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# GENERAL DECAY OF SOLUTIONS FOR VISCOELASTIC EQUATION WITH NONLINEAR SOURCE TERMS 

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#### Abstract

A viscoelastic wave equation in canonical form weakly nonlinear time dependent dissipation and source terms is investigated in this paper. And we establish a general decay result which is not necessarily of exponential or polynomial type.


## 1. Introduction

Recently, the authors have studied the general decay of solutions of the following problem.[10]

$$
\begin{align*}
& u_{t t}-\Delta u+\int_{0}^{t} g(t-\tau) \Delta u d \tau+a(x) u_{t}=|u|^{p} u, \quad x \in \Omega, t \geq 0 \\
& u(x, t)=0, \quad x \in \partial \Omega, t \geq 0  \tag{1}\\
& u(x, 0)=u_{0}(x), u_{t}(x, 0)=u_{1}(x), \quad x \in \Omega
\end{align*}
$$

Many different forms of viscoelastic wave equations have been considered with various methods by many authors. For related works, we refer the readers [1], [3], [5], [6] and [12]

On the other hand, Cavalcanti et el.[2] have been considered the following problem ;

$$
\begin{align*}
& \left|u_{t}\right|^{\rho} u_{t t}-\Delta u-\Delta u_{t t}+\int_{0}^{t} g(t-\tau) \Delta u d \tau-\gamma \Delta u_{t}=0, \quad x \in \Omega, t \geq 0 \\
& u(x, t)=0, \quad x \in \partial \Omega, t \geq 0  \tag{2}\\
& u(x, 0)=u_{0}(x), u_{t}(x, 0)=u_{1}(x), \quad x \in \Omega
\end{align*}
$$

The equation (1) is considered as the absence of the dispersion term $\left|u_{t}\right|^{\rho}$ is missing in some manner. A global existence result for $\gamma \geq 0$ as well as an exponential decay for $\gamma>0$ has been established. These results have been extended by Messaoudi and Tatar [7] to a situation where a source term is competing with the dissipation terms induced by both the viscoelasticity and the

[^0]viscosity. In [7]. the authors combined well known methods with perturbation techniques to show that solution having positive and small initial energy exists globally and decays to the rest state exponentially.

With the above results, we consider the following problem in this paper ;

$$
\begin{align*}
& \left|u_{t}\right|^{\rho} u_{t t}-\Delta u-\Delta u_{t t}+\int_{0}^{t} g(t-\tau) \Delta u d \tau+a(x) u_{t}=b|u|^{p} u, x \in \Omega, t \geq 0 \\
& u(x, t)=0, \quad x \in \partial \Omega, t \geq 0  \tag{3}\\
& u(x, 0)=u_{0}(x), u_{t}(x, 0)=u_{1}(x), \quad x \in \Omega
\end{align*}
$$

where $\rho, b>0, a(x)=1$ and $p>0$ is a constant. And, $g$ is positive function satisfying some conditions to be satisfied later and $\Omega$ is bounded domain of $\mathbb{R}^{n}$ ( $n \geq 1$ ) with a smooth boundary $\partial \Omega$. This type of problems usually appear as a model in nonlinear viscoelasticity (refer [2], [8]).

Recently, Messaoudi and Tatar [8] studied the problem (3) with $a(x)=0$, in which the source term competes with only the viscoelastic dissipation induced by the memory term. They showed that there exists an appropriate set $S$ (called a stable set) such that if the initial datum is in $S$ then the solution continues to live there forever. They also showed that the solution goes to zero with an exponential or polynomial rate of the relaxation function $g$.

In the present paper we are concerned with problem (3) with $a(x)=1$, in which both the weakly nonlinear time-dependent dissipation and source terms are contained. Also, our intention is to show that, for a certain class of relaxation functions and certain initial data in the stable set, the solution energy decays at a similar rate of decay of the relaxation function $g$, which is not necessarily decaying in a polynomial or exponential fashion.

This paper is organized as follows. In section 2, we present some notations and material needed for our work and section 3 contains the statement and the proof of our main result.

## 2. Preliminaries

In this section, we present some materials needed in the proof of our main results. Also, for the sake of completeness we state, without a proof, the global existence result of Messaoudi and Tatar [8].

For the relaxation function $g$, we assume the followings ;
(H1) $g: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$is $C^{1}$-function satisfying $1-\int_{0}^{\infty} g(s) d s=l>0$
(H2) There exists a positive differentiable function $\xi(t)$ satisfying
i) $g^{\prime}(t) \leq-\xi(t) g(t)$ for $t \geq 0$,
ii) $\left|\xi^{\prime}(t) / \xi(t)\right| \leq k, \xi(t)>0$, and $\xi^{\prime}(t) \leq 0$ for $t>0$.
(H3) For the nonlinear term, we assume
i) $p>0$, for $n=1,2$ and $0<p \leq \frac{2(n-1)}{n-2}$, for $n \geq 3$.
ii) $\rho>0$, for $n=1,2$ and $0<\rho \leq \frac{2}{n-2}$, for $n \geq 3$.

Remark 1. Since $\xi$ is nonincreasing, $\xi(t) \leq \xi(0):=M$
We will use the embeddings $H_{0}^{1} \hookrightarrow L^{p}$ for $p \leq \frac{2 n}{n-2}(n \geq 3), p \geq 2(n=1,2)$ and $L^{q} \hookrightarrow L^{p}(p<q)$ with the same embedding constant $C$.

We introduce the modified energy functional
$E(t)=\frac{1}{\rho+2}\left\|u_{t}\right\|_{\rho+2}^{\rho+2}+\frac{1}{2}\left\|\nabla u_{t}\right\|_{2}^{2}+\frac{1}{2}\left(1-\int_{0}^{t} g(s) d s\right)\|\nabla u(t)\|_{2}^{2}+\frac{1}{2}(g * \nabla u)(t)-\frac{b}{p}\|u(t)\|_{p}^{p}$,
where $(g * u)(t)=\int_{0}^{t} g(t-\tau)\|u(t)-u(\tau)\|_{2}^{2} d \tau$.
And, we set $I(t)=\left\|\nabla u_{t}\right\|_{2}^{2}+\left(1-\int_{0}^{t} g(s) d s\right)\|\nabla u(t)\|_{2}^{2}+(g * \nabla u)(t)-b\|u(t)\|_{p}^{p}$.

We also assume the following ;
(H4) $E(0)<d_{1}=\frac{p-2}{2 p}\left(\frac{l}{b^{2 / p} c^{2}}\right)^{\frac{p}{p-2}}$ and $I(0)>0$.
Proposition 2.1. Suppose (H1)-(H4) hold. If $u_{0}, u_{1} \in H_{0}^{1}(\Omega)$, then the solution of (3) is global and bounded in time and satisfies $l\|\nabla u(t)\|_{2}^{2}+\left\|\nabla u_{t}(t)\right\|_{2}^{2} \leq$ $\frac{2 p}{p-2} E(0)$.
Proof. See [9] and [11].
Lemma 2.2. If $u$ is the solution of (3), then the energy functional $E$ satisfies

$$
E^{\prime}(t)=\frac{1}{2}\left(g^{\prime} * \nabla u\right)(t)-\frac{1}{2} g(t)\|\nabla u(t)\|_{2}^{2}-\left\|u_{t}\right\|_{2}^{2} \leq \frac{1}{2}\left(g^{\prime} * \nabla u\right)(t) \leq 0
$$

for almost every $t \in[0, T]$.
Proof. Multiplying (3) with $u_{t}$ and integrating over $\Omega$, we get

$$
\begin{align*}
& \frac{d}{d t}\left\{\frac{1}{\rho+2}\left\|u_{t}\right\|_{\rho+2}^{\rho+2}+\frac{1}{2}\|\nabla u\|_{2}^{2}+\frac{1}{2}\left\|\nabla u_{t}\right\|_{2}^{2}\right\} \\
& -\int_{0}^{t} g(t-\tau) \int_{\Omega} \nabla u(\tau) \nabla u_{t}(t) d x d \tau+\left\|u_{t}\right\|_{2}^{2}-\frac{b}{p} \frac{d}{d t}\|u\|_{p}^{p}=0 \tag{4}
\end{align*}
$$

For the second term on the left side of (4), we obtain

$$
\begin{align*}
& \int_{0}^{t} g(t-\tau) \int_{\Omega} \nabla u(\tau) \nabla u_{t}(t) d x d \tau \\
& =-\frac{1}{2} \frac{d}{d t}\left[\int_{0}^{t} g(t-\tau) \int_{\Omega}|\nabla u(\tau)-\nabla u(t)|^{2} d x d \tau\right] \\
& \quad+\frac{1}{2} \frac{d}{d t}\left[\int_{0}^{t} g(\tau) \int_{\Omega}|\nabla u(t)|^{2} d x d \tau\right]  \tag{5}\\
& \quad+\frac{1}{2} \int_{0}^{t} g^{\prime}(t-\tau) \int_{\Omega}|\nabla u(\tau)-\nabla u(t)|^{2} d x d \tau-\frac{1}{2} g(t) \int_{\Omega}|\nabla u(t)|^{2} d x d \tau
\end{align*}
$$

By inserting (5) into (4), we get

$$
E^{\prime}(t)=\frac{1}{2}\left(g^{\prime} * \nabla u\right)(t)-\frac{1}{2} g(t)\|\nabla u(t)\|_{2}^{2}-\left\|u_{t}\right\|_{2}^{2} \leq \frac{1}{2}\left(g^{\prime} * \nabla u\right)(t) \leq 0 .
$$

## 3. Decay of solutions

In this section, we state and prove main result. For this purpose, we set $L(t)=N E(t)+\varepsilon \psi(t)+\chi(t)$, where $\varepsilon$ and $N$ are positive constants and $\psi(t)=\frac{1}{\rho+1} \xi(t) \int_{\Omega}\left|u_{t}\right|^{\rho} u_{t} u d x+\xi(t) \int_{\Omega} \nabla u_{t} \nabla u d x$,
$\chi(t)=\xi(t) \int_{\Omega}\left(\Delta u_{t}-\frac{\left|u_{t}\right|^{\rho} u_{t}}{\rho+1}\right) \int_{0}^{t} g(t-\tau)(u(t)-u(\tau)) d \tau d x$.
Lemma 3.1. Let $u \in L^{\infty}\left(0, T ; H_{0}^{1}(\Omega)\right)$ be a solution of (3), we have
$\int_{\Omega}\left(\int_{0}^{t} g(t-\tau)(u(t)-u(\tau)) d \tau\right)^{\rho+2} d x \leq(1-l)^{\rho+2} C^{\rho+2}\left(\frac{4 p E(0)}{(p-2) l}\right)^{\rho / 2}(g * \nabla u)(t)$.
Proof. By the Holder's inequality and Poincaré's constant $C$, we get the result.

Lemma 3.2. Suppose that $u$ is a solution of (3), we have $\alpha_{1} L(t) \leq E(t) \leq$ $\alpha_{2} L(t)$ for two positive constants $\alpha_{1}$ and $\alpha_{2}$.

Proof. By Lemma 3.1, (H1) and Remark 1, we obtain the result. [4]
By the above Lemma, we note that $L(t)$ and $E(t)$ have very close relation. Since $L(t)=N E(t)+\varepsilon \psi(t)+\chi(t)$, we are interested in the play of two functions $\psi(t)$ and $\chi(t)$. The next two Lemmas are concerned about these functions.

Lemma 3.3. Suppose (H1)-(H4) and $u_{0}, u_{1} \in H_{0}^{1}(\Omega)$ hold. If $u$ is a solution of (3), then $\psi(t)$ satisfies

$$
\begin{aligned}
\psi^{\prime}(t) \leq & -\xi(t)\left[\frac{l}{2}-\alpha\left(k+\frac{k C}{\rho+1}+C\right)\right]\|\nabla u\|_{2}^{2} \\
& +\frac{(1-l) \xi(t)}{2 l}(g * \nabla u)(t)+\frac{\xi(t)}{\rho+1}\left\|u_{t}\right\|_{\rho+2}^{\rho+2} \\
& +\left\{1+\frac{C}{4 \alpha}+\frac{k}{4 \alpha}\left[1+\frac{C^{2(\rho+1)}}{\rho+1}\left(\frac{2 p E(0)}{(p-2) l}\right)^{\rho}\right]\right\} \xi(t)\left\|\nabla u_{t}\right\|_{2}^{2} \\
& +b \xi(t)\|u\|_{p}^{p} .
\end{aligned}
$$

Proof. We recall that $\psi(t)=\frac{1}{\rho+1} \xi(t) \int_{\Omega}\left|u_{t}\right|^{\rho} u_{t} u d x+\xi(t) \int_{\Omega} \nabla u_{t} \nabla u d x$.
By multiplying (3) with $u$ and integrating over $\Omega$, we obtain

$$
\begin{aligned}
& \int_{\Omega}\left|u_{t}\right|^{\rho} u_{t t} u d x+\int_{\Omega}|\nabla u|\left|\nabla u_{t t}\right| d x \\
& =-\|\nabla u\|_{2}^{2}+b\|u\|_{p}^{p}-\int_{\Omega} u_{t} u d x+\int_{\Omega} \nabla u(t) \int_{0}^{t} g(t-\tau) \nabla u(\tau) d \tau d x .
\end{aligned}
$$

By (H1)-(H3) and Young's inequality,

$$
\begin{aligned}
\psi^{\prime}(t)= & \frac{\xi(t)}{\rho+1}\left\|u_{t}\right\|_{\rho+2}^{\rho+2}+\xi(t)\left\|\nabla u_{t}\right\|_{2}^{2}+\frac{\xi^{\prime}(t)}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho} u_{t} u d x \\
& +\xi^{\prime}(t) \int_{\Omega} \nabla u_{t} \nabla u d x-\xi(t) \int_{\Omega} u u_{t} d x-\xi(t)\|\nabla u\|_{2}^{2} \\
& +\xi(t) \int_{\Omega} \nabla u_{t} \int_{0}^{t} g(t-\tau) \nabla u(\tau) d \tau d x+b\|u\|_{\rho}^{\rho} \xi(t) \\
\leq- & \xi(t)\left[\frac{l}{2}-\alpha\left(k+\frac{k C}{\rho+1}+C\right)\right]\|\nabla u\|_{2}^{2}+\frac{(1-l) \xi(t)}{2 l}(g * \nabla u)(t) \\
& +\frac{\xi(t)}{\rho+1}\left\|u_{t}\right\|_{\rho+2}^{\rho+2} \\
& +\left\{1+\frac{C}{4 \alpha}+\frac{k}{4 \alpha}\left[1+\frac{C^{2(\rho+1)}}{\rho+1}\left(\frac{2 p E(0)}{(p-2) l}\right)^{\rho}\right]\right\} \xi(t)\left\|\nabla u_{t}\right\|_{2}^{2} \\
& +b \xi(t)\|u\|_{p}^{p} .
\end{aligned}
$$

Lemma 3.4. Suppose (H1)-(H4) and $u_{0}, u_{1} \in H_{0}^{1}(\Omega)$ hold. If $u$ is the solution of (3), then, for $\delta>0, \chi(t)$ satisfies

$$
\begin{aligned}
\chi^{\prime}(t) \leq & \delta \xi(t)\left[1+2(1-l)^{2}+b C^{2 p-2}\left(\frac{2(p) E(0)}{(p-2) l}\right)^{p-2}\right] \int_{\Omega}|\nabla u|^{2} d x \\
& +(1-l)\left[2 \delta+\frac{3}{4 \delta}+\frac{b C^{2}}{4 \delta}+\frac{b C^{2}}{4 \delta(\rho+1)}\right] \xi(t)(g * \nabla u)(t) \\
& -\frac{g(0)}{4 \delta}\left(1+\frac{C^{2}}{\rho+1}\right) \xi(t)\left(g^{\prime} * \nabla u\right)(t) \\
& -\frac{\xi(t)}{\rho+1}\left(\int_{0}^{t} g(s) d s\right)\left\|u_{t}\right\|_{\rho+2}^{\rho+2} \\
& +\left\{\delta(k+1)\left[1+C^{2}+\frac{C^{2(\rho+1)}}{\rho+1}\left(\frac{2(p) E(0)}{(p-2) l}\right)^{\rho}\right]\right. \\
& \left.-\int_{0}^{t} g(s) d s\right\} \xi(t)\left\|\nabla u_{t}\right\|_{2}^{2} .
\end{aligned}
$$

Proof. Since $\chi(t)=\xi(t) \int_{\Omega}\left(\Delta u_{t}-\frac{\left|u_{t}\right|^{\rho} u_{t}}{\rho+1}\right) \int_{0}^{t} g(t-\tau)(u(t)-u(\tau)) d \tau d x$,

$$
\begin{align*}
\chi^{\prime}(t)= & \xi(t) \int_{\Omega} \nabla u(t)\left(\int_{0}^{t} g(t-\tau)(\nabla u(t)-\nabla u(\tau)) d \tau\right) d x \\
& -\xi(t) \int_{\Omega} \int_{0}^{t} g(t-\tau) \nabla u(\tau) d \tau \int_{0}^{t} g(t-\tau)(\nabla u(t)-\nabla u(\tau)) d \tau d x \\
& -\xi(t) \int_{0}^{t} g(s) d s\left\|\nabla u_{t}\right\|_{2}^{2}-\xi(t) \int_{\Omega} \nabla u_{t} \int_{0}^{t} g^{\prime}(t-\tau)(u(t)-u(\tau)) d \tau d x \\
& -\frac{\xi(t)}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho} u_{t} \int_{0}^{t} g^{\prime}(t-\tau)(u(t)-u(\tau) d \tau) d x \\
& -\frac{\xi(t)}{\rho+1}\left\|u_{t}\right\|_{\rho+2}^{\rho+2} \int_{0}^{t} g(s) d s \\
& -b \xi(t) \int_{\Omega}|u|^{p-2} u \int_{0}^{t} g(t-\tau)(u(t)-u(\tau)) d \tau d x \\
& -\xi^{\prime}(t) \int_{\Omega} \nabla u_{t} \int_{0}^{t} g(t-\tau)(\nabla u(t)-\nabla u(\tau)) d \tau d x \\
& -\frac{\xi^{\prime}(t)}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho} u_{t} \int_{0}^{t} g(t-\tau)(u(t)-u(\tau)) d \tau d x \\
& +\xi(t) \int_{\Omega} u_{t} \int_{0}^{t} g(t-\tau)(u(t)-u(\tau)) d \tau d x . \tag{6}
\end{align*}
$$

The first term of (6) gives
$\int_{\Omega} \nabla u(t) \int_{0}^{t} g(t-\tau)(\nabla u(t)-\nabla u(\tau)) d \tau d x \leq \delta \int_{\Omega}|\nabla u(t)|^{2} d x+\frac{1-l}{4 \delta}(g * \nabla u)(t)$.
The second term of (6) gives

$$
\begin{aligned}
& \int_{\Omega} \int_{0}^{t} g(t-\tau) \nabla u(\tau) d \tau \int_{0}^{t} g(t-\tau)(\nabla u(t)-\nabla u(\tau)) d \tau d x \\
& \quad \leq 2 \delta(1-l)^{2} \int_{\Omega}|\nabla u(t)|^{2} d x+\left(2 \delta+\frac{1}{4 \delta}\right)(1-l)(g * \nabla u)(t)
\end{aligned}
$$

The third term of (6) gives

$$
\int_{\Omega} \nabla u_{t}\left(\int_{0}^{t} g^{\prime}(t-\tau)(u(t)-u(\tau)) d \tau\right) d x \leq \delta \int_{\Omega}\left|u_{t}\right|^{2} d x-\frac{g(0)}{4 \delta}\left(g^{\prime} * \nabla u\right)(t)
$$

The fourth term of (6) gives

$$
\begin{aligned}
& \frac{1}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho} u_{t} \int_{0}^{t} g^{\prime}(t-\tau)(u(t)-u(\tau) d \tau) d x \\
& \quad \leq \frac{\delta C^{2 \rho+2}}{\rho+1}\left(\frac{2 p}{(p-2) l} E(0)\right)^{\rho} \int_{\Omega}\left|\nabla u_{t}\right|^{2} d x+\frac{g(0) C^{2}}{4 \delta(\rho+1)}\left(-g^{\prime} * \nabla u\right)(t)
\end{aligned}
$$

The sixth term of (6) gives

$$
\begin{aligned}
& -b \int_{\Omega}|u|^{p-2} u \int_{0}^{t} g(t-\tau)(u(t)-u(\tau)) d \tau d x \\
& \quad \leq b \delta C^{2 p-2}\left(\frac{2 p}{(p-2) l} E(0)\right)^{p-2} \int_{\Omega}|\nabla u(t)|^{2} d x \\
& \quad+\frac{b(1-l) C^{2}}{4 \delta}(g * \nabla u)(t)
\end{aligned}
$$

The seventh term of (6) gives

$$
\begin{aligned}
& \int_{\Omega} \nabla u_{t} \int_{0}^{t} g(t-\tau)(\nabla u(t)-\nabla u(\tau)) d \tau d x \\
& \quad \leq \delta \int_{\Omega}\left|\nabla u_{t}\right|^{2} d x+\frac{1-l}{4 \delta}(g * \nabla u)(t) .
\end{aligned}
$$

The eighth term of (6) gives

$$
\begin{aligned}
& \frac{1}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho} u_{t} \int_{0}^{t} g(t-\tau)(u(t)-u(\tau)) d \tau d x \\
& \quad \leq \frac{\delta C^{2 \rho+2}}{\rho+1}\left(\frac{2 p}{(p-2) l} E(0)\right)^{\rho} \int_{\Omega}\left|\nabla u_{t}\right|^{2} d x+\frac{(1-l) C^{2}}{4 \delta(\rho+1)}(g * \nabla u)(t)
\end{aligned}
$$

The ninth term of (6) gives

$$
\begin{array}{rl}
\int_{\Omega} u_{t} \int_{0}^{t} & g(t-\tau)(\nabla u(t)-\nabla u(\tau)) d \tau d x \\
& \leq \delta C^{2} \int_{\Omega}\left|\nabla u_{t}\right|^{2} d x+\frac{1-l}{4 \delta}(g * \nabla u)(t)
\end{array}
$$

By adding all the 10 terms, we have the result.

Theorem 3.5. Suppose (H1)-(H4) and $u_{0}, u_{1} \in H_{0}^{1}(\Omega)$ hold. If $u$ is the solution of (3), then for each $t_{0}>0$ there exist positive constants $K$ and $\lambda$ such that the solution of (3) satisfies $E(t) \leq K e^{-\lambda \int_{t_{0}}^{t} \xi(s) d s}, t \geq t_{0}$.

Proof. First of all, since $g$ is positive and continuous with $g(0)>0$,

$$
\begin{equation*}
\int_{0}^{t} g(s) d s \geq \int_{0}^{t_{0}} g(s) d s=g_{0}>0, t \geq t_{0} \tag{7}
\end{equation*}
$$

We recall that $L(t)=N E(t)+\varepsilon \psi(t)+\chi(t)$. By Proposition 2.2, Lemma 3.3 and Lemma 3.4,

$$
\begin{align*}
L^{\prime}(t) \leq & -\left[\left\{g_{0}-\varepsilon\left(1+\frac{C}{4 \alpha}+\frac{k}{4 \alpha}\left(1+\frac{C^{2(\rho+1)}}{\rho+1}\left(\frac{2 p E(0)}{(p-2) l}\right)^{\rho}\right)\right)\right\}\right. \\
& \left.\left.-(k+1) \delta\left\{1+C^{2}+\frac{C^{2(\rho+1)}}{\rho+1}\left(\frac{2 p E(0)}{(p-2) l}\right)^{\rho}\right)\right\}\right] \xi(t)\left\|\nabla u_{t}\right\|^{2} \\
& -\left\{\varepsilon\left[\frac{l}{2}-\alpha\left(k+\frac{k C^{2}}{\rho+1}+C^{2}\right)\right]-\delta\left[1+2(1-l)^{2}\right.\right. \\
& \left.\left.\left.+b C^{2 p-2}\left(\frac{2 p E(0)}{(p-2) l}\right)^{p-2}\right)\right]\right\} \xi(t)\|\nabla u\|_{2}^{2} \\
& +\left\{\frac{\varepsilon}{2 l}+\frac{3+8 \delta^{2}+b C^{2}}{4 \delta}+\frac{b C^{2}}{4 \delta(\rho+1)}\right\}(1-l) \xi(t)(g * \nabla u)(t) \\
& +\left\{\frac{N}{2}-\frac{g(0)}{4 \delta}\left(1+\frac{C^{2}}{\rho+1}\right) M\right\}\left(g^{\prime} * \nabla u\right)(t) \\
& -\left(g_{0}-\varepsilon\right) \frac{\xi(t)}{\rho+1}\left\|u_{t}\right\|_{\rho+2}^{\rho+2}+\varepsilon b \xi(t)\|u\|_{p}^{p} . \tag{8}
\end{align*}
$$

At this point, we choose $\alpha>0$ so small that $\frac{l}{2}-\alpha\left(k+\frac{k C^{2}}{\rho+1}+C^{2}\right)>\frac{l}{4}$. For a fixed $\alpha$, we choose $\varepsilon>0$ sufficiently small enough that
$\varepsilon<g_{0} /\left[2\left(1+\frac{C}{4 \alpha}+\frac{k}{4 \alpha}\left(1+\frac{C^{2(\rho+1)}}{\rho+1}\left(\frac{2 p E(0)}{(p-2) l}\right)^{\rho}\right)\right)\right]$. Once $\alpha$ and $\varepsilon$ are fixed, we choose a positive constant $\delta$ satisfying $\delta<\min \left\{\delta_{1}, \delta_{2}\right\}$ where

$$
\begin{aligned}
& \delta_{1}=\frac{g_{0}}{2(k+1)\left[1+C^{2}+\frac{C^{2(\rho+1)}}{\rho+1}\left(\frac{2 p E(0)}{(p-2)}\right)^{\rho}\right)}, \\
& \delta_{2}=\frac{\varepsilon l}{\left.4\left[1+2(1-l)^{2}+b C^{2 p-2}\left(\frac{2 p E(0)}{(p-2) l}\right)^{p-2}\right)\right]} .
\end{aligned}
$$

Now, we pick $N$ sufficiently large using the above $\delta$ and $\varepsilon$ that

$$
\left\{\frac{N}{2}-\frac{g(0)}{4 \delta}\left(1+\frac{C^{2}}{\rho+1}\right) M\right\}-\left\{\frac{3+8 \delta^{2}+b C^{2}}{4 \delta}+\frac{b C^{2}}{4 \delta(\rho+1)}\right\}(1-l)>0
$$

Then, by (8) and Lemma 3.2, we have

$$
L^{\prime}(t) \leq-\beta \xi(t) L(t), \quad \forall t \geq t_{0}, \beta>0
$$

Integrating the above inequality, by Lemma 3.2, we get the desired main result,

$$
E(t) \leq K e^{-\lambda \int_{t_{0}}^{t} \xi(s) d s}
$$

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