–24]. From now on, unless additional conditions are imposed, all the functions γ, β and θ will satisfy the conditions given in Proposition 2.2.

Lemma 3.4. *Assume that conditions of Proposition 2.2 are satisfied. Let* $u \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ *and* $\lambda_u := [u]_{S_{\gamma,\beta,\theta}}$. *Then the following inequality*

$$\max\{\lambda_u^{\gamma^-+\beta^-}, \lambda_u^{\theta^+}\} \geq \Re^{\gamma,\beta,\theta}(u) \geq \min\{\lambda_u^{\gamma^-+\beta^-}, \lambda_u^{\theta^+}\}$$

holds.

Proof. For $u \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$,

$$\begin{aligned}\Re^{\gamma,\beta,\theta}(u) &= \int_\Omega |u|^{\theta(x)} dx + \sum_{i=1}^n \int_\Omega |u|^{\gamma(x)} |D_i u|^{\beta(x)} dx \\ &= \int_\Omega \lambda_u^{\theta(x)} \left|\frac{u}{\lambda_u}\right|^{\theta(x)} dx + \sum_{i=1}^n \int_\Omega \lambda_u^{\gamma(x)+\beta(x)} \left| \frac{|u|^{\frac{\gamma(x)}{\beta(x)}} D_i u}{\lambda_u^{\frac{\gamma(x)}{\beta(x)}+1}} \right|^{\beta(x)} dx,\end{aligned}$$

if $\lambda_u \geq 1$, we have

$$\begin{aligned}&\geq \lambda_u^{\theta^-} \int_\Omega \left|\frac{u}{\lambda_u}\right|^{\theta(x)} dx + \lambda_u^{\gamma^-+\beta^-} \sum_{i=1}^n \int_\Omega \left| \frac{|u|^{\frac{\gamma(x)}{\beta(x)}} D_i u}{\lambda_u^{\frac{\gamma(x)}{\beta(x)}+1}} \right|^{\beta(x)} dx \\ &\geq \lambda_u^{\gamma^-+\beta^-} \left(\int_\Omega \left|\frac{u}{\lambda_u}\right|^{\theta(x)} dx + \sum_{i=1}^n \int_\Omega \left| \frac{|u|^{\frac{\gamma(x)}{\beta(x)}} D_i u}{\lambda_u^{\frac{\gamma(x)}{\beta(x)}+1}} \right|^{\beta(x)} dx \right) \\ &= \lambda_u^{\gamma^-+\beta^-} \Re^{\gamma,\beta,\theta}(u/\lambda_u) = \lambda_u^{\gamma^-+\beta^-}.\end{aligned}$$

Since by virtue of Proposition 2.2 and the definition of $[\cdot]_{S_{\gamma,\beta,\theta}}$, $\Re^{\gamma,\beta,\theta}(u/\lambda_u) = 1$. We deduce from the last equality that $\Re^{\gamma,\beta,\theta}(u) \geq \lambda_u^{\gamma^-+\beta^-}$. (Note that, $\lambda \mapsto \Re^{\gamma,\beta,\theta}(u/\lambda)$ is a convex continuous and decreasing function of λ on $[1, \infty)$.)

Obviously, if $0 < \lambda_u < 1$, then $\Re^{\gamma,\beta,\theta}(u) \geq \lambda_u^{\theta^+}$. We can show the other side of the inequality similarly that yields the proof is complete. $\square$

Theorem 3.5. *Assume that conditions of Proposition 2.2 are satisfied and let* $p \in M_0(\Omega)$, $p(x) \geq \theta(x)$ *a.e.* $x \in \Omega$. *Then, the embedding*

$$W^{1,\,p(x)}(\Omega) \subset S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega) \tag{3.4}$$

holds.

Proof. Let $u \in W^{1,\,p(x)}(\Omega)$, as a consequence of Lemma 3.4 to attain the embedding (3.4), it is sufficient to show that $\Re^{\gamma,\beta,\theta}(u)$ is finite (i.e., $\Re^{\gamma,\beta,\theta}(u) < \infty$)

$$\Re^{\gamma,\beta,\theta}(u) = \int_\Omega |u|^{\theta(x)}\,dx + \sum_{i=1}^{n} \int_\Omega |u|^{\gamma(x)} |D_i u|^{\beta(x)}\,dx,$$

by Lemma 3.1 and using Young's inequality we get

$$\leq \int_\Omega |u|^{p(x)}\,dx + |\Omega| + \sum_{i=1}^{n} \int_\Omega |D_i u|^{p(x)}\,dx + n \int_\Omega |u|^{\frac{p(x)\gamma(x)}{p(x)-\beta(x)}}\,dx,$$

estimating the third integral on the right side of the last inequality by using Lemma 3.1, we obtain

$$\Re^{\gamma,\beta,\theta}(u) \leq (n+1)\left(\sigma_p(u) + |\Omega|\right) + \sum_{i=1}^{n} \sigma_p(D_i u),$$

thus we attain the desired result from this inequality and (2.2).

If $p(x) = \theta(x)$ a.e. $x \in \Omega$, by using the above operations we can obtain (3.4) in a similar way. $\square$

We omit the proof of the following lemma as it is straightforward.

Lemma 3.6. *Let* $\gamma, \beta : \Omega \longrightarrow [1, \infty)$ *be functions satisfying* $1 \leq \gamma^- \leq \gamma(x) \leq \gamma^+ < \infty$, $1 \leq \beta^- \leq \beta(x) \leq \beta^+ < \infty$ *a.e.* $x \in \Omega$ *and* $\gamma, \beta \in C^1(\bar{\Omega})$. *Then the function* $\varphi : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$, $\varphi(x,t) := |t|^{\frac{\gamma(x)}{\beta(x)}}\, t$ *satisfies the following*:

(i) *For every fixed* $x_0 \in \Omega$, $\varphi(x_0, \cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ *is continuously differentiable*; *has an inverse and inverse function is also continuously differentiable.*
(ii) *For every fixed* $t_0 \in \mathbb{R} - \{0\}$, φ *and* φ^{-1} *is continuous on* Ω, *and for* $i = \overline{1,n}$, *partial derivatives* $\varphi_{x_i}(x, t_0)$, $\varphi^{-1}_{x_i}(x, t_0)$ *exist and are continuous.*

Definition 3.7. Let $\eta \in M_0(\Omega)$, we introduce $L^{1,\,\eta(x)}(\Omega)$ the class of functions $u : \Omega \to \mathbb{R}$

$$L^{1,\,\eta(x)}(\Omega) \equiv \left\{ u \in L^1(\Omega) \mid D_i u \in L^{\eta(x)}(\Omega),\ i = \overline{1,n} \right\}.[1]\ [9]$$

---

[1] This space is not Banach differently from the space $W^{1,\,\eta(x)}(\Omega)$.

Theorem 3.8. *Let the functions γ, β and φ satisfy the conditions of Lemma 3.6 and $L^{1,\ \beta(x)}(\Omega)$ be the space given in Definition 3.7. Then, φ is a bijective mapping between $S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ and $L^{1,\ \beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)$ where $\psi(x) := \frac{\theta(x)\beta(x)}{\gamma(x)+\beta(x)}$.*

Proof. First, let us verify that for all $u \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$,

$$v := |u|^{\frac{\gamma(x)}{\beta(x)}} u = \varphi(u) \in L^{1,\ \beta(x)}(\Omega) \cap L^{\psi(x)}(\Omega)\,^{2}$$

to show this, by the definition of the spaces $L^{1,\ \beta(x)}(\Omega)$ and $L^{\psi(x)}(\Omega)$, it is sufficient to prove that $\sigma_\beta(D_i v)$ and $\sigma_\psi(v)$ are finite. As

$$\sigma_\psi(v) = \int_\Omega |v|^{\psi(x)}\,dx = \int_\Omega |u|^{\frac{\psi(x)(\gamma(x)+\beta(x))}{\beta(x)}}\,dx = \int_\Omega |u|^{\theta(x)}\,dx,$$

the above equation ensures that $\sigma_\psi(v)$ is finite.

For $i = \overline{1,n}$, let us show that $\sigma_\beta(D_i v)$ is finite.

$$\begin{aligned}\sigma_\beta(D_i v) &= \int_\Omega |D_i v|^{\beta(x)}\,dx = \int_\Omega \left|D_i\left(|u|^{\frac{\gamma(x)}{\beta(x)}} u\right)\right|^{\beta(x)} dx\\ &= \int_\Omega \left|\left(\tfrac{\gamma(x)+\beta(x)}{\beta(x)}\right)|u|^{\frac{\gamma(x)}{\beta(x)}} D_i u + D_i\left(\tfrac{\gamma(x)}{\beta(x)}\right)|u|^{\frac{\gamma(x)}{\beta(x)}} u\ln|u|\right|^{\beta(x)} dx,\end{aligned}$$

by applying Corollary 3.3 to estimate the integral on the right hand side of the above equation and estimate the corresponding coefficients, we obtain

$$\sigma_\beta(D_i v) \le C_1\int_\Omega |u|^{\gamma(x)}|D_i u|^{\beta(x)}\,dx + C_2\int_\Omega |u|^{\theta(x)}\,dx + C_3,$$

here $C_1 = C_1(\beta^\pm,\gamma^+)$, $C_2 = C_2\left(\beta^\pm,\gamma^+,\ \|\gamma\|_{C^1(\bar\Omega)}, \|\beta\|_{C^1(\bar\Omega)}, \varepsilon_0\right)$ and $C_3 = C_3(\beta^+,|\Omega|,\varepsilon_0) > 0$ are constants. ($\varepsilon_0 > 0$, comes from Proposition 2.2 which satisfy $\theta(x) \ge \gamma(x) + \beta(x) + \varepsilon_0$.)

For $C_4 := \max\{C_1, C_2\}$, we have

$$\begin{aligned}\sigma_\beta(D_i v) &\le C_4\left(\int_\Omega |u|^{\theta(x)}\,dx + \sum_{i=1}^{n}\int_\Omega |u|^{\gamma(x)}|D_i u|^{\beta(x)}\,dx\right) + C_3\\ &= C_4\Re^{\gamma,\beta,\theta}(u) + C_3 \qquad (3.5)\end{aligned}$$

since $u \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$, we get the desired result by (3.5).

Conversely, we need to show that for all $v \in L^{1,\ \beta(x)}(\Omega) \cap L^{\psi(x)}(\Omega)$

$$w := |v|^{-\frac{\gamma(x)}{\gamma(x)+\beta(x)}} v = \varphi^{-1}(v) \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega).$$

From the definition of the space $S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$, it is sufficient to prove $\Re^{\gamma,\beta,\theta}(w)$ is finite. By using the similar process and results as mentioned

[2]From now on, we denote $\varphi(x,u) := \varphi(u) = |u|^{\frac{\gamma(x)}{\beta(x)}} u$ for simplicity.

above, we attain

$$\begin{aligned}\Re^{\gamma,\beta,\theta}(w) &= \int_\Omega |w|^{\theta(x)}\,dx + \sum_{i=1}^n \int_\Omega |w|^{\gamma(x)}\,|D_i w|^{\beta(x)}\,dx \\ &= \int_\Omega |v|^{\frac{\beta(x)\theta(x)}{\gamma(x)+\beta(x)}}\,dx + \sum_{i=1}^n \int_\Omega |v|^{\frac{\gamma(x)\beta(x)}{\gamma(x)+\beta(x)}} \left| D_i \left(|v|^{-\frac{\gamma(x)}{\gamma(x)+\beta(x)}}\, v\right)\right|^{\beta(x)} dx \\ &\le C_5 \sum_{i=1}^n \int_\Omega |D_i v|^{\beta(x)}\,dx + C_6 \int_\Omega |v|^{\psi(x)}\,dx + C_7, \end{aligned} \tag{3.6}$$

here $C_5 = C_5\left(\beta^+\right) > 0$, $C_6 = C_6\left(\beta^+, \varepsilon_1, \|\gamma\|_{C^1(\bar\Omega)}, \|\beta\|_{C^1(\bar\Omega)}\right) > 0$ and $C_7 = C_7\left(\beta^+, \varepsilon_1, |\Omega|\right) > 0$ are constants. Since $v \in L^{1,\,\beta(x)}(\Omega) \cap L^{\psi(x)}(\Omega)$, $w \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ by (3.6).

To complete the proof, it remains to verify that φ is a bijective mapping, by Lemma 3.6 for fixed $x_0 \in \Omega$, $\varphi(t) := \varphi(x_0, t)$ and $\varphi^{-1}(\tau) := \varphi^{-1}(x_0, \tau)$ are strictly monotone that yields for every $v \in L^{1,\,\beta(x)}(\Omega) \cap L^{\psi(x)}(\Omega)$, there exists an unique $u \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ such that $u = \varphi^{-1}(v)$, and for every $u \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$, there exists a unique $v \in L^{1,\,\beta(x)}(\Omega) \cap L^{\psi(x)}(\Omega)$ such that $v = \varphi(u)$ which verifies the bijectivity of φ. □

Now, we give two important results of Theorem 3.8 which help us to understand the topology of the space $S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$.

Corollary 3.9. *Let β, γ and ψ satisfy the conditions of Theorem 3.8. Then $S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ is a complete metric space with the metric which is defined below. $\forall u,\ v \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$,*

$$d_{S_1}(u,v) := \|\varphi(u) - \varphi(v)\|_{L^{\psi(x)}(\Omega)} + \sum_{i=1}^n \|\varphi'_t(u)\,D_i u - \varphi'_t(v)\,D_i u\|_{L^{\beta(x)}(\Omega)},$$

here $\varphi(x,t) = |t|^{\frac{\gamma(x)}{\beta(x)}}\, t$ and for every fixed $x \in \Omega$, $\varphi'_t(t) = \left(\frac{\gamma(x)}{\beta(x)} + 1\right) |t|^{\frac{\gamma(x)}{\beta(x)}}$.

Corollary 3.10. *Under the conditions of Corollary 3.9, φ is a homeomorphism between the spaces $S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ and $L^{1,\,\beta(x)}(\Omega) \cap L^{\psi(x)}(\Omega)$.*

Proof. (Sketch of the proof) Since we have showed that φ is a bijection between $S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ and $L^{1,\,\beta(x)}(\Omega) \cap L^{\psi(x)}(\Omega)$, it is sufficient to prove the continuity of φ as well as φ^{-1} in the sense of topology induced by the metric $d_{S_1}(\cdot,\cdot)$. For this, we need to show that

$$\text{(a)} \qquad d_{S_1}(u_m, u_0) \underset{m\nearrow\infty}{\longrightarrow} 0 \Rightarrow \varphi(u_m) \overset{L^{1,\,\beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)}{\underset{m\nearrow\infty}{\longrightarrow}} \varphi(u_0)$$

for all $\{u_m\}_{m=1}^\infty \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ which converges to u_0 and

$$\text{(b)} \qquad v_m \overset{L^{1,\,\beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)}{\underset{m\nearrow\infty}{\longrightarrow}} v_0 \Rightarrow d_{S_1}\left(\varphi^{-1}(v_m), \varphi^{-1}(v_0)\right) \underset{m\nearrow\infty}{\longrightarrow} 0$$

for all $\{v_m\}_{m=1}^{\infty} \in L^{1,\ \beta(x)}(\Omega) \cap L^{\psi(x)}(\Omega)$ which converges to v_0.

Since for all v_m and v_0, there exist unique u_m and $u_0 \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ such that $\varphi(u_m) = v_m$ and $\varphi(u_0) = v_0$, the implication (b) can be written equivalently, $\varphi(u_m) \underset{m\nearrow\infty}{\overset{L^{1,\ \beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)}{\longrightarrow}} \varphi(u_0) \Rightarrow d_{S_1}(u_m, u_0) \underset{m\nearrow\infty}{\longrightarrow} 0$ for all $u_m \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$ which converges to u_0. Since the proofs of (a) and (b) are similar, we only prove (b): Let v_0, $\{v_m\}_{m=1}^{\infty} \in L^{1,\ \beta(x)}(\Omega) \cap L^{\psi(x)}(\Omega)$ and $v_m \overset{L^{1,\ \beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)}{\longrightarrow} v_0 \Leftrightarrow \varphi(u_m) \overset{L^{1,\ \beta(x)}(\Omega)\cap L^{\psi(x)}(\Omega)}{\longrightarrow} \varphi(u_0)$.

To verify $d_{S_1}(u_m, u_0) \to 0$, by definition of d_{S_1} it is sufficient to establish that

$$\|\varphi_t'(u_m) D_i u_m - \varphi_t'(u_0) D_i u_0\|_{L^{\beta(x)}(\Omega)} \to 0 \text{ and } \|\varphi(u_m) - \varphi(u_0)\|_{L^{\psi(x)}(\Omega)} \to 0$$

as $m \nearrow \infty$. The second convergence above is obvious by definition of d_{S_1} and the first one can be proved by applying Theorem 3.8 and Vitali convergence theorem by virtue of the equivalence $\|\varphi_t'(u_m) D_i u_m - \varphi_t'(u_0) D_i u_0\|_{L^{\beta(x)}(\Omega)} \to 0 \Leftrightarrow \sigma_\beta(\varphi_t'(u_m) D_i u_m - \varphi_t'(u_0) D_i u_0) \to 0$. □

Theorem 3.11. *Suppose that conditions of Theorem 3.8 are satisfied. Let $p \in M_0(\Omega)$, additionally β satisfies $1 \leq \beta^- \leq \beta(x) < n$, $x \in \Omega$ and for $\varepsilon > 0$, the inequality*

$$p(x) + \varepsilon < \tfrac{n(\gamma(x)+\beta(x))}{n-\beta(x)},\ x \in \Omega$$

is satisfied. Then the compact embedding

$$S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$$

holds.

Proof. First, we show that $S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega) \subset L^{p(x)}(\Omega)$, then we prove the compactness of this embedding.

For every $u \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$, by Theorem 3.8

$$\varphi(u) = |u|^{\frac{\gamma(x)}{\beta(x)}} u = v \in L^{1,\ \beta(x)}(\Omega) \cap L^{\psi(x)}(\Omega).$$

Since $L^{1,\ \beta(x)}(\Omega) \cap L^{\psi(x)}(\Omega) \subset W^{1,\ \beta(x)}(\Omega)$ and the embedding $W^{1,\ \beta(x)}(\Omega) \subset L^{\beta^*(x)}(\Omega)$ exists for $\beta^*(x) = \frac{n\beta(x)}{n-\beta(x)}$, we get $v \in L^{\beta^*(x)}(\Omega)$. From the definition of v and the space $L^{\beta^*(x)}(\Omega)$ we attain

$$v \in L^{\beta^*(x)}(\Omega) \Leftrightarrow u \in L^{\frac{n(\gamma(x)+\beta(x))}{n-\beta(x)}}(\Omega).$$

On the other side, by the conditions of theorem, we have

$$L^{\frac{n(\gamma(x)+\beta(x))}{n-\beta(x)}}(\Omega) \subset L^{p(x)+\varepsilon}(\Omega) \subset L^{p(x)}(\Omega),$$

which yields $u \in L^{p(x)}(\Omega)$.

To prove the compactness of this embedding, let

$$\{u_m\}_{m=1}^{\infty} \in S_{1,\gamma(x),\beta(x),\theta(x)}(\Omega)$$

be a bounded sequence (i.e., $[u_m]_{\gamma,\beta,\theta} < \infty$, $\forall m \geq 1$). From Theorem 3.8, we have

$$\{\varphi(u_m)\} = \{v_m\}_{m=1}^{\infty} \in W^{1,\,\beta(x)}(\Omega).$$

It follows from the compact embedding result

$$W^{1,\,\beta(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega),$$

where $q(x) < \beta^*(x) - \tilde{\varepsilon}$, $x \in \Omega$ and $\tilde{\varepsilon} \in (0, n/n-1)$ [9,11] ($\beta \in C^1(\bar{\Omega})$ yields β is log-Hölder continuous) that, there exists a subsequence $\{v_{m_j}\} \subset \{v_m\}$ such that

$$v_{m_j} \overset{L^{q(x)}(\Omega)}{\longrightarrow} v_0, \tag{3.7}$$

we attain by (3.7),

$$v_{m_j} \xrightarrow[\Omega]{a.e.} v_0.$$

From Lemma 3.6, $\varphi^{-1}(x,\tau) = |\tau|^{\frac{\gamma(x)}{\gamma(x)+\beta(x)}}\tau$ is continuous (with respect to τ and x) so we have

$$\varphi^{-1}(v_{m_j}) \xrightarrow[\Omega]{a.e.} \varphi^{-1}(v_0).$$

To end the proof, we use Lemma 3.12[3] (Theorem 7 in [16]).

Denote $u_0 := \varphi^{-1}(v_0)$ and the set

$$\Lambda := \left\{ f_j \mid f_j(x) = \left|u_{m_j}(x) - u_0(x)\right|^{p(x)} \right\}$$

and the function

$$\Phi(t) := t^{\bar{\varepsilon}},\ t \geq 0,\ \bar{\varepsilon} = \frac{\varepsilon}{p^+}\ .$$

Clearly, $\Phi : [0,\infty) \to [0,\infty)$ is increasing and $\lim\limits_{t\to+\infty} \Phi(t) = +\infty$.

Furthermore for all $f_j \in \Lambda$ we have,

$$\begin{aligned}\int_\Omega |f_j(x)|\,\Phi(|f_j(x)|)\,dx &= \int_\Omega \left|u_{m_j} - u_0\right|^{p(x)} \left|u_{m_j} - u_0\right|^{\bar{\varepsilon}p(x)} dx \\ &= \int_\Omega \left|u_{m_j} - u_0\right|^{(\bar{\varepsilon}+1)p(x)} dx\ .\end{aligned}$$

Estimating the last integral by using Lemma 3.1, we arrive at

$$\leq \int_\Omega \left|u_{m_j} - u_0\right|^{p(x)+\varepsilon} dx + |\Omega|,$$

---

[3]**Lemma 3.12** Let Λ be a family of real functions defined on bounded domain Ω. If there is an increasing function $\Phi : [0,\ \infty) \to [0,\ \infty)$ that satisfies

$$\lim_{t\to+\infty} \Phi(t) = +\infty$$

and there is a positive constant L such that

$$\int_\Omega |f_\alpha(x)|\,\Phi(|f_\alpha(x)|)\,dx \leq L, \quad \forall f_\alpha \in \Lambda,$$

then every function in Λ is Lebesgue integrable, and the functions family Λ possesses absolutely equicontinuous integrals on Ω.

by using the well known inequality for the absolute value above, we get

$$\leq 2^{p^{+}+\varepsilon-1}\left(\int_{\Omega}\left|u_{m_j}\right|^{p(x)+\varepsilon}dx+\int_{\Omega}\left|u_0\right|^{p(x)+\varepsilon}dx\right)+\left|\Omega\right|. \tag{3.8}$$

Since $u_0, \left\{u_{m_j}\right\} \subset L^{p(x)+\varepsilon}(\Omega)$ are bounded, from (3.8), there exists a number $L>0$ such that

$$\int_{\Omega}\left|f_j(x)\right|\Phi\left(\left|f_j(x)\right|\right)dx \leq L, \tag{3.9}$$

here $L = L\left(\left|\Omega\right|, p^{+}, \varepsilon, \left\|u_0\right\|_{L^{p(x)+\varepsilon}(\Omega)}, \left\|u_{m_j}\right\|_{L^{p(x)+\varepsilon}(\Omega)}\right)$.

Consequently it follows from (3.9) that the family of functions Λ have absolutely equicontinuous integrals on Ω. Hence using this and $u_{m_j} \underset{\Omega}{\xrightarrow{a.e.}} u_0$, we obtain [16]

$$\int_{\Omega}\left|u_{m_j}(x)-u_0(x)\right|^{p(x)}dx \longrightarrow 0,\ m_j \nearrow \infty,$$

that implies $\left\|u_{m_j}-u_0\right\|_{L^{p(x)}(\Omega)} \rightarrow 0$ which ends the proof. □

4. Proof of the existence theorem

The proof is based on Theorem 2.5. We introduce the following spaces and mappings in order to apply Theorem 2.5 to prove Theorem 2.4.

$$\begin{aligned}
&S_{gY_0} := \mathring{S}_{1,q_0(x)(p_0(x)-2),q_0(x),\alpha(x)}(\Omega),\ X := W_0^{1,\,p_1(x)}(\Omega),\\
&Y := W^{-1,\,q_0(x)}(\Omega) + L^{\alpha'(x)}(\Omega),\\
&X_0 := W_0^{1,\,p_0(x)}(\Omega) \cap W_0^{1,\,p_1(x)}(\Omega) \cap L^{\alpha(x)}(\Omega),\\
&Y_{2_1} := W^{-1,\,2}(\Omega),\quad Y_{2_2} := L^2(\Omega)
\end{aligned}$$

and

$$A(u) := -div\left(\left|\nabla u\right|^{p_1(x)-2}\nabla u\right), \tag{4.1}$$

$$B_1(u) := -\sum_{i=1}^{n} D_i\left(\left|u\right|^{p_0(x)-2}D_i u\right),\quad B_2(u) := c(x,u), \tag{4.2}$$

$$B := B_1 + B_2 \text{ and } T := A + B. \tag{4.3}$$

We show that all the conditions of Theorem 2.5 are satisfied by proving some lemmas. Then, we establish the proof of Theorem 2.4 based on these lemmas.

Lemma 4.1. *Under the conditions of Theorem 2.4, the operator T defined by (4.3) is coercive in the generalized sense on X_0.*

Proof. For all $u \in W_0^{1,\,p_0(x)}(\Omega) \cap W_0^{1,\,p_1(x)}(\Omega) \cap L^{\alpha(x)}(\Omega)$, we have

$$\begin{aligned}
\left\langle T(u), u\right\rangle &= \left\langle A(u), u\right\rangle + \left\langle B(u), u\right\rangle\\
&= \int_{\Omega}\left|\nabla u\right|^{p_1(x)}dx
\end{aligned}$$

$$+\sum_{i=1}^{n}\int_{\Omega}|u|^{p_0(x)-2}|D_iu|^2\,dx+\int_{\Omega}c(x,u)\,udx. \tag{4.4}$$

If we take account the condition (U2) into the last integral of (4.4) and apply the following simple calculated inequality

$$\int_{\Omega}|u|^{\alpha(x)}\,dx+\sum_{i=1}^{n}\int_{\Omega}|u|^{q_0(x)(p_0(x)-2)}|D_iu|^{q_0(x)}\,dx$$
$$\leq (n+1)\left(\int_{\Omega}|u|^{\alpha(x)}\,dx+\sum_{i=1}^{n}\int_{\Omega}|u|^{p_0(x)-2}|D_iu|^2\,dx+|\Omega|\right), \tag{4.5}$$

we get

$$\langle T(u),u\rangle \geq C_8\left(\int_{\Omega}|u|^{\alpha(x)}\,dx+\sum_{i=1}^{n}\int_{\Omega}|u|^{q_0(x)(p_0(x)-2)}|D_iu|^{q_0(x)}\right)$$
$$+\int_{\Omega}|\nabla u|^{p_1(x)}\,dx-C_9, \tag{4.6}$$

here $C_8=C_8\left(n,\tilde{C}\right)>0$ and $C_9=C_9\left(n,|\Omega|,\tilde{C}\right)>0$ are constants.

Applying Lemma 3.4 to estimate the right-hand side of the inequality (4.6), we obtain

$$\langle T(u),u\rangle \geq C_{10}\left([u]^{q_0^-+1}_{S_{q_0(p_0-2),q_0,\alpha}}+\|u\|^{p_1^-}_{W_0^{1,\,p_1(x)}(\Omega)}\right)-C_{11}, \tag{4.7}$$

by the definitions of $[\cdot]_{S_{q_0(p_0-2),q_0,\alpha}}$ and $\|.\|_{W_0^{1,\,p_1(x)}(\Omega)}$.

Since $q_0^-+1>2$ and $p_1^->1$, $\lambda_0(\tau)=\tau^{q_0^-}$ and $\lambda_1(\tau)=\tau^{p_1^--1}$ tends to infinity when $\tau\nearrow\infty$, (see Theorem 2.5) we deduce that operator T is coercive in the generalized sense on X_0. □

Lemma 4.2. *Under the conditions of Theorem 2.4, the operator A is monotone and bounded from $W_0^{1,\,p_1(x)}(\Omega)$ into $W^{-1,\,q_1(x)}(\Omega)$.*

Proof. First we prove that $A: W_0^{1,\,p_1(x)}(\Omega)\to W^{-1,\,q_1(x)}(\Omega)$ is bounded. For this, it is sufficient to investigate the dual form $\langle A(u),v\rangle$ for every $v\in W_0^{1,\,p_1(x)}(\Omega)$,

$$|\langle A(u),v\rangle|=\left|\int_{\Omega}|\nabla u|^{p_1(x)-2}\nabla u\cdot\nabla vdx\right|,$$

by using (2.1) to the right hand side of the above equation, we get

$$|\langle A(u),v\rangle|\leq 2\left\||\nabla u|^{p_1(x)-1}\right\|_{L^{q_1(x)}(\Omega)}\|v\|_{W_0^{1,\,p_1(x)}(\Omega)}. \tag{4.8}$$

From (4.8), we demonstrate the boundedness of A from $W_0^{1,\,p_1(x)}(\Omega)$ into $W^{-1,\,q_1(x)}(\Omega)$.

Now let us show that $A : W_0^{1,\ p_1(x)}(\Omega) \to W^{-1,\ q_1(x)}(\Omega)$ is a monotone operator.

Indeed for all $u, v \in W_0^{1,\ p_1(x)}(\Omega)$ we have,

$$\begin{aligned} &\langle A(u) - A(v), u - v\rangle \\ &= \int_\Omega \left(|\nabla u|^{p_1(x)-2} \nabla u - |\nabla v|^{p_1(x)-2} \nabla v\right) \cdot (\nabla u - \nabla v)\, dx. \end{aligned}$$

Since the inequality $\left(|a|^{p-2} a - |b|^{p-2} b\right) \cdot (a-b) \geq 0$ is valid for $1 < p < \infty$, $a, b \in \mathbb{R}^n$ from the last equality, we attain

$$\langle A(u) - A(v), u - v\rangle \geq 0$$

that completes the proof. [4] □

Lemma 4.3. *Under the conditions of Theorem 2.4, B is a bounded operator from $\mathring{S}_{1,q_0(x)(p_0(x)-2),q_0(x),\alpha(x)}(\Omega)$ into $W^{-1,\ q_0(x)}(\Omega) + L^{\alpha'(x)}(\Omega)$.*

Proof. Since $B = B_1 + B_2$, we shall show that both B_1 and B_2 are bounded.

First let us verify $B_2 : \mathring{S}_{1,q_0(x)(p_0(x)-2),q_0(x),\alpha(x)}(\Omega) \to W^{-1,\ q_0(x)}(\Omega) + L^{\alpha'(x)}(\Omega)$ is bounded.

As $\mathring{S}_{1,q_0(x)(p_0(x)-2),q_0(x),\alpha(x)}(\Omega) \subset L^{\alpha(x)}(\Omega)$, it is sufficient to show the boundedness of B_2, from $L^{\alpha(x)}(\Omega)$ to $L^{\alpha'(x)}(\Omega)$.

For every $u \in L^{\alpha(x)}(\Omega)$

$$\sigma_{\alpha'}(B_2(u)) = \int_\Omega |B_2(u)|^{\alpha'(x)} dx = \int_\Omega |c(x,u)|^{\alpha'(x)} dx,$$

by taking the conditions of Theorem 2.4 into account and estimating the above integral, we obtain

$$\sigma_{\alpha'}(B_2(u)) \leq 2\left(\|c_0\|^2_{L^\infty(\Omega)} \sigma_\alpha(u) + \sigma_{\alpha'}(c_1)\right). \tag{4.9}$$

We deduce from (4.9) that B_2 is bounded.

Let us prove that $B_1 : \mathring{S}_{1,q_0(x)(p_0(x)-2),q_0(x),\alpha(x)}(\Omega) \to W^{-1,\ q_0(x)}(\Omega) + L^{\alpha'(x)}(\Omega)$ is bounded. For $i = \overline{1,n}$, denote $b_i(x) := |u|^{p_0(x)-2} |D_i u(x)|$ and for all $v \in W_0^{1,\ p_0(x)}(\Omega)$

$$|\langle B_1(u), v\rangle| = \left| -\sum_{i=1}^n \int_\Omega D_i\left(|u|^{p_0(x)-2} D_i u\right) v dx\right|,$$

by applying (2.1) to the right hand side of the above equation, we arrive at

$$|\langle B_1(u), v\rangle| \leq 2\left(\sum_{i=1}^n \|b_i\|_{L^{q_0(x)}(\Omega)}\right) \|v\|_{W_0^{1,\ p_0(x)}(\Omega)}.$$

[4] Here, we specify that since A is monotone and hemicontinuous, it is pseudo-monotone [25].

Since $u \in \mathring{S}_{1,q_0(x)(p_0(x)-2),q_0(x),\alpha(x)}(\Omega)$, from (2.3) and the definition of the functions $b_i(x)$, obviously $\sum\limits_{i=1}^{n} \|b_i\|_{L^{q_0(x)}(\Omega)} < \infty$. Thus, we verify that B_1 is bounded which concludes that B is a bounded operator from

$$\mathring{S}_{1,q_0(x)(p_0(x)-2),q_0(x),\alpha(x)}(\Omega)$$

into $W^{-1,\, q_0(x)}(\Omega) + L^{\alpha'(x)}(\Omega)$. □

Lemma 4.4. *Under the conditions of Theorem 2.4, B is a weakly compact operator from $\mathring{S}_{1,q_0(x)(p_0(x)-2),q_0(x),\alpha(x)}(\Omega)$ into $W^{-1,\, q_0(x)}(\Omega) + L^{\alpha'(x)}(\Omega)$.*

Proof. Since $B = B_1 + B_2$, we shall show that both B_1 and B_2 are weakly compact. First, we show the weak compactness of B_1. Let $\{u_m\}_{m=1}^{\infty}$, $u_0 \in \mathring{S}_{1,q_0(x)(p_0(x)-2),q_0(x)}(\Omega) \cap L^{\alpha(x)}(\Omega)$ and $u_m \overset{S_{gY_0}}{\rightharpoonup} u_0$. By Theorem 3.8 we have

$$\{w_m\}_{m=1}^{\infty} := \{\varphi(u_m)\}_{m=1}^{\infty} = \left\{|u_m|^{p_0(x)-2} u_m\right\}_{m=1}^{\infty} \subset W_0^{1,\, q_0(x)}(\Omega).$$

As $q_0^- > 1$, $W_0^{1,\, q_0(x)}(\Omega)$ is a reflexive space. Thus, there exists a subsequence $\{w_{m_j}\}_{j=1}^{\infty}$ of $\{w_m\}$ such that

$$w_{m_j} = \left|u_{m_j}\right|^{p_0(x)-2} u_{m_j} \overset{W_0^{1,\, q_0(x)}(\Omega)}{\rightharpoonup} \xi.$$

Let us verify $\xi = |u_0|^{p_0(x)-2} u_0$. Since $W_0^{1,\, q_0(x)}(\Omega) \hookrightarrow L^{q_0(x)}(\Omega)$, there exists a subsequence $\left\{w_{m_{j_k}}\right\} \subset \{w_{m_j}\}$ (denote this subsequence by w_{m_j} in order to avoid notation confusion) such that

$$\left|u_{m_j}\right|^{p_0(x)-2} u_{m_j} \overset{L^{q_0(x)}(\Omega)}{\longrightarrow} \xi$$

yields

$$\varphi\left(u_{m_j}\right) = \left|u_{m_j}\right|^{p_0(x)-2} u_{m_j} \underset{a.e.}{\overset{\Omega}{\longrightarrow}} \xi. \tag{4.10}$$

From Lemma 3.6, $\varphi^{-1}(x,\tau) = |\tau|^{-\frac{p_0(x)-2}{p_0(x)-1}} \tau$ is continuous (with respect to τ and x) by using this and (4.10) we obtain

$$u_{m_j} \underset{a.e.}{\overset{\Omega}{\longrightarrow}} \varphi^{-1}(x,\xi) = \varphi^{-1}(\xi), \tag{4.11}$$

by (4.11), we arrive at $\varphi^{-1}(\xi) = u_0$, equivalently $\xi = |u_0|^{p_0(x)-2} u_0$.

To verify the weak compactness of B_1, we must show that for arbitrary $v \in W_0^{1,\, p_0(x)}(\Omega)$

$$\langle B_1\left(u_{m_j}\right), v\rangle \to \langle B_1\left(u_0\right), v\rangle, \quad j \nearrow \infty.$$

By the definition of operator B_1,

$$\langle B_1\left(u_{m_j}\right), v\rangle = \sum_{i=1}^{n} \langle -D_i\left(\left|u_{m_j}\right|^{p_0(x)-2} D_i u_{m_j}\right), v\rangle$$

$$= \sum_{i=1}^{n} \langle \left| u_{m_j} \right|^{p_0(x)-2} D_i u_{m_j},\ D_i v \rangle. \tag{4.12}$$

We have the following equality by using Lemma 3.6 and chain rule

$$\begin{aligned} D_i \left(\left| u_{m_j} \right|^{p_0(x)-2} u_{m_j} \right) = &\ (p_0(x) - 2) \left| u_{m_j} \right|^{p_0(x)-2} D_i u_{m_j} \\ &+ (D_i p_0) \left| u_{m_j} \right|^{p_0(x)-2} u_{m_j} \ln \left| u_{m_j} \right|. \end{aligned} \tag{4.13}$$

If we insert the equality (4.13) into (4.12), we obtain

$$\begin{aligned} \langle B_1 \left(u_{m_j} \right), v \rangle = &\ \sum_{i=1}^{n} \langle \left(\tfrac{1}{p_0(x)-2} \right) D_i \left(\left| u_{m_j} \right|^{p_0(x)-2} u_{m_j} \right), D_i v \rangle \\ &- \sum_{i=1}^{n} \langle \left(\tfrac{D_i p_0}{p_0(x)-2} \right) \left| u_{m_j} \right|^{p_0(x)-2} u_{m_j} \ln \left| u_{m_j} \right|, D_i v \rangle. \end{aligned} \tag{4.14}$$

Let us denote the first sum in (4.14) by I_1 and the second one by I_2, i.e.,

$$\langle B_1 \left(u_{m_j} \right), v \rangle = I_1 - I_2,$$

if we use the same manner in [20] (Lemma 3.3) and pass to the limit in I_1, we obtain

$$I_1 \underset{j \nearrow \infty}{\longrightarrow} \sum_{i=1}^{n} \langle \left(\tfrac{1}{p_0(x)-2} \right) D_i \left(|u_0|^{p_0(x)-2} u_0 \right), D_i v \rangle. \tag{4.15}$$

Considering Lemma 3.2 together with Theorem 3.11 and continuity of the function $|t|^{p_0(x)-2} t \ln |t|$ with respect to t and pass to the limit in I_2, we obtain

$$I_2 \underset{j \nearrow \infty}{\longrightarrow} \sum_{i=1}^{n} \langle \left(\tfrac{D_i p_0}{p_0(x)-2} \right) |u_0|^{p_0(x)-2} u_0 \ln |u_0|, D_i v \rangle. \tag{4.16}$$

From (4.15) and (4.16), we have

$$\begin{aligned} \langle B_1 \left(u_{m_j} \right), v \rangle &\underset{j \nearrow \infty}{\longrightarrow} \sum_{i=1}^{n} \langle \left(\tfrac{1}{p_0(x)-2} \right) D_i \left(|u_0|^{p_0(x)-2} u_0 \right), D_i v \rangle \\ &- \sum_{i=1}^{n} \langle \left(\tfrac{D_i p_0}{p_0(x)-2} \right) |u_0|^{p_0(x)-2} u_0 \ln |u_0|, D_i v \rangle \\ = &\sum_{i=1}^{n} \langle \tfrac{1}{p_0(x)-2} \left[D_i \left(|u_0|^{p_0(x)-2} u_0 \right) - (D_i p_0) |u_0|^{p_0(x)-2} u_0 \ln |u_0| \right], D_i v \rangle \\ &\text{by (4.13),} \\ = &\sum_{i=1}^{n} \langle |u_0|^{p_0(x)-2} D_i u_0,\ D_i v \rangle = \langle B_1 (u_0), v \rangle. \end{aligned}$$

Therefore, we prove the weak compactness of B_1 from $\mathring{S}_{1,q_0(x)(p_0(x)-2),q_0(x),\alpha(x)}(\Omega)$ to $W^{-1,\ q_0(x)}(\Omega) + L^{\alpha'(x)}(\Omega)$.

Now, we prove the weak compactness of B_2. As $(\alpha')^- > 1$, $L^{\alpha'(x)}(\Omega)$ is a reflexive space and $\{B_2(u_m)\}_{m=1}^{\infty} := \{\eta_m\}_{m=1}^{\infty} \subset L^{\alpha'(x)}(\Omega)$ is bounded (see, Lemma 4.3), so there exists a subsequence $\left\{\eta_{m_j}\right\} \subset \{\eta_m\}$ such that

$$\eta_{m_j} = B_2\left(u_{m_j}\right) \overset{L^{\alpha'(x)}(\Omega)}{\rightharpoonup} \psi.$$

The embedding

$$\mathring{S}_{1,q_0(x)(p_0(x)-2),q_0(x),\alpha(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega) \tag{4.17}$$

is compact by Theorem 3.11 for $s(\cdot)$ which satisfies the inequality $s(x) < \frac{np_0(x)}{n-q_0(x)}$, $x \in \Omega$.

By (4.17), there exists a subsequence $\left\{u_{m_{j_k}}\right\} \subset \left\{u_{m_j}\right\}$ (let us denote this subsequence by u_{m_j} in order to avoid notation confusion) such that

$$u_{m_j} \overset{L^{s(x)}(\Omega)}{\longrightarrow} u_0$$

yields

$$u_{m_j} \xrightarrow[a.e.]{\Omega} u_0. \tag{4.18}$$

Since the function $c(x,\tau)$ is continuous with respect to variable τ ($c(x,\tau)$ is Carathèdory function), by (4.18) we get

$$B_2\left(u_{m_j}\right) = c\left(x, u_{m_j}\right) \xrightarrow[a.e.]{\Omega} c\left(x, u_{m_j}\right) = B_2(u_0). \tag{4.19}$$

We deduce from (4.19) that $\psi = B_2(u_0)$.

Finally, we arrive at

$$B_2\left(u_{m_j}\right) \overset{L^{\alpha'(x)}(\Omega)}{\rightharpoonup} B_2(u_0)$$

which implies that

$$B_2\left(u_{m_j}\right) \overset{W^{-1,\,q_0(x)}(\Omega)+L^{\alpha'(x)}(\Omega)}{\rightharpoonup} B_2(u_0). \tag{4.20}$$

As a result, by (4.20) we obtain the weak compactness of B_2 which provides the weak compactness of the operator B. ☐

It remains to define the corresponding operators for B in condition "2)" of Theorem 2.5 to apply this theorem to the problem (1.1).

Since $B = B_1 + B_2$, according to condition "2)" we define corresponding B_{01} with regard to B_1 and corresponding B_{02} with regard to B_2 as below:

$$B_{01}(u) := -\sum_{i=1}^{n} D_i\left(|u|^{\frac{p_0(x)-2}{2}} D_i u\right)$$

and

$$B_{02}(u) := \left[c(x,u)\,u\right]^{\frac{1}{2}}.$$

Note that here $c(x,\tau)\,\tau > 0$ by the condition (U2).

By the same arguments which are used in the proof of Lemma 4.3 to establish the boundedness of the operators B_1 and B_2, we can show that the operators B_{01} and B_{02} are bounded between the spaces which are introduced below:

$$B_{01} : X_0 \subset \overset{\circ}{S}_{1,p_0(x)-2,2,\alpha(x)}(\Omega) \longrightarrow W^{-1,\,2}(\Omega)$$

and

$$B_{02} : X_0 \subset \overset{\circ}{S}_{1,q_0(x)(p_0(x)-2),q_0(x),\alpha(x)}(\Omega) \longrightarrow L^2(\Omega).$$

Here, we have to prove the weak compactness of B_{01} and B_{02} to show that condition "2)" in Theorem 2.5 is satisfied.

By using the similar manner which has been presented in the proof of Lemma 4.4, and by the definition of the functionals corresponding to operators B_{01} and B_{02} (see (4.21), (4.22)), following lemmas can be proved straightforwardly, so we omit the proofs of them.

Lemma 4.5. *Under the conditions of Theorem 2.4, B_{01} is weakly compact operator from X_0 into $W^{-1,\,2}(\Omega)$. Moreover, for the function $\mu(\tau) = \tau^2$ and for all $u \in X_0$, the equality*

$$\langle B_1(u), u\rangle \equiv \mu\left(\|B_{01}(u)\|_{W^{-1,\,2}(\Omega)}\right) \tag{4.21}$$

holds.

Lemma 4.6. *Under the conditions of Theorem 2.4, B_{02} is weakly compact operator from X_0 into $L^2(\Omega)$. Moreover, for all $u \in X_0$, the equality*

$$\langle B_2(u), u\rangle \equiv \mu\left(\|B_{02}(u)\|_{L^2(\Omega)}\right) \tag{4.22}$$

holds.

Now we can give the proof of Theorem 2.4.

Proof of Theorem 2.4. From Lemma 4.1-Lemma 4.6, we show that all the conditions of Theorem 2.5 are satisfied for problem (1.1) under the conditions of Theorem 2.4. Thus, we conclude that Theorem 2.5 can be applied to the problem (1.1). By using this theorem, we prove the existence of a weak solution of problem (1.1) in the sense of Definition 2.3. □

Remark 4.7 (Uniqueness). Our purpose in this article is basically to answer the question of solvability of the equation with variable exponent nonlinearities when the exponents are independent of each others. Although there exist methods for examining the uniqueness of the equations including only $A(u)$ or only $B(u)$ which are defined by (4.1) and (4.3), uniqueness of the solution is a quite difficult problem in our case when the equation is sum of these operators and exponents are not dependent.

If we use the well-known and common approach to prove the uniqueness of problem (1.1), then it is emerged that there must be very strong and certain conditions on the exponents and on the norm of the solution in the proper spaces. (For example, the exponents $p_0(\cdot)$ and $p_1(\cdot)$ must be dependent each other, the norm of the solution in the proper space need to be sufficiently small etc.) Since obtained results would be worthless and weak under the above mentioned conditions, in the present paper we are not concerned with the uniqueness of the problem (1.1). We have been studying on this problem and as a future work, we plan to verify the uniqueness under more general and weak conditions.

References

[1] E. Acerbi and G. Mingione, *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal. **164** (2002), no. 3, 213–259.
[2] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
[3] S. N. Antontsev and J. F. Rodrigues, *On stationary thermo-rheological viscous flows*, Ann. Univ. Ferrara Sez. VII Sci. Mat. **52** (2006), no. 1, 19–36.
[4] S. N. Antontsev and S. I. Shmarev, *A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions*, Nonlinear Anal. **60** (2005), no. 3, 515–545.
[5] ———, *On the localization of solutions of elliptic equations with nonhomogeneous anisotropic degeneration*, Siberian Math. J. **46** (2005), no. 5, 765–782; translated from Sibirsk. Mat. Zh. **46** (2005), no. 5, 963–984.
[6] ———, *Elliptic equations and systems with nonstandard growth conditions: existence, uniqueness and localization properties of solutions*, Nonlinear Anal. **65** (2006), no. 4, 728–761.
[7] Y. Chen, S. Levine, and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. **66** (2006), no. 4, 1383–1406.
[8] L. Diening, *Theoretical and numerical results for electrorheological fluids*. Ph.D. Thesis, 2002.
[9] L. Diening, P. Harjulehto, P. Hastö, and M. Ružička, *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics, **2017**, Springer, Heidelberg, 2011.
[10] X. Fan, J. Shen, and D. Zhao, *Sobolev embedding theorems for spaces* $W^{k,p(x)}(\Omega)$, J. Math. Anal. Appl. **262** (2001), no. 2, 749–760.
[11] X. Fan and D. Zhao, *On the spaces* $L^{p(x)}(\Omega)$ *and* $W^{m,p(x)}(\Omega)$, J. Math. Anal. Appl. **263** (2001), no. 2, 424–446.
[12] O. Kovacik and J. Rakosnik, *On spaces* $L^{p(x)}$ *and* $W^{k,p(x)}$, Czechoslovak Math. J. **41** (1991), 592–618.
[13] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, 1969.
[14] G. de Marsily, *Quantitative Hydrogeology. Groundwater Hydrology for Engineers*. Academic Press, London, 1986.
[15] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, **1034**, Springer-Verlag, Berlin, 1983.
[16] I. P. Natanson, *Theory of Functions of a Real Variable*, Moscow-Leningrad, 1950.
[17] V. Rădulescu and D. Repovš, *Partial differential equations with variable exponents*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2015.
[18] K. Rajagopal and M. Ruzicka, *Mathematical modeling of electro-rheological fluids*, Contin. Mech. Thermodyn. **13** (2001), 59–78.
[19] M. Ruzicka, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, **1748**, Springer-Verlag, Berlin, 2000.
[20] U. Sert and K. Soltanov, *On solvability of a class of nonlinear elliptic type equation with variable exponent*, J. Appl. Anal. Comput. **7** (2017), no. 3, 1139–1160.

[21] K. N. Soltanov, *Solvability of nonlinear equations with operators in the form of the sum of a pseudomonotone and a weakly compact operator*, Russian Acad. Sci. Dokl. Math. **45** (1992), no. 3, 676–681 (1993); translated from Dokl. Akad. Nauk **324** (1992), no. 5, 944–948.

[22] ______, *Some imbedding theorems and nonlinear differential equations*, Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **19** (1999), no. 5, Math. Mech., 125–146 (2000).

[23] ______, *Some nonlinear equations of the nonstable filtration type and embedding theorems*, Nonlinear Anal. **65** (2006), no. 11, 2103–2134.

[24] K. N. Soltanov and J. Sprekels, *Nonlinear equations in non-reflexive Banach spaces and strongly nonlinear differential equations*, Adv. Math. Sci. Appl. **9** (1999), no. 2, 939–972.

[25] E. Zeidler, *Nonlinear functional analysis and its applications. II/B*, translated from the German by the author and Leo F. Boron, Springer-Verlag, New York, 1990.

[26] V. V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR. Izv. **29** (1987), 33–36.

[27] ______, *On some variational problems*, Russian J. Math. Phys. **5** (1997), no. 1, 105–116 (1998).

[28] ______, *On the technique for passing to the limit in nonlinear elliptic equations*, Funct. Anal. Appl. **43** (2009), no. 2, 96–112; translated from Funktsional. Anal. i Prilozhen. **43** (2009), no. 2, 19–38.

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