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MINIMAL AND HARMONIC REEB VECTOR FIELDS ON TRANS-SASAKIAN 3-MANIFOLDS

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Abstract. In this paper, we obtain some necessary and sufficient conditions for the Reeb vector field of a trans-Sasakian 3-manifold to be minimal or harmonic. We construct some examples to illustrate main results. As applications of the above results, we obtain some new characteristic conditions under which a compact trans-Sasakian 3-manifold is homothetic to either a Sasakian or cosymplectic 3-manifold.

1. Introduction

In differential geometry of almost contact metric manifolds, trans-Sasakian manifolds are an important field of research because such manifolds include the well known α-Sasakian manifolds (see [29]), β-Kenmotsu manifolds (see [29]) and cosymplectic manifolds (see [2]) as their special cases. In [25], the notion of trans-Sasakian manifolds M was proposed for the first time which is an almost contact metric manifold such that $M \times \mathbb{R}$ belongs to the class W_4 of Hermitian manifolds (see [20]). Note that Hermitian manifolds of class W_4 are closely related to locally conformally Kähler manifolds.

The local structures of trans-Sasakian manifolds were classified by Marrero in [21], namely a connected trans-Sasakian manifold of dimension greater than three is of class either C_5 or C_6. In general, a trans-Sasakian manifold of type (α, β) is said to be proper if it is of class either C_5 or C_6, or equivalently, it is of type either $(\alpha, 0)$, or $(0, \beta)$ or $(0, 0)$. However, there exist many trans-Sasakian 3-manifolds which are not proper (see [1], [3], [23] and [24]). Therefore, to find on what condition a trans-Sasakian 3-manifold is proper is an interesting problem. S. Deshmukh et al. in [11], [12], [13], [14] and [15] gave various conditions under which a compact trans-Sasakian 3-manifold is homothetic to either a Sasakian 3-manifold or a cosymplectic 3-manifold. Trans-Sasakian 3-manifolds under some curvature restrictions were also studied in [6], [7], [8] and [9].

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It is well known that the Reeb vector field plays important role in geometry of trans-Sasakian 3-manifolds. In this paper, we start to investigate the minimality and harmonicity of the Reeb vector field of a trans-Sasakian 3-manifold M. In fact, we prove that the Reeb vector field of M is minimal if and only if it is harmonic. After giving some equivalent conditions for the Reeb vector field of M to be minimal or harmonic, we also construct several concrete examples to illustrate our main results. At last, as an application of the above results, we give some characteristic conditions for a compact trans-Sasakian 3-manifold being homothetic to either a Sasakian or a cosymplectic manifold. These results can be regarded as generalizations of those in [6], [7], [12] and [15].

2. Preliminaries

2.1. Minimal and harmonic vector fields

Let (M, g) be a Riemannian manifold of dimension m and (T^1M, g_S) its unit tangent sphere bundle furnished with the standard Sasakian metric g_S. Let V be a unit vector field of M. Then there exists a metric g on M induced from g_S via V which can be written as follows:

$$(V^*g_S)(X, Y) = g(X, Y) + g(\nabla_X V, \nabla_Y V) \tag{2.1}$$

for any vector fields X and Y on M, where ∇ denotes the Levi-Civita connection of the metric g. We define a $(1, 1)$-type tensor field L_V on M by

$$L_V = \mathrm{id} + (\nabla V)^t \circ \nabla V, \tag{2.2}$$

where id is the identity map and $(\nabla V)^t$ is the transpose of ∇V. Now (2.1) can be written as $V^*g_S = g(L_V\cdot\ , \cdot)$. When M is compact and orientable, the volume of V is defined as the volume of the corresponding submanifold (M, V^*g_S) of (T^1M, g_S) and can be written as

$$\mathrm{Vol}(V) = \int_M f(V) dv_g,$$

where $f(V) = \sqrt{\det(L_V)}$. We define another $(1, 1)$-type tensor field K_V by

$$K_V = f(V)(L_V)^{-1} \circ (\nabla V)^t. \tag{2.3}$$

According to Gil-Medrano and Llinares-Fuster [17], V is a critical point for the volume function if and only if the following 1-form

$$\omega_V(X) = \mathrm{trace}\{Y \to (\nabla_Y K_V)X\} \tag{2.4}$$

vanishes on the distribution $\mathcal{D}^V$ determined by all vector fields orthogonal to V. Following Gil-Medrano [16], such a critical point is said to be a *minimal vector field* even when M is non-compact and non-orientable.

The energy of V is defined as the energy of the map from (M, g) into (T^1M, g_S) and can be written as

$$E(V) = \frac{m}{2}\mathrm{Vol}(M, g) + \frac{1}{2}\int_M \|\nabla V\|^2 dv_g.$$

Following Gil-Medrano [16], a unit vector field V is a critical point for the energy function if and only if the following 1-form

$$\rho_V(X) = \mathrm{trace}\{Y \to (\nabla_Y(\nabla V)^t)X\} \tag{2.5}$$

vanishes on the distribution $\mathcal{D}^V$ (see [31, 32]). A unit vector field V satisfying this condition is said to be *harmonic*.

Moreover, the map $V : (M, g) \to (T^1M, g_S)$ defines a *harmonic map* if and only if V is a harmonic vector field and, in addition, the following 1-form

$$\bar{\rho}_V(X) = \mathrm{trace}\{Y \to R(\nabla_Y V, V)X\} \tag{2.6}$$

vanishes for any vector field X on M (see [16]), where R denotes the Riemannian curvature tensor defined by $R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

2.2. Trans-Sasakian manifolds

According to D. E. Blair [2], an almost contact metric structure defined on a smooth differentiable manifold M of dimension $2n+1$ is a (ϕ, ξ, η, g)-structure satisfying

$$\begin{gathered}\phi^2 = -\,\mathrm{id} + \eta \otimes \xi,\ \eta(\xi) = 1,\\ \phi^* g = g - \eta \otimes \eta,\end{gathered} \tag{2.7}$$

where ϕ is a $(1, 1)$-type tensor field, ξ is a tangent vector field called the characteristic or the Reeb vector field and η is a 1-form called the almost contact form. A Riemannian manifold M furnished with an almost contact metric structure is said to be an *almost contact metric manifold*, denoted by (M, ϕ, ξ, η, g).

Let M be an almost contact metric manifold of dimension $2n + 1$. On the product $M \times \mathbb{R}$ there exists an almost complex structure J defined by

$$J\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

where X denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a $\mathcal{C}^\infty$-function on $M^{2n+1} \times \mathbb{R}$.

An almost contact metric manifold is said to be *normal* if the above almost complex structure J is integrable. An almost contact metric manifold is said to be a *trans-Sasakian manifold* (see [21]) if it is normal and $\mathrm{d}\eta = \alpha\Phi$, $\mathrm{d}\Phi = 2\beta\eta \wedge \Phi$, where $\alpha = \frac{1}{2n}\mathrm{tr}(\phi\nabla\xi)$, $\beta = \frac{1}{2n}\mathrm{div}\xi$ and $\Phi(\cdot, \cdot) = g(\cdot, \phi\cdot)$. It is known that an almost contact metric manifold M is trans-Sasakian if and only if there exist two smooth functions α and β satisfying

$$(\nabla_X\phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.8}$$

for any vector fields X and Y.

Usually, a trans-Sasakian manifold is denoted by $(M, \phi, \xi, \eta, \alpha, \beta)$ and is called a trans-Sasakian manifold of type (α, β). From the definition of tran-Sasakian manifolds, putting $Y = \xi$ in (2.8) and using (2.7) we have

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi) \tag{2.9}$$

for any vector field X.

By Propositions 1, 2 and Corollary 1 of [22], we observe that a normal almost contact metric 3-manifold is always trans-Sasakian. Therefore, by the definition of trans-Sasakian manifolds, we state that an almost contact metric 3-manifold is trans-Sasakian if and only if it is normal.

Note that a trans-Sasakian 3-manifold is an α-Sasakian manifold if $\alpha \in \mathbb{R}^*$ and $\beta = 0$ (see [29]), a β-Kenmotsu manifold if $\beta \in \mathbb{R}^*$ and $\alpha = 0$ (see [29]), or a cosymplectic manifold if $\alpha = \beta = 0$ (see [2]).

In this paper, all manifolds are assumed to be connected.

3. Minimal and harmonic Reeb vector fields on trans-Sasakian 3-manifolds

A trans-Sasakian manifold of type (α, β) is of C_6-class if $\beta = 0$ (see [5]). As seen in [21], α on a trans-Sasakian manifold of C_6-class of dimension greater than three is a constant. Then the trans-Sasakian manifolds of C_6-class of dimension greater than 3 are just α-Sasakian manifolds. However, α on a trans-Sasakian 3-manifold of C_6-class is not necessarily a constant.

A trans-Sasakian manifold of type (α, β) is of C_5-class if $\alpha = 0$ (see [5]). On such manifolds of dimension greater than three there holds naturally $\mathrm{d}\beta \wedge \eta = 0$ (see [24]). However, the above equation does not necessarily hold for dimension three. The set of all trans-Sasakian manifolds of C_5-class contains the set of all β-Kenmotsu manifolds as its proper subset. For trans-Sasakian manifolds of C_5-class with non-constant function β we refer the reader to [1], [3], [6] and [24]. Note that a trans-Sasakian manifold of C_5-class of dimension greater than three is also called a f-cosymplectic manifold (see [1]) or a f-Kenmotsu manifold (see [24]).

It has been proved that the Reeb vector field of a Sasakian manifold is harmonic (see [31]) and also is a harmonic map (see [19]). It has been proved in [18, Theorem 2.2] that every unit strongly normal geodesic vector field is minimal. Applying this one knows that the Reeb vector field of a cosymplectic manifold is always minimal. Also, the Reeb vector of a β-Kenmotsu or a cosymplectic manifold is also harmonic (see [26]). For the minimality of the Reeb vector fields of almost Kenmotsu 3-manifolds and almost cosymplectic 3-manifolds we refer the reader to [30] and [27] respectively. In view of the above statements, in this paper we concentrate only on the study of the minimality and harmonicity of the Reeb vector fields on trans-Sasakian manifolds of dimension three.

In what follows, let M be a trans-Sasakian 3-manifold of type (α, β). In this section, we aim to give several equivalent conditions for the Reeb vector field of

M to be minimal or harmonic. As an application, we also present a sufficient and necessary condition for the Reeb vector field of M defining a harmonic map.

The following lemma was proved in [9, Theorem 3.2] (see also [15]).

Lemma 3.1 ([9]). *On a trans-Sasakian 3-manifold of type (α,β) we have*

$$\xi(\alpha)+2\alpha\beta=0. \tag{3.1}$$

In this paper, we denote by ∇f the gradient of a smooth function f on M. Moreover, putting $n=1$ in [9, Proposition 3.4] we obtain the following lemma.

Lemma 3.2 ([9]). *On a trans-Sasakian 3-manifold of type (α,β) we have*

$$Q\xi=\phi(\nabla\alpha)-\nabla\beta+(2(\alpha^2-\beta^2)-\xi(\beta))\xi, \tag{3.2}$$

where Q denotes the Ricci operator associated with the Ricci tensor S which is defined by $S(\cdot,\cdot)=\mathrm{trace}\{X\to R(X,\cdot)\cdot\}$.

On an n-dimensional Riemannian manifold (M,g), the rough Laplacian operator $\bar\Delta$ acting on any smooth vector field X is defined by

$$\bar\Delta X=\sum_{i=1}^{n}(\nabla_{e_i}\nabla_{e_i}X-\nabla_{\nabla_{e_i}e_i}X),$$

where $\{e_i: i=1,\dots,n\}$ is a local orthonormal frame on the manifold. If there exist a vector field V and a smooth function f such that $\bar\Delta V=fV$, we say that V is an eigenvector field of $\bar\Delta$ with eigenfunction f.

Lemma 3.3 ([12]). *On a trans-Sasakian 3-manifold of type (α,β) we have*

$$\bar\Delta\xi=-\phi(\nabla\alpha)+\nabla\beta-(2(\alpha^2+\beta^2)+\xi(\beta))\xi. \tag{3.3}$$

Now we are ready to prove the following.

Theorem 3.1. *On a trans-Sasakian 3-manifold M the following six conditions are equivalent to each other.*

1. *The Reeb vector field is minimal.*
2. *The Reeb vector field is harmonic.*
3. *There hold $e(\alpha)-\phi e(\beta)=0$ and $\phi e(\alpha)+e(\beta)=0$ for any vector field e orthogonal to the Reeb vector field.*
4. *There holds $\nabla\alpha+\phi(\nabla\beta)+2\alpha\beta\xi=0$ ($\Leftrightarrow \phi(\nabla\alpha)-\nabla\beta+\xi(\beta)\xi=0$).*
5. *The Reeb vector field is an eigenvector field of the Ricci operator.*
6. *The Reeb vector field is an eigenvector field of the rough Laplacian $\bar\Delta$.*

Proof. For each point p on M, we may choose a local orthonormal frame $\{\xi,e,\phi e\}$ on certain neighbourhood U of p. Using (2.7)-(2.9), we see that the Levi-Civita connection ∇ of M can be written as the following (see [9]):

$$\begin{aligned}&\nabla_\xi\xi=0,\ \nabla_\xi e=\lambda\phi e,\ \nabla_\xi\phi e=-\lambda e,\\&\nabla_e\xi=\beta e-\alpha\phi e,\ \nabla_e e=-\beta\xi+\gamma\phi e,\ \nabla_e\phi e=\alpha\xi-\gamma e,\\&\nabla_{\phi e}\xi=\alpha e+\beta\phi e,\ \nabla_{\phi e}e=-\alpha\xi-\delta\phi e,\ \nabla_{\phi e}=-\beta\xi+\delta e,\end{aligned} \tag{3.4}$$

where λ, γ and δ are smooth functions on U.

From the last line of (3.4), $\nabla\xi$ and its transpose can be written as the following forms:

$$\nabla\xi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & \alpha \\ 0 & -\alpha & \beta \end{pmatrix} \text{ and } (\nabla\xi)^t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & -\alpha \\ 0 & \alpha & \beta \end{pmatrix} \tag{3.5}$$

with respect to $\{\xi, e, \phi e\}$, respectively. From (2.2), L_ξ can be written as the following

$$L_\xi = E + (\nabla\xi)^t \circ \nabla\xi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^2+\beta^2+1 & 0 \\ 0 & 0 & \alpha^2+\beta^2+1 \end{pmatrix} \tag{3.6}$$

with respect to $\{\xi, e, \phi e\}$. Therefore, we have $f(\xi) = \sqrt{\det(L_\xi)} = \alpha^2+\beta^2+1$. Consequently, by (2.3) we have

$$K_\xi = f(\xi)(L_\xi)^{-1} \circ (\nabla\xi)^t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & -\alpha \\ 0 & \alpha & \beta \end{pmatrix}. \tag{3.7}$$

In view of (3.5) and (3.7), we see that the two operators $(\nabla\xi)^t$ and K_ξ are the same. Then, the equivalence between (1) and (2) follows from (2.4) and (2.5).

Now we compute the covariant derivative of K_ξ with respect to $\{\xi, e, \phi e\}$.

$$\begin{aligned} (\nabla_\xi K_\xi)e &= \xi(\beta)e + \xi(\alpha)\phi e, \\ (\nabla_e K_\xi)e &= (\alpha^2-\beta^2)\xi + e(\beta)e + e(\alpha)\phi e, \\ (\nabla_{\phi e} K_\xi)e &= -2\alpha\beta\xi + \phi e(\beta)e + \phi e(\alpha)\phi e, \end{aligned} \tag{3.8}$$

where we have used (3.4). From (2.4) we have

$$\omega_\xi(e) = \text{trace}\{Y \to (\nabla_Y K_\xi)e\} = e(\beta) + \phi e(\alpha). \tag{3.9}$$

Similarly, we continue to compute the derivative of K_ξ with respect to $\{\xi, e, \phi e\}$.

$$\begin{aligned} (\nabla_\xi K_\xi)\phi e &= -\xi(\alpha)e + \xi(\beta)\phi e, \\ (\nabla_e K_\xi)\phi e &= 2\alpha\beta\xi - e(\alpha)e + e(\beta)\phi e, \\ (\nabla_{\phi e} K_\xi)\phi e &= (\alpha^2-\beta^2)\xi - \phi e(\alpha)e + \phi e(\beta)\phi e, \end{aligned} \tag{3.10}$$

where we have used (3.4). From (2.4) we also have

$$\omega_\xi(\phi e) = \text{trace}\{Y \to (\nabla_Y K_\xi)\phi e\} = -e(\alpha) + \phi e(\beta). \tag{3.11}$$

Thus, from (2.4) (resp. (2.5)), the equivalence between (1) and (3) (resp. (2) and (3)) follows from (3.9) and (3.11).

Let e be a vector field orthogonal to ξ. Thus, $e(\beta)+\phi e(\alpha) = 0$ is equivalent to $g(\nabla\alpha+\phi\nabla\beta, \phi e) = 0$ and $e(\alpha)-\phi e(\beta) = 0$ is equivalent to $g(\nabla\alpha+\phi\nabla\beta, e) = 0$. Thus, we see that the Reeb vector field ξ is minimal or harmonic if and only if $\nabla\alpha + \phi\nabla\beta$ is collinear with ξ and this is equivalent to $\nabla\alpha + \phi\nabla\beta = \eta(\nabla\alpha + \phi\nabla\beta)\xi = -2\alpha\beta\xi$, where we have used Lemma 3.1. By (2.7), the action of ϕ on the previous relation gives $\phi(\nabla\alpha) - \nabla\beta + \xi(\beta)\xi = 0$. Conversely, the action of ϕ on the previous relation gives that $\nabla\alpha + \phi(\nabla\beta) + 2\alpha\beta\xi = 0$.

If the Reeb vector field ξ is an eigenvector field of the Ricci operator, from Lemma 3.2 we have

$$Q\xi = \eta(Q\xi)\xi = 2(\alpha^2 - \beta^2 - \xi(\beta))\xi. \tag{3.12}$$

Comparing the above relation with (3.2) we obtain $\phi(\nabla\alpha) - \nabla\beta + \xi(\beta)\xi = 0$. Conversely, it is easy to check that the minimality or harmonicity of ξ, together with (3.2), implies (3.12).

If the Reeb vector field ξ is an eigenvector field of the rough Laplacian operator $\bar{\Delta}$, from Lemma 3.3 we have

$$\bar{\Delta}\xi = \eta(\Delta\xi)\xi = -2(\alpha^2 + \beta^2)\xi. \tag{3.13}$$

Comparing the above relation with (3.3) we obtain $\phi(\nabla\alpha) - \nabla\beta + \xi(\beta)\xi = 0$. Conversely, it is easy to check that the minimality or harmonicity of ξ, together with (3.3), implies (3.13). This completes the proof. □

An almost contact metric manifold is said to be η-Einstein if the Ricci operator is given by $Q = a\mathrm{id} + b\eta \otimes \xi$, where a, b are smooth functions.

It has been proved in [9, Theorem 4.1] that the Ricci operator Q of a trans-Sasakian 3-manifold is given by

$$\begin{aligned} Q = &\Big(\frac{r}{2} + \xi(\beta) - \alpha^2 + \beta^2\Big)\,\mathrm{id} - \Big(\frac{r}{2} + \xi(\beta) - 3\alpha^2 + 3\beta^2\Big)\,\eta \otimes \xi \\ &+ \eta \otimes (\phi(\nabla\alpha) - \nabla\beta) - g(\nabla\beta - \phi(\nabla\alpha), \cdot) \otimes \xi. \end{aligned} \tag{3.14}$$

Corollary 3.1. *On a trans-Sasakian* 3*-manifold the following three statements are equivalent to each other.*

(1) *The Reeb vector field is minimal or harmonic.*
(2) *The manifold is η-Einstein.*
(3) *The Ricci operator is given as the following*:

$$Q = \Big(\frac{r}{2} + \xi(\beta) - \alpha^2 + \beta^2\Big)\,\mathrm{id} - \Big(\frac{r}{2} + 3\xi(\beta) - 3\alpha^2 + 3\beta^2\Big)\,\eta \otimes \xi.$$

Proof. The proof follows directly from Theorem 3.1 and (3.14). □

As another application of Theorem 3.1, we have:

Theorem 3.2. *The Reeb vector field of a trans-Sasakian* 3*-manifold defines a harmonic map if and only if* $\nabla\alpha + \phi(\nabla\beta) + 2\alpha\beta\xi = 0$ *and* $\beta(\alpha^2 - \beta^2 - \xi(\beta)) = 0$.

Proof. Applying relation (2.9) and Lemma 3.1, we compute

$$R(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{X,Y}\xi$$

as the following:

$$\begin{aligned} R(X,Y)\xi = &\, Y(\alpha)\phi X - X(\alpha)\phi Y + X(\beta)(Y - \eta(Y)\xi) \\ &- Y(\beta)(X - \eta(X)\xi) + (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) \\ &+ 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) \end{aligned} \tag{3.15}$$

for any vector fields X, Y. Using (3.15) and Lemma 3.1, we obtain $R(\phi X, \xi)\xi = (\alpha^2 - \beta^2 - \xi(\beta))\phi X$ for any vector field X. Thus, applying again (2.9), from (2.6) and Lemma 3.1 we obtain

$$\rho_\xi(\xi) = -\alpha \mathrm{trace}\{Y \to R(\phi Y, \xi)\xi\} + \beta g(Q\xi, \xi) = 2\beta(\alpha^2 - \beta^2 - \xi(\beta)). \tag{3.16}$$

Similarly, applying (3.15) we obtain $R(X, e)\xi = e(\alpha)\phi X - e(\beta)X - (X(\alpha) + 2\alpha\beta\eta(X))\phi e + (X(\beta) + (\beta^2 - \alpha^2)\eta(X))e + e(\beta)\eta(X)\xi$. Thus, applying again (2.9), from (2.6) and Lemma 3.1 we obtain

$$\begin{aligned} \rho_\xi(e) &= -\alpha \mathrm{trace}\{Y \to R(\phi Y, \xi)e\} + \beta g(Q\xi, e) \\ &= -\alpha \sum_{i=1}^{3} g(R(E_i, e)\xi, \phi E_i) + \beta(Q\xi, e) \\ &= \alpha(\phi e(\beta) - e(\alpha)) - \beta(\phi e(\alpha) + e(\beta)), \end{aligned} \tag{3.17}$$

where $\{E_1, E_2, E_3\}$ is a local orthonormal frame of the tangent space at a point of the manifold.

Similarly, using (3.15) we obtain $R(X, \phi e)\xi = \phi e(\alpha)\phi X - \phi e(\beta)X + (X(\alpha) + 2\alpha\beta\eta(X))e + (X(\beta) + (\beta^2 - \alpha^2)\eta(X))\phi e + \phi e(\beta)\eta(X)\xi$. Thus, applying again (2.9), from (2.6) and Lemma 3.1 we obtain

$$\begin{aligned} \rho_\xi(\phi e) &= -\alpha \mathrm{trace}\{Y \to R(\phi Y, \xi)\phi e\} + \beta g(Q\xi, \phi e) \\ &= -\alpha \sum_{i=1}^{3} g(R(E_i, \phi e)\xi, \phi E_i) + \beta(Q\xi, \phi e) \\ &= -\alpha(\phi e(\alpha) + e(\beta)) + \beta(e(\alpha) - \phi e(\beta)). \end{aligned} \tag{3.18}$$

Thus, the proof follows from (2.6), (3.16)-(3.18) and Theorem 3.1. □

The following two corollaries follows directly from Theorems 3.1 and 3.2.

Corollary 3.2. *The Reeb vector field of a* 3*-dimensional* β*-Kenmotsu manifold is harmonic but is never a harmonic map.*

Corollary 3.3. *The Reeb vector field of a* 3*-dimensional* α*-Sasakian manifold or cosymplectic manifold is a harmonic map.*

4. Examples

Except for the above three typical examples of trans-Sasakian manifolds, one would like to know the minimality and harmonicity of non-proper trans-Sasakian 3-manifolds. Next, we construct some concrete examples to illustrate our main results.

Example 4.1. Let (x, y, z) be the standard Cartesian coordinates of $\mathbb{R}^3$. We consider a manifold M defined by $M := \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$. Let

$$e_1 = z\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right),\ e_2 = z\frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined on M by $g(e_i, e_j) = \delta_{ij}$, $i, j \in \{1, 2, 3\}$, where δ_{ij} denotes the Kronecker symbol. On M we can define an almost contact metric structure (ϕ, ξ, η, g) as the following:

$$\xi = e_3,\ \eta(\cdot) = g(e_3, \cdot),\ \phi e_1 = e_2,\ \phi e_2 = -e_1,\ \phi\xi = 0.$$

It has been proved in [6, p. 798] that M is a trans-Sasakian 3-manifold of type $(-\frac{1}{2}z^2, -\frac{1}{z})$. From Theorem 3.1, it is easy to see that the Reeb vector field of M is neither harmonic nor a harmonic map.

Example 4.2. Let (x, y, z) be the canonical Cartesian coordinates in $\mathbb{R}^3$. We can define on $\mathbb{R}^3$ as almost contact metric structure (ϕ, ξ, η, g) as the following:

$$\xi = \frac{\partial}{\partial z},\ \eta = dz - ydx,$$

$$\phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -y & 0 \end{pmatrix} \text{ and } g = \begin{pmatrix} e^z + y^2 & 0 & -y \\ 0 & e^z & 0 \\ -y & 0 & 1 \end{pmatrix}.$$

It has been proved in [3, p. 202] that $(\mathbb{R}^3, \phi, \xi, \eta, g)$ is a trans-Sasakian 3-manifold of type $(-\frac{1}{2e^z}, \frac{1}{2})$. Obviously, from Theorem 3.2, we see that the Reeb vector field of this structure is neither harmonic nor a harmonic map.

Lemma 4.1 ([21])**.** *Let (M, ϕ, ξ, η, g) be a Sasakian 3-manifold and f be a non-constant positive function on M. Then, (M, ϕ, ξ, η, g') is a trans-Sasakian 3-manifold of type $(\frac{1}{f}, \frac{1}{2f}\xi(f))$, where the Riemannian metric g' is defined by $g' = fg + (1 - f)\eta \otimes \eta$.*

Applying the above lemma, now we construct a large class of trans-Sasakian 3-manifolds whose Reeb vector fields may be harmonic or harmonic maps. Firstly, let us recall the following well known examples of trans-Sasakian 3-manifolds.

Example 4.3. Let (x, y, z) be the canonical Cartesian coordinates in $\mathbb{R}^3$. On $\mathbb{R}^3$ there exists a standard Sasakian structure (see Blair [2, p. 60]) defined as the following:

$$\xi = 2\frac{\partial}{\partial z},\ \eta = \frac{1}{2}(dz - ydx),$$

$$\phi = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{pmatrix} \text{ and } g = \frac{1}{4}\begin{pmatrix} 1 + y^2 & 0 & -y \\ 0 & 1 & 0 \\ -y & 0 & 1 \end{pmatrix}.$$

The orthonormal ϕ-basis is given by $\{\xi, e_1 := 2\frac{\partial}{\partial y}, e_2 := \phi e_1 = 2(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z})\}$. Let f be a positive function on $\mathbb{R}^3$. From Lemma 4.1, $(\mathbb{R}^3, \phi, \xi, \eta, g')$ is a trans-Sasakian 3-manifold of type $(\frac{1}{f}, \frac{1}{2f}\xi(f))$, where $g' = fg + (1 - f)\eta \otimes \eta$.

Proposition 4.1. *The Reeb vector field of the trans-Sasakian* 3*-manifold defined in Example* 4.3 *is minimal or harmonic if and only if the following system of partial differential equations hold*:

$$y\left(\frac{\partial f}{\partial z}\right)^2 - fy\frac{\partial^2 f}{\partial z^2} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial z} - f\frac{\partial^2 f}{\partial z\partial x} - \frac{\partial f}{\partial y} = 0, \tag{4.1}$$

$$f\frac{\partial^2 f}{\partial z\partial y} - \frac{\partial f}{\partial x} - y\frac{\partial f}{\partial z} - \frac{\partial f}{\partial y}\frac{\partial f}{\partial z} = 0. \tag{4.2}$$

Proof. As seen before, the trans-Sasakian 3-manifold defined in Example 4.3 is of type $(\frac{1}{f}, \frac{1}{2f}\xi(f))$. Applying $\alpha = \frac{1}{f}$ and $\beta = \frac{1}{2f}\xi(f) = \frac{1}{f}\frac{\partial f}{\partial z}$, and choosing a local orthogonal ϕ-basis $\{\xi, e := \frac{\partial}{\partial y}, \phi e = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\}$, then the remaining proof follows directly from Theorem 3.1. Notice that the above basis is not necessarily orthonormal for the metric g'. □

Proposition 4.2. *The Reeb vector field of the trans-Sasakian* 3*-manifold defined in Example* 4.3 *is a harmonic map if and only if either* (4.1)*,* (4.2) *and the following partial differential equation*

$$1 + \left(\frac{\partial f}{\partial z}\right)^2 - 2f\frac{\partial^2 f}{\partial z^2} = 0 \tag{4.3}$$

hold or the manifold is α-Sasakian or a cosymplectic manifold.

Proof. Suppose that the Reeb vector field ξ of the trans-Sasakian 3-manifold is a harmonic map. Notice that the second condition of Theorem 3.2 is equivalent to either $\beta = 0$ or $\alpha^2 - \beta^2 - \xi(\beta) = 0$ holds on certain open subset of the manifold. From the first condition of Theorem 3.2, $\beta = 0$ implies that α is a constant. In this context, the manifold is α-Sasakian if $\alpha \in \mathbb{R} - \{0\}$ or cosymplectic if $\alpha = 0$. The remaining proof follows from the Theorem 3.2 and Proposition 4.1. □

We now construct some concrete non-proper trans-Sasakian 3-manifolds with minimal or harmonic Reeb vector fields.

Example 4.4. Let $M := \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$ and (x, y, z) be the standard Cartesian coordinates of $\mathbb{R}^3$. On M we consider three vector fields defined as the following:

$$e_1 = \frac{x}{z}\frac{\partial}{\partial x},\ e_2 = \frac{y}{z}\frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z}.$$

Now we define a trans-Sasakian structure (M, ϕ, ξ, η, g) on M as the following:

$$\xi = e_3,\ \eta = g(e_3, \cdot),$$

$$\phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$.

It has been proved in [10, p. 262] that (M, ϕ, ξ, η, g) is a trans-Sasakian 3-manifold of type $(-1, \frac{1}{z})$. Obviously, by Theorems 3.1 and 3.2, one observes that the Reeb vector field of M is minimal (or harmonic) but is never a harmonic map.

Example 4.5. Let $M := \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$ and (x, y, z) be the standard Cartesian coordinates of $\mathbb{R}^3$. On M we consider three vector fields defined as the following:

$$e_1 = \frac{e^x}{z^2}\frac{\partial}{\partial x},\ e_2 = \frac{e^y}{z^2}\frac{\partial}{\partial y},\ e_3 = -\frac{\partial}{\partial z}.$$

Now we define a trans-Sasakian structure (M, ϕ, ξ, η, g) on M as the following:

$$\xi = e_3,\ \eta = g(e_3, \cdot),$$

$$\phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$.

It has been proved in [28, p. 161] that (M, ϕ, ξ, η, g) is a trans-Sasakian 3-manifold of type $(\frac{1}{z}, -\frac{1}{z})$. Obviously, from Theorems 3.1 and 3.2, it is easy to see that the Reeb vector field of M is minimal (or harmonic) but is never a harmonic map.

Finally, before closing this section, we construct a large class of trans-Sasakian 3-manifolds whose Reeb vector fields are either harmonic or harmonic maps.

Example 4.6. Let (x, y, z) be the standard Cartesian coordinates of $\mathbb{R}^3$. On $\mathbb{R}^3$ we consider a Riemannian metric g as the following:

$$g = \frac{1}{e^{2f(z)}} dx \otimes dx + \frac{1}{e^{2f(z)}} dy \otimes dy + dz \otimes dz,$$

where $f(z)$ is a non-constant smooth function on $\mathbb{R}^3$. From the above metric, an orthonormal ϕ-basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ is given by

$$e_1 = e^{f(z)}\frac{\partial}{\partial x},\ e_1 = e^{f(z)}\frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z}.$$

Now we define a trans-Sasakian structure (ϕ, ξ, η, g) on $\mathbb{R}^3$ as the following:

$$\xi = e_3,\ \eta = g(e_3, \cdot),\ \phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the ϕ-basis $\{e_1, e_2, e_3\}$. According to a direct calculation, we see that $(\mathbb{R}^3, \phi, \xi, \eta, g)$ is a trans-Sasakian 3-manifold of type $(0, -f'(z))$.

Remark 4.1. Example 4.6 corrects a little mistake in [4, Example 3.9]. Moreover, if we take $f(z) = 2 \ln z$, Example 4.6 is just that of in [33, Section 3]. Also, if we take $f(z) = z^2$, Example 4.6 is just that of in [1, Section 5.2].

Applying Theorems 3.1 and 3.2, we have the following two propositions.

Proposition 4.3. *The Reeb vector field of the trans-Sasakian structure defined in Example* 4.6 *is always minimal or harmonic.*

Proposition 4.4. *The Reeb vector field of the trans-Sasakian structure defined in Example* 4.6 *defines a harmonic map if and only if*

$$(f'(z))^2 - f''(z) = 0,$$

or equivalently, $f(z) = -\ln(c_1 + c_2 z)$*, where* c_1 *and* c_2 *are arbitrary constants.*

5. Trans-Sasakian 3-manifolds homothetic to Sasakian or cosymplectic manifolds

In this section, we obtain some applications of main results shown in Section 3 and characterize some new conditions for a compact trans-Sasakian 3-manifold being proper.

First, we need the following lemma which is useful in proofs of main results.

Lemma 5.1. *Let* M *be a compact trans-Sasakian* 3*-manifold of type* (α, β) *such that the Reeb vector field is minimal or harmonic. Then,* α *is a constant.*

Proof. Suppose that the Reeb vector field ξ is minimal or harmonic, by Theorem 3.1 we have

$$e(\alpha) - \phi e(\beta) = 0 \text{ and } \phi e(\alpha) + e(\beta) = 0 \tag{5.1}$$

for any unit vector field e orthogonal to ξ. Applying (3.4), (5.1) and Lemma 3.1, now we compute the usual Laplacian of α as the following:

$$\begin{aligned}\Delta\alpha &= \nabla_e\nabla_e\alpha - (\nabla_e e)\alpha + \nabla_{\phi e}\nabla_{\phi e}\alpha - (\nabla_{\phi e}\phi e)\alpha + \nabla_\xi\nabla_\xi\alpha - (\nabla_\xi\xi)\alpha\\ &= e(\phi e(\beta)) - (\gamma\phi e - \beta\xi)(\alpha) - \phi e(e(\beta)) - (\delta e - \beta\xi)(\alpha) + \xi(-2\alpha\beta)\\ &= (\nabla_e\phi e - \nabla_{\phi e}e)(\beta) + \gamma\phi e(\alpha) - \delta e(\alpha) - 2\alpha\xi(\beta)\\ &= 0.\end{aligned} \tag{5.2}$$

Since M is assumed to be compact, from (5.2) we see that α is a constant. □

From Theorem 3.1 and Lemma 5.1, we have:

Corollary 5.1. *The Reeb vector field* ξ *of a compact trans-Sasakian* 3*-manifold is minimal or harmonic if and only if* β *is invariant along the distribution orthogonal to* ξ*.*

In order to give applications, we need the following lemma.

Lemma 5.2. *Let* (M, g) *be a Riemannian manifold. If* M *admits a Killing vector field* ξ *of constant length satisfying*

$$k^2(\nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi) = g(Y, \xi)X - g(X, Y)\xi$$

for a non-zero constant k *and any vector fields* X, Y*, then* M *is homothetic to a Sasakian manifold.*

Theorem 5.1. *Let M be a compact trans-Sasakian 3-manifold of type (α, β). Then the Reeb vector field defines a harmonic map if and only if M is homothetic to either a Sasakian or a cosymplectic manifold.*

Proof. Let the Reeb vector field ξ of M be a harmonic map. By Lemma 5.1 and Theorem 3.2 we have that α is a constant and $\beta(\alpha^2 - \beta^2 - \xi(\beta)) = 0$. If $\alpha \neq 0$, by Lemma 3.1 we have $\beta = 0$ and hence by Lemma 5.2 we see that M is homothetic to a Sasakian manifold.

Next we consider the other case $\alpha = 0$. Suppose that $\beta = 0$, then in this context M is homothetic to a cosymplectic manifold. Now suppose that M is not a cosymplectic manifold, by Theorem 3.2 we have

$$\xi(\beta) = -\beta^2. \tag{5.3}$$

From (2.9) we see that $\mathrm{div}\xi = 2\beta$. Thus, using (5.3) we compute the divergence of $\beta\xi$ as the following:

$$\mathrm{div}(\beta\xi) = \xi(\beta) + \beta\mathrm{div}\xi = \beta^2. \tag{5.4}$$

Applying the divergence theorem on compact manifold M, we observe that β is zero. By the above analyses, we see that $\alpha = 0$ implies only that M is homothetic to a cosymplectic manifold.

The converse follows from Corollary 3.3. This completes the proof. □

Remark 5.1. Theorem 5.1 generalizes Theorem 4.1 of [15] because the vanishing of $Q\xi$ implies that ξ defines a harmonic map. Moreover, Theorem 5.1 extends Theorem 3.1 of [7] because ξ-projective flatness of trans-Sasakian 3-manifolds implies that ξ is minimal or harmonic. From Corollary 3.1, since the minimality or harmonicity of ξ implies that the manifold is η-Einstein, then Theorem 5.1 extends also Theorem 6.1 of [6].

In fact, Theorem 5.1 can be improved as the following two forms.

Corollary 5.2. *Let M be a compact trans-Sasakian 3-manifold of type (α, β). The Reeb vector field is an eigenvector field of the Ricci operator satisfying $Q\xi = \lambda\xi$ and $\xi(\lambda) = 0$ for a smooth function λ if and only if M is homothetic to either a Sasakian or a cosymplectic manifold.*

Proof. Suppose that ξ is an eigenvector field of the Ricci operator satisfying $Q\xi = \lambda\xi$ and $\xi(\lambda) = 0$. From Theorem 3.1, $Q\xi = \lambda\xi$ is equivalent to that ξ is minimal or harmonic and $Q\xi = 2(\alpha^2 - \beta^2 - \xi(\beta))\xi$. On the other hand, by Theorem 5.1, we know α is a constant. Thus, $\xi(\lambda) = 0$ is equivalent to

$$2\beta\xi(\beta) + \xi(\xi(\beta)) = 0. \tag{5.5}$$

Since ξ is harmonic and α is a constant, from Theorem 3.1 we have $\nabla\beta = \xi(\beta)\xi$. Using this and (5.5) we compute the usual Laplacian of β as the following:

$$\begin{aligned} \Delta\beta = \mathrm{div}(\nabla\beta) &= \sum_{i=1}^{3} g(E_i(\xi(\beta))\xi + \xi(\beta)(-\phi E_i + \beta E_i - \beta\eta(E_i)\xi), E_i) \\ &= \xi(\xi(\beta)) + 2\beta\xi(\beta) = 0. \end{aligned} \tag{5.6}$$

Since M is compact, then the harmonicity of β means that it is a constant. By Lemma 3.1 we have $\alpha\beta = 0$ and hence $\beta = 0$. In fact, if $\beta \neq 0$ we have that $\mathrm{div}\xi = 2\beta$ is a non-zero constant, this contradicts the compactness of M. The remaining proof follows from Lemma 5.2. The converse follows from Corollary 3.3. This completes the proof. □

The above result generalizes Theorem 3.2 of [15].

Corollary 5.3. *Let M be a compact trans-Sasakian 3-manifold of type (α, β). The Reeb vector field is an eigenvector field of the rough Laplacian operator $\bar{\Delta}$ satisfying $\bar{\Delta}\xi = \lambda\xi$ and $\xi(\lambda) = 0$ for a smooth function λ if and only if M is homothetic to either a Sasakian or a cosymplectic manifold.*

Proof. Suppose that ξ is an eigenvector field of the rough Laplacian operator $\bar{\Delta}$ satisfying $\bar{\Delta}\xi = \lambda\xi$ and $\xi(\lambda) = 0$. From Theorem 3.1, $\bar{\Delta}\xi = \lambda\xi$ is equivalent to that ξ is minimal or harmonic and $\bar{\Delta}\xi = -2(\alpha^2 + \beta^2)\xi$. Thus, by Theorem 5.1, we know α is a constant. Now, $\xi(\lambda) = 0$ is equivalent to

$$2\beta\xi(\beta) = 0. \tag{5.7}$$

Using this and (5.7) we compute the divergence of $\beta^3\xi$ as the following:

$$\mathrm{div}(\beta^3\xi) = 3\beta^2\xi(\beta) + \beta^3\mathrm{div}\xi = 2\beta^4, \tag{5.8}$$

where we have used $\mathrm{div}\xi = 2\beta$. Since M is compact, applying the divergence theorem on (5.8), we obtain $\beta = 0$. The remaining proof follows from Lemma 5.2. The converse follows from Corollary 3.3. This completes the proof. □

The above result generalizes Theorem 3.1 of [12].

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