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Stochastic Properties of Life Distribution with Increasing Tail Failure Rate and Nonparametric Testing Procedure

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Purpose: The purpose of this study is to investigate the tail behavior of the life distribution which exhibits an increasing failure rate or other positive aging effects after a certain time point.
Methods: We characterize the tail behavior of the life distribution with regard to certain reliability measures such as failure rate, mean residual life and reliability function and derive several stochastic properties regarding such life distributions. Also, utilizing an L-statistic and its asymptotic normality, we propose new nonparametric testing procedures which verify if the life distribution has an increasing tail failure rate.
Results: We propose the IFR-Tail (Increasing Failure Rate in Tail), DMRL-Tail (Decreasing Mean Residual Life in Tail) and NBU-Tail (New Better than Used in Tail) classes, all of which represent the tail behavior of the life distribution. And we discuss some stochastic properties of these proposed classes. Also, we develop a new nonparametric test procedure for detecting the IFR-Tail class and discuss its relative efficiency to explore the power of the test.
Conclusion: The results of our research could be utilized in the study of wide range of applications including the maintenance and warranty policy of the second-hand system.

Keywords: Failure Rate, Mean Residual Life, Tail Behavior, Nonparametric Class, L-Statistics

1. Introduction

Most of the repairable systems possess the positive aging property and thus, the functional performance of such a system deteriorates as it ages. Furthermore, the failure rate of the system increases monotonically at the later stage of its life cycle and thus the system is more likely to fail during the wear out phase after reaching a certain age. The tail behavior of the life distribution that the system exhibits not only affects the failure process of the system at the later stage, but also becomes a significant factor to determine an optimal maintenance policy to minimize the system failures during its life cycle. Recently, the second-hand system receives an increasing attention in the maintenance policy of the repairable system due to the fact that the second-hand system would offer an option for many customers, especially for quite expensive systems. For the second-hand system, the user is mainly interested in the residual life beyond the purchasing age of the system, rather than its entire life. Thus, the tail behavior of the life distribution may play a crucial role to analyze the aging property of the second-hand system and to find an optimal maintenance policy to keep the system failure-free during its remaining life cycle. Several articles dealing with the maintenance policy for the second-hand system exist in the literature. For instance,

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Shafiee Finkelstein and Chukova [1] discuss an optimal upgrade strategy for the second- hand system by studying the tail behavior of the system and Kim, Lim and Park [2] investigate the effects of periodic inspection for the second-hand system with increasing failure rate.

As the criteria to quantify the tail behavior of the system, two reliability measures of failure rate and mean residual life have been widely used to quantify the aging effect and several nonparametric classes of life distributions are defined according to the direction of monotonicity of such measures. Let X denote the life of a system with cumulative distribution function F and probability density function f. Then, the failure rate and mean residual life of X are defined as $r(u)=f(u)/\overline{F}(u)$ and $m(u)=\int_0^\infty \overline{F}(u+x)dx/\overline{F}(u)$ for $u \ge 0$, respectively. It is well known that the classes of life distributions with non-decreasing failure rate and non-increasing mean residual life are defined as an IFR and a DMRL class, respectively. The dual classes of DFR and IMRL are defined similarly by reversing the direction of monotonicity of these two measures, respectively. In addition, the NBU and $NBU-t_0$ classes, each of which plays an important role in developing the maintenance policies, are defined based on the reliability function and have been discussed by many authors, including Hollander and Proschan [3] and Hollander, Park and Proschan [4]. In the followings, we give the definitions for the NBU and $NBU-t_0$ classes.

Definition 1. A random variable X or F is said to be new better than used (NBU) if $\overline{F}(x+t) \le \overline{F}(x)\overline{F}(t)$ for all $x \ge 0$ and $t \ge 0$.

Definition 2. A random variable X or F is said to be new better than used of specified age t_0 ($NBU-t_0$) if $\overline{F}(x+t_0) \le \overline{F}(x)\overline{F}(t_0)$ for all $x \ge 0$ and fixed $t_0 \ge 0$.

The main objective of this paper is to characterize the tail behavior of the life distribution with respect to its failure rate, mean residual life and reliability function and obtain several stochastic properties regarding such life distributions.

Let $t \ge 0$ be fixed and denote

$$X_t = [X-t \mid X>t] \quad for\ t \in (\alpha, \beta),$$

where $\alpha = sup\{x : F_X(x)=0\}$ and $\beta = inf\{x : F_X(x)=1\}$. Then X_t can be interpreted as a random residual life at time t and its reliability function (survival function) can be expressed as

$$\overline{F_t}(u) = \frac{\overline{F}(u+t)}{\overline{F}(t)}, \quad u \ge 0.$$

The mean residual life, denoted by $m(t)$, can be obtained by taking an expectation for X_t and the failure rate of X_t at time x equals the failure rate of X at time $t+x$. That is, $m(t)=EX_t$ and $r_t(x)=r(t+x)$, where $r(\bullet)$ is the failure rate function of X. Although the focus of research regarding the life distribution of a system mostly concerns the entire life, more important aspect in the aging phenomenon for the deteriorating system is the aging effect on the system performance at the later stage, rather than the effect of an earlier stage. Ahmad [5] points out the fact that the entire life experiences several stages in sequence, among which the stage of "wear out age" comes last. It is in this old age stage that the maintenance of the system can be more costly and needs more attention. Hence, the old age characteristics of a life distribution can be considered quite important subject of research and consequently, the old age behavior of a system has been extensively studied by many researchers in the literature.

Several nonparametric classes of life distributions have been proposed based on the tail behavior of the distribution. Alzaid [6] proposes the "used better than aged" (UBA) class and the "used better than aged in expectation" ($UBAE$) class. The "new better than used from t_0" ($NBU-[t_0, \infty)$) class is proposed by Ahmad [7]. These nonparametric classes are summarized in the following definitions. Let $t_0 \ge 0$ be a fixed constant.

Definition 3. A random variable X or F is said to be UBA if $0 < m(\infty) < \infty$ and for all $t \ge 0$ and $x \ge 0$, $\overline{F_t}(x) \le e^{-x/m(\infty)}$.

Definition 4. A random variable X or F is said to be $UBAE$ if $0 < r(\infty) < \infty$ and for all $t \geq 0$, $m(t) \leq m(\infty)$.

Definition 5. A random variable X or F is said to be $NBU-[t_0, \infty)$ if for all $x \geq 0$ and $y \geq t_0$, $\overline{F_y}(x) \leq \overline{F}(x)$.

These classes of life distributions are defined based on the tail behavior of the distribution reaching a certain time point. Willmot and Cai [8] discuss the implications between the *UBA* class and some other classes to show that the *UBA* class contains the DMRL class as its subclass. Ahmad [9] presents some properties of the *UBA* and *UBAE* classes of life distributions and develops a nonparametric test for detecting each of the *UBA*-ness and the *UBAE*-ness. Later, Ahmad [5] introduces a concept of old age and investigates the moment inequalities and moment generating function of the random variable representing the old age.

This paper derives several monotonicity properties that the failure rate, mean residual life and reliability function of the distribution exhibit beyond a specified age, disregarding the age patterns preceding such an age. In Section 2, we consider the tail behavior of the distribution with regard to its failure rate, mean residual life and reliability function to define the *IFR-Tail* (Increasing Failure Rate in Tail), *DMRL-Tail* (Decreasing Mean Residual Life in Tail) and *NBU-Tail* (New Better than Used in Tail) classes of life distributions. Section 3 discusses some stochastic properties of these proposed classes and explains their mutual implications with other existing classes of life distributions in depth. In Section 4, we propose a new nonparametric test procedure for detecting the *IFR-Tail* class utilizing an L-statistic and discuss its relative efficiency to explore the power of the test. Concluding remarks are given in Section 5.

2. Classes of Life Distributions Characterizing the Tail Behavior

Regardless of the aging pattern of the life distribution at the earlier stage, each of the failure rate, mean residual life and reliability function may be used to classify the life distributions by consideration of its tail behavior at the later stage. The following nonparametric classes of life distributions are defined based on the tail behavior of the distribution as follows.

Definition 6. A random variable X or F is said to belong to the *IFR-Tail* (Increasing Failure Rate in Tail) class if there exists a value s_0 such that $r(t)$ is non-decreasing for $t \geq s_0 \geq 0$.

Definition 7. A random variable X or F is said to belong to the *DMRL-Tail* (Decreasing Mean Residual Life in Tail) class if there exists a value s_0 such that $m(t)$ is non-increasing for $t \geq s_0 \geq 0$.

Definition 8. A random variable X or F is said to belong to the *NBU-Tail* (New Better than Used in Tail) class if there exists a value s_0 such that $\overline{F}(x+t) \leq \overline{F}(x)\overline{F}(t)$ for all $x \geq 0$ and $t \geq s_0 \geq 0$.

The dual classes of *DFR-Tail* (Decreasing Failure Rate in Tail), *DMRL-Tail* (Decreasing Mean Residual Life in Tail) and *NWU-Tail* (New Worse than Used in Tail) are defined similarly by replacing 'non-decreasing', 'non-increasing' and '$\leq$' in Definitions 6, 7 and 8 by 'non-increasing', 'non-decreasing' and '$\geq$', respectively.

Most of the repairable systems are known to exhibit the age pattern with the bathtub-shaped failure rate and have the increasing tail failure rate beyond a certain age. Thus, the life distribution of many repairable systems may belong to the *IFR-Tail* class, which obviously contains the IFR class. <Fig. 1> shows various patterns of failure rates for the life distributions which belong to the *IFR-Tail* class. We note that a life distribution which has either bathtub-shaped or hockey stick-shaped failure rate belongs to the *IFR-Tail* class. Note also that the value of s_0 is unknown, but arbitrary in the definitions of the *IFR- Tail*, *DMRL-Tail* and *NBU-Tail* classes, while the value of t_0 is assumed to be fixed for the $NBU-t_0$ and $NBU-[t_0, \infty)$ classes, Throughout this paper, the value of s_0 in the definitions of the *IFR-Tail*, *DMRL-Tail* and *NBU-Tail* classes is referred to as a trend-change point of the life distribution.

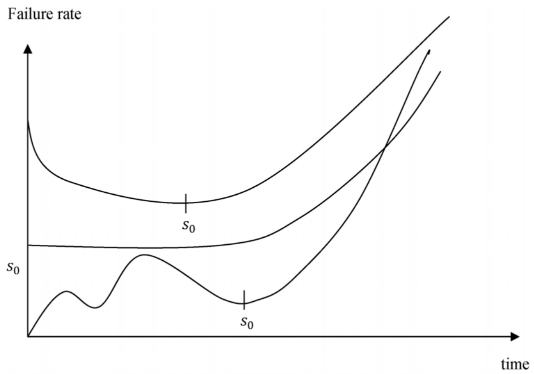

Fig. 1 Failure rate of a life distribution which belongs to IFR−Tail class

3. Some Properties of IFR-Tail, DMRL-Tail and NBU-Tail Classes

It is obvious from Definitions 6, 7 and 8 that the IFR, *DMRL* and *NBU* classes are the subclasses of the *IFR-Tail*, *DMRL-Tail* and *NBU-Tail* classes. Guess, Nam and Park (GNP) [10] discuss the trend changes of failure rate and mean residual life for the *IDF* (Increasing then Decreasing Failure Rate) and *DIMRL* (Decreasing then Increasing Mean Residual Life) classes and investigate the relation of trend-change points for both classes. It is also well known that the *IFR* class is a subclass of the *DMRL* class, which is again the subclass of the *NBU* class. The following theorem states some implications among the *IFR-Tail*, *DMR.L-Tail* and *NBU-Tail* classes.

Theorem 1.
(a) The *IFR-Tail* class implies the *DMRL-Tail* class
(b) The $NBU-[t_0, \infty)$ class implies the NBU-Tail class

Proof.

(a) Denote $N(t)=r(t)\int_t^{\infty}\overline{F}(x)dx$. From the definition of $m(t)$, it can easily be obtained that $\frac{d}{dt}m(t) = \frac{N(t)}{\overline{F}(t)}$. Since F belongs to the *IFR-Tail* class, there exists a $t_0 \geq 0$ such that $r(t)$ increases in $t \geq t_0 \geq 0$. Thus, we have

$$N(t) \leq \int_t^{\infty} r(x)\overline{F}(x)dx - \overline{F}(t) = \int_t^{\infty} f(x)dx - \overline{F}(t) = 0$$

for $t \geq t_0$. Thus, we have $\frac{d}{dt}m(t) < 0$ and hence $m(t)$ decreases for $t \geq t_0$. This completes the proof.

(b) If F belongs to the $NBU-[t_0, \infty)$ class, then the *NBU-ness* property holds for all $t \geq t_0 \geq 0$ and thus F belongs to the *NBU-Tail* class by the definition.

Theorem 2. The trend-change point of the *IFR-Tail* distribution is always greater than the trend- change point of the *DMRL-Tail* distribution.

Proof. Note that the *IFR-Tail* class contains the *IFR* class. If *F* belongs to the *IFR* class, the result follows directly from the GNP's [10] results. Now we consider the case when *F* is IFR-Tail, but not *IFR*. Let s_0 be a change point of the *IFR-Tail* distribution. Then $r(t)$ increases for $t \geq s_0$ and thus $m(t)$ decreases for $t \geq s_0$. Consider an interval $[w, s_0))$ in which $r(t)$ is monotonically decreasing. Then it is noted that $r'(t) \leq 0$ and $N'(t) = r'(t)\int_t^{\infty}\overline{F}(x)dx \leq 0$ for $t \in [w, s_0)$. It follows that $N(t)$ is decreasing for $t \in [w, s_0)$. If $r(w) \leq \left[\frac{\int_w^{\infty}\overline{F}(x)dx}{\overline{F}(w)}\right]^{-1} = m(w)^{-1}$, then $N(w) \leq 0$ which implies that F is DMRL for $t \geq w$. If $r(w) \geq \left[\frac{\int_w^{\infty}\overline{F}(x)dx}{\overline{F}(w)}\right]^{-1} = m(w)^{-1}$, then $N(w) > 0$ and thus there exists a point $t^* \in [w, s_0)$ such that $N(t) \geq 0$ for $t \in [w, t^*)$, $N(t)=0$ for $t=t^*$, and $N(t)<0$ for $t \in (t^*, s_0)$. Note that if $r(t)$ is strictly decreasing for $t \in [w, s_0)$, then t^* is uniquely determined. This implies that $m(t)$ increases for $t \in [w, t^*)$ and decreases for $t \in (t^*, \infty)$. The result follows. ■

We note that if F belongs to the *IFR-Tail* class, it does not necessarily imply that F belongs to the *NBU* class, although the *IFR* class is a subclass of the *NBU* class. However, it can be shown that if F is an *IFR-Tail* distribution, then there exists a s_0 such that the residual life at s_0 is stochastically greater than the residual life at t for $t \geq s_0 \geq 0$. Note that the *NBU-ness* property holds only for the special case when $s_0 = 0$. These results are stated in the following Theorem.

Theorem 3. Suppose that F belongs to the *IFR-Tail* class. Then there exists a $s_0 \geq 0$ such that $X_t \leq X_{s_0}$ for all $t \geq s_0 \geq 0$.

Proof. Let $x > 0$ be given. Then, since F is an *IFR-Tail* distribution, there exists a $s_0 \geq 0$ such that $r(t)$ is increasing in $t \geq s_0 \geq 0$ and consequently, $\int_{s_0}^{s_0+x} r(u)du \leq \int_t^{t+x} r(u)du$ for all $t \geq s_0 \geq 0$. Hence, for all $t \geq s_0 \geq 0$,

$$\overline{F}_t(x) = \frac{\overline{F}(x+t)}{\overline{F}(t)} = \mathrm{Exp}\frac{\left[-\int_0^{t+x} r(u)du\right]}{\exp\left[-\int_0^{t} r(u)du\right]}$$

$$= \exp\left[-\int_t^{t+x} r(u)du\right]$$

$$\leq \exp\left[-\int_{s_0}^{s_0+x} r(u)du\right]$$

$$= \frac{\exp\left[-\int_0^{s_0+x} r(u)du\right]}{\exp\left[-\int_0^{s_0} r(u)du\right]}$$

$$= \frac{\overline{F}(x+s_0)}{\overline{F}(s_0)} = \overline{F}_{s_0}(x),$$

which implies that $X_t \leq X_{s_0}$ for all $t \geq s_0 \geq 0$. ■

4. Nonparametric Test Procedure for Increasing Tail Failure Rate

It is true in general that most of the repairable systems deteriorate as it ages, regardless of the aging patterns at the earlier stage. Thus, the tail behavior of the life distribution should be considered as an important factor to analyze the aging phenomenon of such a repairable system and to develop an optimal maintenance strategy, especially when purchasing the second-hand system. The *IFR-tail* distribution characterizes the tail behavior of the distribution with respect to its failure rate and thus, it is an interesting problem to verify that the life distribution of a repairable system exhibits the *IFR-tail* property based on the failure data.

In this section, we consider a problem of testing

H_0: F is an exponential distribution

against

H_1: F is an *IFR-Tail* distribution with the trend-change point $s_0 > 0$ known,

based on a random sample $X_1, X_2, \cdots, X_n$ obtained from a continuous distribution F. Many researchers in this field have considered this type of hypothesis.

Mitra and Anis [11] propose a nonparametric procedure for testing exponentiality against the IFR alternatives based on an L-statistic and more recently, Annis [12] extends Mitra and Anis's [11] test to propose a family of test statistics for testing the same hypotheses. Motivated by Mitra and Anis's [11] test statistic, we consider the following parameter as a measure of deviation of F from exponentiality in favor of *IFR-Tail* alternatives.

$$\Delta(F) = \iint [r(x) - r(y)]\overline{F}(y)\overline{F}(x)dydx$$

where the integration is defined on the set $\{(x, y) : 0 \leq s_0 \leq y \leq x < \infty\}$. Here s_0 denotes a known trend-change point for the failure rate of the distribution under H_1. Under H_0, $\triangle(F) = 0$ and under H_1, $\triangle(F) > 0$. Thus, the greater $\triangle(F)$ is, the greater is the evidence in favor of H_1.

More convenient form of $\triangle(F)$ can be obtained by integration by parts as

$$\begin{aligned}\Delta(F) &= 2\int_{s_0}^{\infty}\overline{F}(t)^2dt - \overline{F}(s_0)\int_{s_0}^{\infty}\overline{F}(t)dt \\ &= \int_{s_0}^{\infty} x[4\overline{F}(x) - \overline{F}(t_0)]dF(x) \\ &\quad - \overline{F}(s_0)^2 \cdot s_0 \\ &= \int_{s_0}^{\infty} xJ(F(x))dF(x) - \overline{F}(s_0)^2 \cdot s_0,\end{aligned}$$

where $J(u) = 4(1-u) - \overline{F}(s_0)$.

To derive the test statistic based on the measure $\triangle(F)$, we replace F of $\triangle(F)$ by F_n, where F_n is an empirical distribution obtained from the random sample $X_1, X_2, \cdots, X_n$.

Define $k = Min\{i | X_{(i)} > t_0\}$. Then

$$\begin{aligned}D_n = \Delta(F_n) &= \int_{s_0}^{\infty} xJ(F_n(x))dF_n(x) \\ &\quad - \overline{F}_n(s_0)^2 \cdot s_0 \\ &= \sum_{j=k}^{n}\frac{1}{n}X_{(j)}J(F_n(X_{(j)})) \\ &\quad - \frac{(n-k+1)^2 s_0}{n^2} \\ &= \sum_{j=k}^{n}\frac{1}{n}X_{(j)}\left[4\left(1-\frac{j}{n}\right) - \frac{n-k+1}{n}\right] \\ &\quad - \frac{(n-k+1)^2 s_0}{n^2} \\ &= \frac{1}{n^2}\sum_{j=k}^{n}\{X_{(j)}[4(n-j)-(n-k+1)] \\ &\quad -(n-k+1)s_0\},\end{aligned}$$

where $0 \equiv X_{(0)} \le X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ are the order statistics corresponding to the random sample $X_1, X_2, \cdots, X_n$. Since D_n is not a scale invariant statistic, we divide D_n by the sample mean $\overline{X}_n$ and use the following scale invariant statistic as our test statistic.

$$D_n^* = \frac{D_n}{\overline{X}_n},$$

where $\overline{X}_n = \dfrac{\sum_{j=1}^{n} X_j}{n}$.

The asymptotic normality of the test statistics D_n^* is established by applying the similar technique as that of Lim and Park's [13]. The asymptotic distribution of D_n^* is stated in the following theorem without proof. See Theorem 3 of Joe and Proschan [14] for detailed proofs of asymptotic normality of the L-statistics, which is similar to our test statistic.

Theorem 4. Let F be a continuous life distribution satisfying the following conditions.

(i) $\int_0^{\infty} x^2 dF(x) < \infty$

(ii) $f(F^{-1}(p))$ exists and is positive,

(iii) $\int (F(x)\overline{F}(x))^{\frac{1}{2}-\delta}dx < \infty$ for some $0 < \delta < 2^{-1}$.

Then

$$\sqrt{n}(D_n^* - \Delta(F)/\mu(F)) \xrightarrow{d} N(0, \sigma^2(J, F)) \text{ as } n \to \infty,$$

where $\mu(F) = \int_0^{\infty}\overline{F}(x)dx$

and

$$\begin{aligned}\sigma^2(J, F) &= \iint J(F(x))J(F(y))(F(x \wedge y) \\ &\quad - F(x)F(y))dxdy \\ &\quad + F(s_0)\overline{F}(s_0)^{-5}f(s_0)^{-2} \\ &\quad - 2\overline{F}(s_0)f(s_0)^{-1} \\ &\quad \int J(F(x))(F(s_0) \wedge F(x) \\ &\quad - F(s_0)F(x)dx.\end{aligned}$$

Here $\wedge$ is the symbol for minimum. Since D_n^* is a scale invariant statistics, the asymptotic null variance of $\sqrt{n}D_n^*$ can be obtained by taking the scale parameter equal to 1 under H_0.

Table 1 Pitman's asymptotic efficacy of the test based on $\Delta(F)$

s_0	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$E_{F_1}(\triangle(F))$	0.750	0.679	0.614	0.556	0.503	0.455	0.412	0.372	0.337	0.305	0.276
$E_{F_2}(\triangle(F))$	0.083	0.062	0.046	0.034	0.025	0.019	0.014	0.010	0.008	0.006	0.004

By taking $F_0(x)=1-\exp(-x)$, we obtain the null mean of 0 and the null variance of $\sigma_0^2=\sigma^2(J,\ F_0)=\frac{1}{3}e^{-3s_0}$. Hence, we have the asymptotic null distribution of $\sqrt{n}\,D_n^*$ as

$$\sqrt{n}\,D_n^* \xrightarrow{d} N(0,\ \sigma_0^2) \text{ as } n\rightarrow\infty.$$

Note that the null variance depends on s_0. When $s_0=0$, the test is reduced to that of Mitra and Anis's [11] and in this case, $\sigma^2(J,\ F_0)=\frac{1}{3}$, which agrees with the asymptotic null variance of Mitra and Anis's [11] test statistic.

The approximate α level test of H_0 against H_1 (which is referred to as an IFR-Tail test in this paper) is to reject H_0 in favor of H_1 if

$$\frac{\sqrt{n}\,D_n^*}{\sigma_0}\geq z_\alpha$$

where z_α is the upper α-quantile of the standard normal distribution.

To study the Pitman's asymptotic efficacy of the proposed test, we consider the linear failure rate distribution and Makeham distribution as alternatives, both of which belong to the IFR-Tail class. The survival functions of these distributions are given as follows.

(i) Linear failure rate distribution:

$$\overline{F}_1(x)=\exp[-(x+\theta x^2/2)],\ \theta\geq 0,\ x\geq 0$$

(ii) Makeham distribution:

$$\overline{F}_2(x)=\exp[-(x+\theta(x+e^{-x}-1)],$$
$$\theta\geq 0,\ x\geq 0.$$

By taking $\theta=0$, both F_1 and F_2 reduce to an exponential distribution, which is hypothesized distribution under H_0 By direct calculations, we obtain the Pitman's asymptotic efficacy for F_1 and F_2 as follows, respectively.

$$E_{F_1}(\triangle(F))=\frac{3}{4}e^{-s_0}$$

$$\text{and } E_{F_2}(\triangle(F))=\frac{1}{12}e^{-3s_0}$$

<Table 1> presents the efficacy values of the proposed *IFR-Tail* test for selected values of s_0. We note that when $s_0=0$, $E_{F_1}(\triangle(F))=0.750$ and $E_{F_2}(\triangle(F))=0.083$ which are the same efficacy values as those of Mitra and Anis's [11] test. It is observed from <Table 1> that the efficacy values decrease steadily as s_0 increases, which may indicate that as the value of s_0 becomes larger (i.e. as the trend-change point occurs later), the more sample size is needed to achieve the same power of the *IFR-Tail* test, which is as anticipated.

5. Conclusion

The main purpose of this paper is to investigate the tail behavior of the life distribution and to derive some stochastic properties with respect to the failure rate, mean residual life and reliability function. There exist a large number of references regarding the life distributions with monotonic failure rate and mean residual life in the literature. However, most of the repairable systems exhibit either bathtub-shaped or hockey stick-shaped failure rate, instead of monotonic failure rate, in real situations. Thus, the aging pattern of the system at a later stage of its life cycle could be more interesting phenomenon, especially for quite expensive equipment that requires very high reliability even at a later stage of its life. Recently, the second-hand system receives an increasing attention in the market and its optimal maintenance policy becomes an important issue to minimize the system's failure and to keep the maintenance cost low. When the second-hand system is purchased, the user is more concerned about the tail distribution of the system, regardless of the system's aging pattern prior to its purchase. The tail behavior of the life distribution is quite an important issue to study for a wide range of applications, including the maintenance and warranty policy of the second-hand system.

In this paper, we characterize the tail behavior of the life distribution by defining new classes of life distributions with respect to its failure rate, mean residual life and reliability function and study several stochastic properties that such tail distributions possesses. These new classes are referred to as *IFR-Tail*, *DMRL-Tail* and *NBU-Tail* classes which specify the aging patterns after reaching a certain old age. The *IFR-Tail*, *DMRL-Tail* and *NBU-Tail* classes contain the well-known *IFR*, *DMRL* and *NBU* classes as their subclasses. This paper also compares the *IFR-Tail* and *DMRL-Tail* classes with respect to their trend-change points. Furthermore, we develop a new nonparametric procedure to test whether the life distribution belongs to the *IFR-Tail* class or not based on the failure data obtained from the distribution. To study the power of our proposed test, we evaluate the efficacies of the test for two *IFR-Tail* alternatives and the result shows that as the trend-change point occurs later, the more sample size is needed to achieve the same power of the test. Our test is a generalized version of the one proposed by Mitra and Anis's [11] test, which was designed to test the monotonic IFR class.

More thorough analysis is necessary to understand the aging pattern of the system at a later stage, which could be critical to establish an optimal maintenance policy and to minimize the operating cost during its life cycle. The maintenance of the old system beyond a certain age is especially important because the system is likely to fail more often at a later stage and so the tail behavior of the life distribution needs more attention in the future research.

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