J. Korean Math. Soc. **55** (2018), No. 6, pp. 1285–1303
https://doi.org/10.4134/JKMS.j170083
pISSN: 0304-9914 / eISSN: 2234-3008

GRADIENT ESTIMATES AND HARNACK INEQUALITES OF NONLINEAR HEAT EQUATIONS FOR THE V-LAPLACIAN

Ha Tuan Dung

Abstract. This note is motivated by gradient estimates of Li-Yau, Hamilton, and Souplet-Zhang for heat equations. In this paper, our aim is to investigate Yamabe equations and a non linear heat equation arising from gradient Ricci soliton. We will apply Bochner technique and maximal principle to derive gradient estimates of the general non-linear heat equation on Riemannian manifolds. As their consequence, we give several applications to study heat equation and Yamabe equation such as Harnack type inequalities, gradient estimates, Liouville type results.

1. Introduction

In the seminal paper [12] by Li and Yau thirty years ago, the authors introduced gradient estimates of the following heat equation

$$\Big(\Delta - q(x,t) - \frac{\partial}{\partial t}\Big)u(x,t) = 0 \tag{1.1}$$

on complete Riemannian manifolds, where the potential $q(x,t)$ is assumed to be C^2 in the first variable and C^1 in the second variables. As a application, they proved an interesting Harnack inequality. Due to Li-Yau's Harnack inequality, we have known that the temperature at a given point in spacetime is controlled from the above by the temperature at a later time.

Later, Hamilton (see [11]) and Souplet-Zhang (see [18]) introduced other kinds of gradient estimates for heat equations of form

$$\frac{\partial}{\partial t}u(x,t) = \Delta u(x,t) \tag{1.2}$$

on compact Riemannian manifolds. It is worth to notice that using Hamilton's gradient estimates we can compare the temperature of two different points at the same time while using Souplet-Zhang's gradient estimate, we can compare temperature distribution instantaneously.

Received January 27, 2017; Revised March 4, 2018; Accepted August 29, 2018.
2010 *Mathematics Subject Classification.* Primary 32M05; Secondary 32H02.
Key words and phrases. gradient estimates, Bakry-Émery curvature, Bochner's technique, Harnack-type inequalities, Liouville-type theorems.

Recently, gradient estimates of Li-Yau, Hamilton, and Souplet-Zhang are generalized in many direction. For example, Ma [14] investigated the following non linear equation

$$\Delta u + au \log u + bu = 0 \tag{1.3}$$

on complete non-compact Riemannian manifolds where $a < 0$ and b are constants. He emphasized that (1.3) has a closed relationship with the so called gradient Ricci solitons. Here by saying a Riemannian manifold (M, g) to be a gradient Ricci soliton, we mean that there is a smooth function f on M and a constant $\lambda \in \mathbb{R}$ such that

$$\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g.$$

On a gradient Ricci soliton, if we set $u = e^f$, then by some direct computations, one can show that u satisfies

$$\Delta u + 2\lambda u \log u + (A_0 + n\lambda)u = 0$$

for some constant A_0; see [14].

Motivated by gradient estimates of Li-Yau, Hamilton, and Souplet-Zhang, it is very natural to look for gradient estimate of the nonlinear equation (1.3). Moreover, an important generalization of the Laplacian is the following operator

$$\Delta_V \cdot = \Delta + \langle V, \nabla \cdot \rangle$$

which can be considered as a special case of V-harmonic maps introduced in [6]. Here V is a smooth vector field on M. Due to [6], V-harmonic maps include Hermitian harmonic maps, Weyl harmonic maps, affine harmonic maps, and Finsler maps from a Finsler manifold into a Riemannian manifold. In [4], the authors introduced existence and uniqueness theorems for V-harmonic maps from complete noncompact manifolds. They also obtained a Liouville type theorem for V-harmonic maps. In particular, a V-Laplacian comparison theorem was pointed out under the Bakry-Émery Ricci condition in [4]. Recall that on a complete Riemannian manifold (M, g), we can define the Bakry-Émery curvature by

$$Ric_V = Ric - \frac{1}{2}\mathcal{L}_V g, \quad Ric_V^N = Ric_V - \frac{1}{N} V \otimes V,$$

where $N > 0$ is a natural number and $\mathcal{L}_V$ is the Lie derivative along the direction V. Later, Chen and Qiu established gradient estimates of Li-Yau type and Harnack inequalities of the following nonlinear parabolic equation for the V-Laplacian

$$u_t = \Delta_V u + au \log u.$$

Considered the same heat equation, Li derived Cheng-Yau, Li-Yau, Hamilton gradient estimates for Riemannian manifolds with Bakry-Emery Ricci curvature bounded from below. He also estimated global and local upper bounds, in terms of Bakry-Emery Ricci curvature for the Hessian of positive and bounded solutions of the linear heat equation. For further result related to V-Laplacian, we refer the reader to [4, 6–8, 10, 16, 19] and the references therein.

On the other hand, our noted is also motivated by works on Yamabe equation. For example, in [1], for some constants $b < 0$ and $p > 1$ the authors considered the following Yamabe-type equation

$$\Delta u + bu + u^p = 0$$

on compact manifolds. They showed that the above Yamabe-type equation has only trivial solution provided that some conditions on the Ricci tensor, the dimension constant, and the ranges of b, p are added. When the underlying Riemannian manifold is complete, non-compact, Brandolini et al. [2] considered the Yamabe-type equation

$$\Delta u + a(x)u + A(x)u^p = 0, \tag{1.4}$$

where $a(x)$ and $A(x)$ are continuous functions on M and $p > 1$. If $A(x) < 0$ everywhere, under some integrable conditions, the authors showed that (1.4) has no positive bounded solution. For further discussion on Yamabe's problem, we refer the reader to [9, 15] and the references therein.

Inspired by (1.3) and (1.4), in this paper, let (M, g) be a Riemannian manifold and V be a smooth vector field on M, we consider the following general heat equation

$$u_t = \Delta u + \langle V, \nabla u\rangle + au^\alpha \log u + bu^\alpha + cu, \tag{1.5}$$

where $\alpha > 0$ is a constant and a, b, c are functions defined on $M \times [0, \infty)$ which are differentiable with respect to the first variable $x \in M$. Our aim is to derive some gradient estimates Souplet-Zhang type for positive bounded solutions of (1.5). Suppose u is a positive solution to (1.5) and $u \leq C$ for some positive constant C. Let $\widetilde{u} := u/C$ then $0 < \widetilde{u} \leq 1$ and $\widetilde{u}$ is a solution to

$$\widetilde{u}_t = \Delta\widetilde{u} + \langle V, \nabla\widetilde{u}\rangle + \widetilde{a}\widetilde{u}^\alpha \log u + \widetilde{b}\widetilde{u}^\alpha + c\widetilde{u},$$

where $\widetilde{a} = aC^{\alpha-1}$, $\widetilde{b} = aC^{\alpha-1}\log C + b$. Due to this resson, without loss of generality, we may assume $0 < u \leq 1$. Throughout this paper, the symbols q^+ and q^- are denoted by

$$q^+ = \max\{q, 0\}\,; q^- = \min\{q, 0\}\,.$$

Our main theorem is as follows.

Theorem 1.1. *Let (M, g) be a complete noncompact n-dimensional Riemannian manifold and V be a smooth vector field on M such that $Ric_V \geq -K$ for some $K \geq 0$ and $|V| \leq L$ for some positive number L. Let $\alpha > 0$ be a constant and a, b, c be functions defined on $M \times [0, \infty)$, which are differentiable with respect to $x \in M$. Suppose that u is a positive solution to the following nonlinear heat equation*

$$u_t = \Delta u + \langle V, \nabla u\rangle + au^\alpha \log u + bu^\alpha + cu \tag{1.6}$$

with $u \leq 1$ for all $(x, t) \in M \times [0, \infty)$. Then

1. *If $\alpha \geq 1$, we have*

$$\frac{|\nabla u|}{u} \leq \left(\frac{1}{t^{\frac{1}{2}}} + \sup_{M\times[0,\infty)} \sqrt{2\left(K + H + a^+ + \alpha b^+ + c^+ + \frac{1}{2}\left(|\nabla a| + |\nabla b| + |\nabla c|\right)\right)} + \sup_{M\times[0,\infty)} \sqrt[4]{|\nabla a| + |\nabla b| + |\nabla c|}\right)(1 - \log u).$$

2. *If $0 < \alpha < 1$, and a, b, c are functions of constant sign on $M \times [0, \infty)$, we have*

$$\frac{|\nabla u|}{u} \leq \left(\frac{1}{t^{\frac{1}{2}}} + \sup_{M\times[0,\infty)} \sqrt{2\left(K + |u^{\alpha-2}|_\infty \left(a + 3|a| + 3|b|\right) + c + |c|\right)} + \sup_{M\times[0,\infty)} \sqrt[4]{|u^{\alpha-2}|_\infty \left(\frac{|\nabla a|^2}{2|a|} + \frac{|\nabla b|^2}{2|b|}\right) + \frac{|\nabla c|^2}{2|c|}}\right)(1 - \log u),$$

where

$$|u|_\infty = \sup_M |u|,$$

$$H = (\alpha - 1)\left|a^-\right| \sup_{M\times[0,\infty)} \log\frac{1}{u}.$$

Remark 1.2. Note that if $\alpha = 1$, (we may assume $c = 0$ in this case), the equation (1.6) becomes $u_t = \Delta u + \langle V, \nabla u\rangle + au\log u + bu$ which was considered by Dung anh Khanh in [10]. By the first conclusion of Theorem 1.1 we obtain

$$\frac{|\nabla u|}{u} \leq \left(\frac{1}{t^{\frac{1}{2}}} + \sup_{M\times[0,\infty)} \left\{\sqrt{2(K + a^+ + b^+) + |\nabla a| + |\nabla b|} + \sqrt[4]{|\nabla a| + |\nabla b|}\right\}\right) \times (1 - \log u).$$

Since

$$|\nabla a| + |\nabla b| \leq \frac{|\nabla a|^2}{2a^+} + \frac{|\nabla b|^2}{2b^+} + (a^+ + b^+).$$

The above gradient estimate is similar to the estimate in Theorem 2.1, in [10]. However, at any point where a, b are small and $|\nabla a|$, $|\nabla b|$ are large, it seems that our estimate is better that in [10]. Moreover, if $\alpha = 1$, $|V| \leq L$ and $a, b, c = 0$, the equation (1.6) reads as $u_t = \Delta u + \langle V, \nabla u\rangle$. Hence, for any positive solution u to this equation $u \leq 1$, we have

$$\frac{|\nabla u|}{u} \leq \left(\frac{1}{t^{\frac{1}{2}}} + \sqrt{2K}\right)(1 - \log u).$$

This inequality is exactly a main result in [19]. We also want to mention that if $V = \nabla f$, where f is a smooth function on M, due to method we used in this paper, we need to assume that $|\nabla f| \leq L$. It turns out that our result is not as good as that in [19]. This is because we do not have a good enough V-Laplacian comparison for general smooth vector field V. Nevertheless, when $V = \nabla f$, we can use a Brighton's proof trick (see [19]) to devire the same result as in [19]. It is also worth to notice that when $V = \nabla f$, using the method given is this paper, the same strategy of applying Brighton's Laplacian comparison as in [19], we still can obtain all results in this paper, without assuming that V is bounded.

The paper is organized as follows. In Section 2, we will introduce a technique lemma. Then we will use it to give a proof of Theorem 1.1. Using similar arguments as is the proof of Theorem 1.1, we show another gradient estimates with assumption on Ric_V^N. This result can be considered as a generalization of Souplet-Zhang and Ruan's gradient estimates (see [17, 18]). In Section 3, we derive several applications of gradient estimates given in Section 2. They are Harnack type inequality, Liouville type theorem and gradient estimates for nonlinear elliptic equations.

2. Gradient estimates for a nonlinear heat equation

To begin with, let us introduce some notations and a technique lemma. Suppose that u is a positive solution of (1.5) with $u \le 1$ for all $(x,t) \in M \times [0,\infty)$. Let

$$\Box = \Delta + \langle V, \nabla\rangle - \partial_t, \;\; f = \log u \le 0.$$

By direct computation, we have

$$\begin{aligned}\Box f &= \Delta f + \langle V, \nabla f\rangle - f_t \\ &= \frac{\Box u}{u} - |\nabla f|^2 \\ &= -ae^{(\alpha-1)f} f - be^{(\alpha-1)f} - c - |\nabla f|^2.\end{aligned}$$

Now, we state the computational lemma.

Lemma 2.1. *Let $w = |\nabla \log(1-f)|^2$, where f is as the above paragraph.*

(i) *If $\alpha \ge 1$, then w satisfies*

$$\begin{aligned}\Box w \ge{}& -2\left[K + H + a^+ + \alpha b^+ + c^+ + \frac{1}{2}(|\nabla a| + |\nabla b| + |\nabla c|)\right] w \\ & - (|\nabla a| + |\nabla b| + |\nabla c|) + 2(1-f)w^2 + 2f|\nabla w| w^{\frac{1}{2}}. \qquad (2.1)\end{aligned}$$

(ii) *If $0 < \alpha < 1$, then w satisfies*

$$\begin{aligned}\Box w \ge{}& -2\left(K + \left|u^{\alpha-2}\right|_\infty (a + 3|a| + 3|b|) + c + |c|\right) w \\ & - \left[\left|u^{\alpha-2}\right|_\infty \left(\frac{|\nabla a|^2}{2|a|} + \frac{|\nabla b|^2}{2|b|}\right) + \frac{|\nabla c|^2}{2|c|}\right] \\ & + 2(1-f)w^2 + 2f\,|\nabla w|\, w^{\frac{1}{2}}. \qquad (2.2)\end{aligned}$$

Here

$$|u|_\infty = \sup_M |u|,$$

$$H = (\alpha - 1)\left|a^-\right| \sup_{M\times[0,\infty)} \log\frac{1}{u}.$$

Proof. By V-Bochner-Weitzenböck formular in [13, 16], we have

$$\frac{1}{2}\Delta_V|\nabla u|^2 \geq |\nabla^2 u|^2 + Ric_V(\nabla u, \nabla u) + \langle \nabla\Delta_V u, \nabla u\rangle .$$

Using this inequality and the assumption $Ric_V \geq -K$, we deduce that

$$\begin{aligned}\Box w \geq\ & 2|\nabla^2\log(1-f)|^2 - 2K|\nabla\log(1-f)|^2\\ &+ 2\big\langle\nabla\Delta_V\log(1-f), \nabla\log(1-f)\big\rangle - w_t.\end{aligned} \tag{2.3}$$

We calculate directly that

$$\begin{aligned}\Delta_V\log(1-f) &= \frac{-\Delta_V f}{1-f} - w\\ &= \frac{-\Box f - f_t}{1-f} - w\\ &= \frac{ae^{(\alpha-1)f}f + be^{(\alpha-1)f} + c + |\nabla f|^2 - f_t}{1-f} - w\\ &= \frac{ae^{(\alpha-1)f}f + be^{(\alpha-1)f} + c}{1-f}\\ &\quad + \left(\log(1-f)\right)_t + (1-f)w - w.\end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), we obtain

$$\Box w \geq -2Kw + 2\left\langle\nabla\left(\tfrac{e^{(\alpha-1)f}(af+b)+c}{1-f} + \left(\log(1-f)\right)_t - fw\right), \nabla\log(1-f)\right\rangle - w_t.$$

Observe that

$$\begin{aligned}2\left\langle\nabla\left(\log(1-f)\right)_t, \nabla\log(1-f)\right\rangle &= \left(|\nabla\log(1-f)|^2\right)_t = w_t,\\ \nabla\left[e^{(\alpha-1)f}(af+b)+c\right] &= (\alpha-1)\nabla f e^{(\alpha-1)f}(af+b)\\ &\quad + e^{(\alpha-1)f}(f\nabla a + a\nabla f + \nabla b) + \nabla c\\ &= \nabla f e^{(\alpha-1)f}[a + (\alpha-1)(af+b)]\\ &\quad + e^{(\alpha-1)f}(f\nabla a + \nabla b) + \nabla c.\end{aligned}$$

Hence,

$$\begin{aligned}\Box w &\geq -2Kw + 2\left\langle\nabla\left(\tfrac{e^{(\alpha-1)f}(af+b)+c}{1-f} - fw\right), \nabla\log(1-f)\right\rangle\\ &= -2Kw - 2\left\langle\tfrac{\nabla f e^{(\alpha-1)f}[a+(\alpha-1)(af+b)]}{1-f} + \tfrac{e^{(\alpha-1)f}(f\nabla a+\nabla b)+\nabla c}{1-f}\right.\\ &\quad \left. + \tfrac{\left[e^{(\alpha-1)f}(af+b)+c\right]\nabla f}{(1-f)^2} - w\nabla f - f\nabla w, \nabla\log(1-f)\right\rangle\\ &= -2Kw - 2e^{(\alpha-1)f}aw - 2\tfrac{e^{(\alpha-1)f}(af+b)[(\alpha-1)(1-f)+1]w}{1-f}\end{aligned}$$

$$- \frac{2c}{1-f} w + 2 \left\langle \frac{e^{(\alpha-1)f}(f\nabla a + \nabla b) + \nabla c}{1-f}, \nabla \log(1-f) \right\rangle$$

$$+ 2(1-f)w^2 - 2f \langle \nabla w, \nabla \log(1-f) \rangle . \tag{2.5}$$

By the Schwartz inequality, we have

$$\begin{aligned}
&\left\langle \frac{e^{(\alpha-1)f}(\nabla a + \nabla b) + \nabla c}{1-f}, \nabla \log(1-f) \right\rangle \\
&\leq \left| \frac{e^{(\alpha-1)f}(f\nabla a + \nabla b) + \nabla c}{1-f} \right| |\nabla \log(1-f)| \\
&\leq \frac{e^{(\alpha-1)f}(-f|\nabla a| + |\nabla b|) + |\nabla c|}{1-f} w^{\frac{1}{2}},
\end{aligned}$$

and

$$- \langle \nabla w, \nabla \log(1-f) \rangle \leq |\nabla w| |\nabla \log(1-f)| = |\nabla w| w^{\frac{1}{2}}.$$

Combining (2.5) and above two estimates, we obtain

$$\begin{aligned}
\Box w \geq & -2Kw - 2e^{(\alpha-1)f} aw - \frac{2e^{(\alpha-1)f} af[(\alpha-1)(1-f)+1]w}{1-f} \\
& - \frac{2e^{(\alpha-1)f} b[(\alpha-1)(1-f)+1]w}{1-f} - \frac{2c}{1-f} w \\
& + 2\frac{e^{(\alpha-1)f}\left(f|\nabla a| - |\nabla b|\right) - |\nabla c|}{1-f} w^{\frac{1}{2}} + 2(1-f)w^2 + 2f|\nabla w| w^{\frac{1}{2}}.
\end{aligned} \tag{2.6}$$

Case 1. If $\alpha \geq 1$, then $0 < e^{(\alpha-1)f} \leq 1$, since $0 < \frac{1}{1-f} \leq 1$, a simple calculation shows

$$\begin{aligned}
-2\frac{c}{1-f} w - 2\frac{|\nabla c|}{1-f} w^{\frac{1}{2}} &\geq -2c^+ w - 2|\nabla c| w^{\frac{1}{2}} \\
&\geq -2c^+ w - |\nabla c| w - |\nabla c|, \\
-2\frac{e^{(\alpha-1)f}|\nabla b|}{1-f} w^{\frac{1}{2}} &\geq -2|\nabla b| w^{\frac{1}{2}} \\
&\geq -|\nabla b| w - |\nabla b|.
\end{aligned}$$

Similarly, since $0 < \frac{-f}{1-f} \leq 1$, we have

$$\begin{aligned}
2\frac{e^{(\alpha-1)f} f|\nabla a|}{1-f} w^{\frac{1}{2}} &\geq -2|\nabla a| w^{\frac{1}{2}} \\
&\geq -|\nabla a| w - |\nabla a|.
\end{aligned}$$

Hence, the inequality (2.6) implies

$$\begin{aligned}
\Box w \geq & -2w \left[K + c^+ + \frac{1}{2} \left(|\nabla a| + |\nabla b| + |\nabla c| \right) \right] - \left(|\nabla a| + |\nabla b| + |\nabla c| \right) \\
& - 2e^{(\alpha-1)f} aw - \frac{2e^{(\alpha-1)f} af[\alpha - (\alpha-1) f]w}{1-f}
\end{aligned}$$

$$-\frac{2e^{(\alpha-1)f}b[(\alpha-1)(1-f)+1]w}{1-f}+2(1-f)w^2+2f|\nabla w|w^{\frac{1}{2}}. \tag{2.7}$$

Observe that

$$a\geq \min\{a,0\}=a^-;\quad 0\leq \frac{-f}{1-f}\leq 1;\quad \alpha-(\alpha-1)\,f\geq\alpha\geq 1.$$

Hence,

$$\begin{aligned}-\frac{2e^{(\alpha-1)f}af[\alpha-(\alpha-1)\,f]w}{1-f} &\geq -\frac{2e^{(\alpha-1)f}a^-f[\alpha-(\alpha-1)\,f]w}{1-f}\\ &\geq 2e^{(\alpha-1)f}a^-[\alpha-(\alpha-1)\,f]w. \end{aligned}\tag{2.8}$$

Note that

$$0<\frac{1}{1-f}\leq 1;\quad b\leq\max\{b,0\}=b^+;\quad e^x(\alpha-x)\leq\alpha,\forall x\leq 0.$$

Hence,

$$\begin{aligned}-\frac{2e^{(\alpha-1)f}b[(\alpha-1)(1-f)+1]w}{1-f} &= -\frac{2e^{(\alpha-1)f}b[\alpha-(\alpha-1)\,f]w}{1-f}\\ &\geq -2\frac{\alpha b^+}{1-f}w\geq -2\alpha b^+w. \end{aligned}\tag{2.9}$$

Plugging inequalities (2.8), (2.9) into (2.7), we infer

$$\begin{aligned}\Box w\geq -2w\Big\{&K+e^{(\alpha-1)f}\left[a^++a^-\,(\alpha-1)\,(f-1)\right]+\alpha b^++c^+\\ &+\frac{1}{2}\left(|\nabla a|+|\nabla b|+|\nabla c|\right)\Big\}\\ &-(|\nabla a|+|\nabla b|+|\nabla c|)+2(1-f)w^2+2f|\nabla w|w^{\frac{1}{2}}.\end{aligned}$$

Using the inequality $1-\log x\leq\frac{1}{x},\forall x\in(0,1)$, we get

$$\begin{aligned}a^-\,(\alpha-1)\,(f-1) &\leq(\alpha-1)\left|a^-\right|\left|f-1\right|\\ &\leq(\alpha-1)\left|a^-\right|(1-\log u)\\ &\leq(\alpha-1)\left|a^-\right|\sup_{M\times[0,\infty)}\log\frac{1}{u}=H.\end{aligned}$$

Since $0<e^{(\alpha-1)f}\leq 1$ and $a^++a^-\,(\alpha-1)\,(f-1)\geq a^+\geq 0$, we have

$$\begin{aligned}\Box w\geq -2&\left[K+H+a^++\alpha b^++c^++\frac{1}{2}\left(|\nabla a|+|\nabla b|+|\nabla c|\right)\right]w\\ &-(|\nabla a|+|\nabla b|+|\nabla c|)+2(1-f)w^2+2f|\nabla w|w^{\frac{1}{2}}.\end{aligned}$$

Case 2. If $\alpha\in(0,1)$, then $e^{(\alpha-1)f}>1$ and $|\alpha-1|<1$, we obtain

$$e^{(\alpha-1)f}(af+b)[(\alpha-1)(1-f)+1]\leq e^{(\alpha-1)f}(-f|a|+|b|)(2-f). \tag{2.10}$$

Using the inequality $\log x \geq \frac{x-1}{x}, \forall x \in (0,1)$, we conclude that

$$2 - f = 2 - \log u \leq 2 - \frac{u-1}{u} = \frac{u+1}{u} \leq \frac{2}{u}.$$

Hence,

$$e^{(\alpha-1)f}(-f|a| + |b|)(2-f) \leq u^{\alpha-1}\frac{2}{u}(-f|a| + |b|) = 2e^{(\alpha-2)f}(-f|a| + |b|).$$

The inequality (2.10) implies

$$e^{(\alpha-1)f}(af+b)[(\alpha-1)(1-f)+1] \leq 2e^{(\alpha-2)f}(-f|a| + |b|).$$

Plugging this inequality into (2.6), we have

$$\begin{aligned}\Box w \geq & -2Kw - 2e^{(\alpha-1)f}aw - \frac{4e^{(\alpha-2)f}(-f|a|+|b|)w}{1-f} - \frac{2c}{1-f}w \\ & + 2\frac{e^{(\alpha-1)f}\left(f|\nabla a| - |\nabla b|\right) - |\nabla c|}{1-f}w^{\frac{1}{2}} + 2(1-f)w^2 + 2f|\nabla w|w^{\frac{1}{2}}. \end{aligned} \tag{2.11}$$

Since $0 < \frac{1}{1-f} \leq 1$, a simple calculation shows

$$\begin{aligned} -2\frac{c}{1-f}w - 2\frac{|\nabla c|}{1-f}w^{\frac{1}{2}} &= -2\frac{c}{1-f}w - \frac{1}{1-f}2\frac{|\nabla c|}{\sqrt{2|c|}}\sqrt{2|c|w} \\ &\geq \frac{1}{1-f}\left(-2cw - \frac{|\nabla c|^2}{2|c|} - 2|c|w\right) \\ &\geq -\frac{|\nabla c|^2}{2|c|} - 2(c+|c|)w, \end{aligned}$$

$$\begin{aligned} -2\frac{e^{(\alpha-1)f}|\nabla b|}{1-f}w^{\frac{1}{2}} &= -\frac{e^{(\alpha-1)f}}{1-f}2\frac{|\nabla b|}{2|b|}\sqrt{2|b|w} \\ &\geq \frac{e^{(\alpha-1)f}}{1-f}\left(-\frac{|\nabla b|^2}{2|b|} - 2|b|w\right) \\ &\geq e^{(\alpha-1)f}\left(-\frac{|\nabla b|^2}{2|b|} - 2|b|w\right). \end{aligned}$$

Similarly, since $0 < \frac{-f}{1-f} \leq 1$, we have

$$\begin{aligned} 2\frac{e^{(\alpha-1)f}f|\nabla a|}{1-f}w^{\frac{1}{2}} &= -\frac{(-f)}{1-f}e^{(\alpha-1)f}2\frac{|\nabla a|}{\sqrt{2|a|}}\sqrt{2|a|w} \\ &\geq \frac{(-f)}{1-f}e^{(\alpha-1)f}\left(-\frac{|\nabla a|^2}{2|a|} - 2|a|w\right) \\ &\geq e^{(\alpha-1)f}\left(-\frac{|\nabla a|^2}{2|a|} - 2|a|w\right). \end{aligned}$$

Hence, the inequality (2.11) implies

$$\Box w \geq -2Kw - 2\left[e^{(\alpha-1)f}\left(a+|a|+|b|\right)+c+|c|\right]w - \frac{4e^{(\alpha-2)f}(-f|a|+|b|)w}{1-f}$$
$$-\left[e^{(\alpha-1)f}\left(\frac{|\nabla a|^2}{2|a|}+\frac{|\nabla b|^2}{2|b|}\right)+\frac{|\nabla c|^2}{2|c|}\right]+2(1-f)w^2+2f|\nabla w|w^{\frac{1}{2}}.$$

Observe that

$$0<\frac{1}{1-f}\leq 1;\quad 0\leq\frac{-f}{1-f}<1;\quad u^{\alpha-1}\leq u^{\alpha-2},$$

therefore, the above inequality reduces to the following

$$\Box w \geq -2Kw - 2\left[e^{(\alpha-1)f}(a+|a|+|b|)+c+|c|\right]w - 4e^{(\alpha-2)f}|a| - 4e^{(\alpha-2)f}|b|$$
$$-\left[e^{(\alpha-1)f}\left(\frac{|\nabla b|^2}{2|b|}+\frac{|\nabla a|^2}{2|a|}\right)+\frac{|\nabla c|^2}{2|c|}\right]+2(1-f)w^2+2f|\nabla w|w^{\frac{1}{2}}$$
$$\geq -2Kw - 2\left[u^{\alpha-2}(a+3|a|+3|b|)+c+|c|\right]w$$
$$-\left[u^{\alpha-2}\left(\frac{|\nabla a|^2}{2|a|}+\frac{|\nabla b|^2}{2|b|}\right)+\frac{|\nabla c|^2}{2|c|}\right]+2(1-f)w^2+2f|\nabla w|w^{\frac{1}{2}}.$$

We obtain

$$\Box w \geq -2\left(K+\left|u^{\alpha-2}\right|_\infty(a+3\,|a|+3\,|b|)+c+|c|\right)w$$
$$-\left[\left|u^{\alpha-2}\right|_\infty\left(\frac{|\nabla a|^2}{2\,|a|}+\frac{|\nabla b|^2}{2\,|b|}\right)+\frac{|\nabla c|^2}{2\,|c|}\right]+2(1-f)w^2+2f\,|\nabla w|\,w^{\frac{1}{2}},$$

where $|u|_\infty=\sup_M|u|$. We complete the proof Lemma 2.1. □

Proof of Theorem 1.1. Choose a smooth function $\eta(r)$ such that $0\leq\eta(r)\leq 1, \eta(r)=1$ if $r\leq 1$, $\eta(r)=0$ if $r\geq 2$ and

$$0\geq\eta(r)^{-\frac{1}{2}}\eta(r)^{'}\geq -c_1,\quad \eta(r)^{''}\geq -c_2$$

for some $c_1,c_2\geq 0$. For a fixed point $p\in M$, let $\rho(x)=dist(p,x)$ and $\psi=\eta\left(\frac{\rho(x)}{R}\right)$. Therefore,

$$\frac{|\nabla\psi|^2}{\psi}=\frac{|\nabla\eta|^2}{\eta}=\frac{1}{\eta(r)}\frac{\left(\eta(r)^{'}\right)^2}{R^2}|\nabla\rho(x)|^2\leq\frac{(-c_1)^2}{R^2}=\frac{c_1^2}{R^2}.$$

Since $|V|\leq L$, the Laplacian comparison theorem in [4] implies

$$\Delta_V\rho\leq\sqrt{(n-1)K}+\frac{n-1}{\rho}+L.$$

Hence,

$$\Delta_V\psi=\frac{\eta(r)^{''}|\nabla\rho|^2}{R^2}+\frac{\eta(r)^{'}\Delta_V\rho}{R}$$

$$\geq \frac{-c_2}{R^2} + \frac{(-c_1)}{R}\sqrt{\eta\left(\frac{\rho}{R}\right)}\mathbf{1}_{B_{2R}(p)\setminus B_R(p)}\Big[\sqrt{(n-1)K} + \frac{n-1}{\rho} + L\Big]$$

$$\geq -\frac{R\Big[\sqrt{(n-1)K} + \frac{n-1}{R} + L\Big]c_1 + c_2}{R^2}. \tag{2.12}$$

Following a Calabi's argument in [3], let $\varphi = t\psi$ and assume that φw obtains its maximal value on $B(p, 2R) \times [0, T]$ at some (x, t), we may assume that x is not in the locus of p. At (x, t), we have

$$\begin{cases} \nabla(\varphi w) = 0, \\ \Delta(\varphi w) \leq 0, \\ (\varphi w)_t \geq 0. \end{cases}$$

Hence,

$$\Box(\varphi w) = \Delta(\varphi w) + \big\langle V, \nabla(\varphi w)\big\rangle - (\varphi w)_t \leq 0.$$

Since $\Box(\varphi w) = \varphi\Box w + w\Box\varphi + 2\langle\nabla w, \nabla\varphi\rangle$, this implies

$$\varphi\Box w + w\Box\varphi + 2\langle\nabla w, \nabla\varphi\rangle \leq 0. \tag{2.13}$$

Case 1. If $\alpha \geq 1$, then combining (2.1), (2.13) and using the fact that

$$\nabla(\varphi w) = \varphi\nabla w + w\nabla\varphi = 0,$$

we obtain

$$\begin{aligned} &\varphi\Bigg\{ -2\left(K + H + a^+ + \alpha b^+ + c^+ + \frac{1}{2}\left(|\nabla a| + |\nabla b| + |\nabla c|\right)\right) w \\ &\quad - \left(|\nabla a| + |\nabla b| + |\nabla c|\right) w + 2(1-f)w^2 + 2f|\nabla w|w^{\frac{1}{2}}\Bigg\} \\ &\quad + w\Box\varphi - 2\frac{|\nabla\varphi|^2}{\varphi}w \leq 0. \end{aligned} \tag{2.14}$$

Using the inequality $2ab \leq a^2 + b^2$, we get

$$\begin{aligned} -2f|\nabla\varphi|w^{\frac{3}{2}} &= (1-f)\varphi\left\{2\frac{(-f)|\nabla\varphi|w^{\frac{1}{2}}}{(1-f)\varphi}w\right\} \\ &\leq (1-f)\varphi\left\{\frac{f^2|\nabla\varphi|^2}{(1-f)^2\varphi^2}w + w^2\right\} \\ &= \frac{f^2|\nabla\varphi|^2 w}{(1-f)\varphi} + (1-f)\varphi w^2. \end{aligned}$$

Plugging this inequality into (2.14), we have

$$\begin{aligned} &-2\varphi w\left[K + H + a^+ + \alpha b^+ + c^+ + \frac{1}{2}\left(|\nabla a| + |\nabla b| + |\nabla c|\right)\right] \\ &-\varphi\left(|\nabla a| + |\nabla b| + |\nabla c|\right) + \varphi(1-f)w^2 - \frac{f^2|\nabla\varphi|^2}{(1-f)\varphi}w + w\Box\varphi \end{aligned}$$

$$-2\frac{|\nabla\varphi|^2}{\varphi}w \le 0. \tag{2.15}$$

Note that

$$0 \le \psi \le 1, \quad 0 \le \frac{1}{1-f} \le 1, \quad 0 \le \frac{f^2}{(1-f)^2} \le 1,$$

multiplying both side of (2.15) by $\frac{\varphi}{1-f}$, we infer

$$\begin{aligned} &-2\varphi w\left[K+H+a^+ + \alpha b^+ + c^+ + \frac{1}{2}\left(|\nabla a| + |\nabla b| + |\nabla c|\right)\right] \\ &-\varphi\left(|\nabla a| + |\nabla b| + |\nabla c|\right) + \varphi w^2 - 3\frac{c_1^2}{R^2}wt + w\Box\varphi \le 0. \end{aligned} \tag{2.16}$$

It is easy to see that $w\Box\varphi = w\big[\Delta_V(t\psi) - (t\psi)_t\big] = tw\Delta_V\psi - \psi w$. Hence, by (2.12) and (2.16), we obtain

$$\begin{aligned} &\varphi w^2 + w\bigg\{-2\left[K+H+a^+ + \alpha b^+ + c^+ + \frac{1}{2}\left(|\nabla a| + |\nabla b| + |\nabla c|\right)\right]\varphi \\ &+t\Big(-A-\frac{\psi}{t}\Big)\bigg\} - \varphi\left(|\nabla a| + |\nabla b| + |\nabla c|\right) \le 0, \end{aligned} \tag{2.17}$$

where

$$A = \frac{R\Big[\sqrt{(n-1)K} + \dfrac{n-1}{R} + L\Big]c_1 + c_2 + 3c_1^2}{R^2}.$$

Multiplying both side of (2.17) by $\varphi = t\psi$, we have at (x,t)

$$\begin{aligned} &(\varphi w)^2 - (\varphi w)T\bigg\{2\left[K+H+a^+ + \alpha b^+ + c^+ + \frac{1}{2}\left(|\nabla a| + |\nabla b| + |\nabla c|\right)\right]\psi \\ &+A+\frac{\psi}{t}\bigg\} - \varphi^2\left(|\nabla a| + |\nabla b| + |\nabla c|\right) \le 0, \end{aligned}$$

where we used $0 \le \psi \le 1, 0 < t < T$. Hence,

$$\begin{aligned} &(\varphi w)^2 - (\varphi w)T\bigg\{2\left[K+H+a^+ + \alpha b^+ + c^+ + \frac{1}{2}\left(|\nabla a| + |\nabla b| + |\nabla c|\right)\right]\psi \\ &+A+\frac{1}{T}\bigg\} - T^2\left(|\nabla a| + |\nabla b| + |\nabla c|\right) \le 0. \end{aligned} \tag{2.18}$$

Before completing the proof, we recall a fact: if $x^2 \le ax + b$ for some $b, x \ge 0$ and $a \in \mathbb{R}$, then

$$x \le \frac{a}{2} + \sqrt{b + \Big(\frac{a}{2}\Big)^2} \le \frac{a}{2} + \sqrt{b} + \frac{a}{2} = a + \sqrt{b}. \tag{2.19}$$

Applying (2.19) to the inequality (2.18), we get

$$\begin{aligned} \varphi w \le T\bigg\{2\left[K+H+a^+ + \alpha b^+ + c^+ + \frac{1}{2}\left(|\nabla a| + |\nabla b| + |\nabla c|\right)\right]\psi + A + \frac{1}{T}\bigg\} \\ + T\sqrt{|\nabla a| + |\nabla b| + |\nabla c|}. \end{aligned}$$

For any $(x_0, T) \in B(p, R) \times [0, T]$ we have at (x_0, T)

$$w \leq \sup_{M\times[0,\infty)} \left\{ \left[K + H + a^+ + \alpha b^+ + c^+ + \frac{1}{2}\left(|\nabla a| + |\nabla b| + |\nabla c|\right) \right] + \sqrt{|\nabla a| + |\nabla b| + |\nabla c|} \right\} + A + \frac{1}{T}.$$

Let R tends to ∞, we obtain at (x_0, T)

$$\frac{|\nabla u|}{u} \leq \left(\frac{1}{T^{\frac{1}{2}}} + \sup_{M\times[0,\infty)} \sqrt{2\left(K + H + a^+ + \alpha b^+ + c^+ + \frac{1}{2}\left(|\nabla a| + |\nabla b| + |\nabla c|\right) \right)} + \sup_{M\times[0,\infty)} \sqrt[4]{|\nabla a| + |\nabla b| + |\nabla c|} \right) (1 - \log u).$$

Since (x_0, T) is arbitrary, the proof is complete.
Case 2. If $0 < \alpha < 1$, repeating the proof of **Case 1** with estimate (2.2) line by line, we arrive

$$\frac{|\nabla u|}{u} \leq \left(\frac{1}{t^{\frac{1}{2}}} + \sup_{M\times[0,\infty)} \sqrt{2\left(K + |u^{\alpha-2}|_\infty \left(a + 3|a| + 3|b|\right) + c + |c| \right)} + \sup_{M\times[0,\infty)} \sqrt[4]{|u^{\alpha-2}|_\infty \left(\frac{|\nabla a|^2}{2|a|} + \frac{|\nabla b|^2}{2|b|} \right) + \frac{|\nabla c|^2}{2|c|}} \right) (1 - \log u).$$

Here

$$|u|_\infty = \sup_M |u|,$$

$$H = (\alpha - 1)\left|a^-\right| \sup_{M\times[0,\infty)} \log\frac{1}{u}.$$

We complete the proof Theorem 1.1. □

Remark 2.2. We would like to notice that the assumption that V is bounded is used only for technique reasons. For example, as in [5] if we assume that $\langle V, \nabla\rho\rangle \leq v(\rho)$ for some non-decreasing function $v(\cdot)$, then a V-Laplacian comparison theorem still holds true, namely

$$\Delta_V \rho \leq \frac{n-1}{\rho} + \sqrt{(n-1)K} + v(\rho).$$

Therefore, when we consider a local estimate, the boundedness of V can be replaced by some suitable condition, saying $\langle V, \nabla\rho\rangle \leq v(\rho)$. Moreover, if $v(\rho)$ is of sub-linear growth, we still have a global estimate.

On the other hand, it is well-known that if Ric_V^N has a lower bound, a V-Laplacian comparison theorem holds true without any additional condition on V. Hence, similarly, we obtain the following theorem.

Theorem 2.3. *Let (M, g) be a complete noncompact n-dimensional Riemannian manifold and V be a smooth vector field on M such that $Ric_V^N \geq -K$ for some $K \geq 0$. Let $\alpha > 0$ be a constant and a, b, c be functions on $M \times [0, \infty)$,* *which are differentiable with respect to $x \in M$. Suppose that u is a positive solution to the following nonlinear heat equation*

$$u_t = \Delta u + \langle V, \nabla u\rangle + au^\alpha \log u + bu^\alpha + cu$$

with $u \le 1$ for all $(x,t) \in M \times [0,\infty)$. Then

1. *If $\alpha \ge 1$, we have*

$$\frac{|\nabla u|}{u} \le \left(\frac{1}{t^{\frac{1}{2}}} + \sup_{M\times[0,\infty)} \sqrt{2\left(K + H + a^+ + \alpha b^+ + c^+ + \frac{1}{2}\left(|\nabla a| + |\nabla b| + |\nabla c|\right)\right)} + \sup_{M\times[0,\infty)} \sqrt[4]{|\nabla a| + |\nabla b| + |\nabla c|}\right)(1-\log u).$$

2. *If $0 < \alpha < 1$, and a, b, c are functions of constant sign on $M \times [0,\infty)$, we have*

$$\frac{|\nabla u|}{u} \le \left(\frac{1}{t^{\frac{1}{2}}} + \sup_{M\times[0,\infty)} \sqrt{2\left(K + |u^{\alpha-2}|_\infty (a + 3|a| + 3|b|) + c + |c|\right)} + \sup_{M\times[0,\infty)} \sqrt[4]{|u^{\alpha-2}|_\infty \left(\frac{|\nabla a|^2}{2|a|} + \frac{|\nabla b|^2}{2|b|}\right) + \frac{|\nabla c|^2}{2|c|}}\right)(1-\log u),$$

where

$$|u|_\infty = \sup_M |u|,$$
$$H = (\alpha - 1)\left|a^-\right| \sup_{M\times[0,\infty)} \log\frac{1}{u}.$$

Proof of Theorem 2.3. Since $Ric_V^N \ge -K$, the Laplacian comparison theorem in [13] implies that

$$\Delta_V \rho \le \sqrt{(n-1)K}\coth\left(\sqrt{\frac{K}{n-1}}\rho\right) \le \sqrt{(n-1)K} + \frac{n-1}{\rho}.$$

Repeating arguments in the proof of Theorem 1.1, we have that in this case, the right hand side of (2.12) does not depend on L. Hence, we have

$$A = \frac{\left(n-1+\sqrt{(n-1)K}R\right)c_1 + c_2 + 3c_1^2}{R^2}.$$

The proof is complete. □

If $V = \nabla\phi$, $a = b = 0$ and c is a negative function in Theorem 2.3 with $\alpha \in (0,1)$, then we recover Ruan's main theorem in [17].

Corollary 2.4 ([17])**.** *Let M be a complete noncompact Riemannian manifold of dimension n and ϕ be a smooth function on M such that $Ric_\phi^N \ge -K$ for some $K \ge 0$. Suppose that c is a non positive function on $M \times [0,\infty)$ and c is differentiable with respect to x. Assume that u is a positive solution of the following heat equation*

$$u_t = \Delta u + \langle\nabla\phi, \nabla u\rangle + cu$$

and $u \le 1$ *on* $M \times [0, \infty)$*. Then*

$$\frac{|\nabla u|}{u} \le \Big(\frac{1}{t^{\frac{1}{2}}} + \sqrt{2K} + \sup_{M\times[0,\infty)} |\nabla \sqrt{-c}|^{\frac{1}{2}} \Big) (1 - \log u) .$$

3. Applications

In this section, we will give several applications of gradient estimates given in Theorems 1.1 and 2.3. The first one is the following Harnack inequality.

Corollary 3.1. *Let* (M, g) *be a complete noncompact* n*-dimensional Riemannian manifold and* V *be a smooth vector field on* M *such that* $Riv_V^N \ge -K$ *for some* $K \ge 0$*. Let* $\alpha \ge 1$ *be a constant and* a, b, c *be functions of constant sign on* $M \times [0, \infty)$*, which are differentiable with respect to* $x \in M$*. Assume that there exist* $C_1, C_2 > 0$ *satisfying*

$$C_1 \ge \max \left\{ H + a^+ + \frac{1}{2}|\nabla a|; \quad \alpha b^+ + \frac{1}{2}|\nabla b|; \quad c^+ + \frac{1}{2}|\nabla c| \right\}$$

and

$$C_2 \ge \max \left\{ \sqrt{|\nabla a|}; \sqrt{|\nabla b|}; \sqrt{|\nabla c|} \right\}.$$

If u *is a positive solution to the general heat equation*

$$u_t = \Delta u + \langle V, \nabla u\rangle + au^\alpha \log u + bu^\alpha + cu$$

and $u \le 1$ *for all* $(x, t) \in M \times [0, \infty)$*, then for any* $x_1, x_2 \in M$ *we have*

$$u(x_2, t) \le u(x_1, t)^\beta e^{1-\beta},$$

where

$$H = (\alpha - 1) \left| a^- \right| \sup_{M\times[0,\infty)} \log \frac{1}{u},$$

$\rho = \rho(x_1, x_2)$ *is the distance between* x_1, x_2 *and*

$$\beta = exp \left(-\frac{\rho}{t^{\frac{1}{2}}} - (\sqrt{2(K + 3C_1)} + \sqrt{3C_2})\rho \right).$$

Proof. Let $\gamma(s)$ be a geodesic of minimal length connecting x_1 and x_2, $\gamma : [0, 1] \to M$, $\gamma(0) = x_2$, $\gamma(1) = x_1$. Let $f = \log u$. Using Theorem 2.3, we have

$$\begin{aligned} \log \frac{1 - f(x_1, t)}{1 - f(x_2, t)} &= \int_0^1 \frac{d \log (1 - f(\gamma(s), t))}{ds} ds \\ &\le \int_0^1 |\dot\gamma| \frac{|\nabla u|}{u(1 - \log u)} ds \\ &\le \frac{\rho}{t^{\frac{1}{2}}} + \big(\sqrt{2(K + 3C_1)} + \sqrt{3C_2}\big)\rho. \end{aligned}$$

Let $\beta = \exp\Big(- \frac{\rho}{t^{\frac{1}{2}}} - \big(\sqrt{2(K + 3C_1)} + \sqrt{3C_2}\big)\rho\Big)$ the above inequality implies

$$\frac{1 - f(x_1, t)}{1 - f(x_2, t)} \le \frac{1}{\beta}.$$

Hence,

$$u(x_2, t) \leq u(x_1, t)^\beta e^{1-\beta}.$$

The proof is complete. □

The second application is a gradient estimate for a non linear heat equation arising from gradient Ricci soliton.

Corollary 3.2. *Let M be a complete noncompact Riemannian manifold of dimension n and V be a smooth vector field on M such that $Riv_V^N \geq -K$ for some $K \geq 0$. Suppose that a, b are real numbers and the positive solution u to the heat equation*

$$u_t = \Delta u + \langle V, \nabla u\rangle + au \log u + bu$$

satisfying $u \leq 1$. Then

$$\frac{|\nabla u|}{u} \leq \left(\frac{1}{t^{\frac{1}{2}}} + \sqrt{2\left(K + a^+ + b^+\right)}\right)(1 - \log u). \tag{3.1}$$

Proof. By the assumption on a, b, we have $\nabla a = 0, \nabla b = 0$. Note that if $\alpha = 1$, then $H = 0$, using Theorem 2.3, we obtain

$$\frac{|\nabla u|}{u} \leq \left(\frac{1}{t^{\frac{1}{2}}} + \sqrt{2\left(K + a^+ + b^+\right)}\right)(1 - \log u).$$

We are done. □

The third application is a Liouville type result.

Corollary 3.3. *Let M be a complete noncompact Riemannian manifold and V be a smooth vector field on M such that $Ric_V^N \geq 0$. Suppose that a, b are nonpositive real numbers. If u is a positive solution to following general elliptic equation*

$$\Delta u + \langle V, \nabla u\rangle + au \log u + bu = 0 \tag{3.2}$$

and $u \leq C$, then $u \equiv e^{-\frac{b}{a}}$.

Proof. Suppose that u is a positive solution of (3.2) with $u \leq C$. Since u does not depend on t, we have $\widetilde{u} := u/C \leq 1$ is a positive solution to the following parabolic equations

$$\widetilde{u}_t = \Delta\widetilde{u} + \langle V, \nabla\widetilde{u}\rangle + au \log \widetilde{u} + \widetilde{b}\widetilde{u},$$

where $\widetilde{b} = b + a \log C$. Since $a \leq 0, b \leq 0$, we have

$$a^+ = \max\{a, 0\} = 0, \widetilde{b}^+ = \max\{\widetilde{b}, 0\} = 0.$$

Using the inequality (3.1), we obtain

$$\frac{|\nabla\widetilde{u}|}{\widetilde{u}} \leq \left(\frac{1}{t^{\frac{1}{2}}} + \sqrt{2K}\right)(1 - \log \widetilde{u}). \tag{3.3}$$

Hence, let t tends to ∞ and $K = 0$ in (3.3), we get

$$\frac{|\nabla\widetilde{u}|}{\widetilde{u}} \leq 0.$$

This implies u must be a constant. Therefore $u = e^{-\frac{b}{a}}$. □

Motivated by studying of Yamabe equation, we show the forth application as follows.

Corollary 3.4. *Let M be a complete noncompact Riemannian manifold and V be a smooth vector field on M such that $Ric_V^N \geq -K$ for some $K \geq 0$. Suppose that α, b, c are real numbers with $\alpha \geq 1$ and the positive solution u to the equations*

$$u_t = \Delta u + \langle V, \nabla u \rangle + bu^\alpha + cu$$

satisfying $u \leq 1$. Then

$$\frac{|\nabla u|}{u} \leq \left(\frac{1}{t^{\frac{1}{2}}} + \sqrt{2\left(K + \alpha b^+ + c^+\right)} \right) (1 - \log u) . \tag{3.4}$$

Proof. If b, c are real numbers, then $\nabla b = 0, \nabla c = 0$. Note that if $a = 0$, then $H = 0$, using Theorem 2.3, we obtain

$$\frac{|\nabla u|}{u} \leq \left(\frac{1}{t^{\frac{1}{2}}} + \sqrt{2\left(K + \alpha b^+ + c^+\right)} \right) (1 - \log u) .$$

The proof is complete. □

We would like to mention that by the scaling argument, all results given in Corollaries 3.3, 3.4, and 3.5 still hold true if u is bounded and positive. However, in this case, our gradient estimates are depended on a upper bound of u.

Finally, we prove a non existence result for Yamabe equation.

Corollary 3.5. *Let M be a complete noncompact Riemannian manifold and V be a smooth vector field on M such that $Ric_V^N \geq 0$. Suppose that α, b, c are real numbers with $\alpha \geq 1$, $b \leq 0$, $c \leq 0$. Then Yamabe-type equation*

$$\Delta u + bu^\alpha + cu = 0 \tag{3.5}$$

has no bounded and positive solution.

Proof. Suppose that u is a positive solution of (3.5) with $u \leq C$. Since u does not depend on t, we have $\widetilde{u} := u/C \leq 1$ is a positive solution to the following parabolic equations

$$\widetilde{u}_t = \Delta \widetilde{u} + \langle V, \nabla \widetilde{u} \rangle + \widetilde{b}\widetilde{u}^\alpha + c\widetilde{u},$$

where $\widetilde{b} = bC^{\alpha - 1}$. Since $b \leq 0, c \leq 0$, we have

$$\widetilde{b}^+ = \max\{\widetilde{b}, 0\} = 0, c^+ = \max\{c, 0\} = 0.$$

Using the inequality (3.4), we obtain

$$\frac{|\nabla \widetilde{u}|}{\widetilde{u}} \leq \left(\frac{1}{t^{\frac{1}{2}}} + \sqrt{2K} \right) (1 - \log \widetilde{u}) . \tag{3.6}$$

Hence, let t tends to ∞ and $K = 0$ in (3.6), we get

$$\frac{|\nabla \widetilde{u}|}{\widetilde{u}} \leq 0.$$

This implies $\widetilde{u}$ must be a constant. Consequently, $u^{\alpha-1} = -\frac{c}{b}$. This gives a contradiction. We are done. □

Acknowledgment. The author would like to express our thanks to an anonymous referee(s) for his/her useful comments and suggestions to improve the presentation of this paper. His thanks also goes to Nguyen Thac Dung for suggesting the problem and useful advice during the preparation of this work. This work is supported by Hanoi Pedagogical University No. 2 (Vietnam).

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Ha Tuan Dung
Department of Mathematics
Hanoi Pedagogical University No. 2
Nguyen Van Linh Road, Xuan Hoa
Phuc Yen District, Vinh Phuc Province, Viet Nam
and
Department of Mathematics
National Tsinghua University
No. 101, Kuangfu Road, Hisnchu, Taiwan
Email address: hatuandung.hpu2@gmail.com