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ON CLASSICAL SOLUTIONS AND THE CLASSICAL LIMIT OF THE VLASOV-DARWIN SYSTEM

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Abstract. In this paper we study the initial value problem of the non-relativistic Vlasov-Darwin system with generalized variables (VDG). We first prove local existence and uniqueness of a nonnegative classical solution to VDG in three space variables, and establish the blow-up criterion. Then we show that it converges to the well-known Vlasov-Poisson system when the light velocity c tends to infinity in a pointwise sense.

1. Introduction

In this paper, we consider the collisionless single particle interacting by the electromagnetic field, more precisely, the non-relativistic Vlasov-Darwin system (VDS) in [17]. Let the function $f = f(t, x, v)$ denote phase space density of particles, in which $t \in \mathbb{R}, x \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ stand for time, position and velocity respectively, then the system reads as:

$$\partial_t f + v \cdot \nabla_x f + \frac{q}{m}(E_L + E_T + v \times B) \cdot \nabla_v f = 0, \tag{1.1}$$

$$\nabla \times B - \frac{1}{c^2}\partial_t E_L = \mu_0 q j_f, \;\; \nabla \cdot B = 0, \tag{1.2}$$

$$\nabla \times E_T + \partial_t B = 0, \;\; \nabla \cdot E_L = \frac{q\rho_f}{\epsilon_0}, \tag{1.3}$$

where q and m are electric charge and mass of particles in the plasma respectively, the constants μ_0 and ϵ_0 are the vacuum permittivity and magnetic permeability respectively, while $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$ is the light speed in vacuum. The macroscopic density and current density corresponding to the f are defined by $\rho_f(t, x) = \int_{R^3} f(t, x, v)dv$ and $j_f(t, x) = \int_{R^3} vf(t, x, v)dv$ respectively. The longitudinal component E_L and transversal component E_T come from the Helmholtz decomposition of the electric field in classical Maxwell's system and the field equation (1.2)-(1.3) neglect the transversal component of the displacement current in the Maxwell-Ampère equation. As an valid approximation of

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the classical Vlasov-Maxwell system (VMS) ([4, 6, 17]), we consider the well-posedness problem for non-relativistic VDS.

For the classical Vlasov-Maxwell system and its relativistic version, the global existence of classical solution remain unsolved up to now. A mass of literatures are contributed to the local existence, the global existence with certain initial condition or in low dimension, continuation criteria, see [1, 7, 9–16, 33] and the other references therein. Particularly, Glassey and Strauss in [15] show that any classical solution exists globally if the momentum support of the distribution function f remains bounded, where they explicitly establish the representation formula of the electromagnetic field and its derivatives by applying the theory of wave equation.

However, for the Darwin equation (1.2)-(1.3), it possesses the underlying elliptic structure. Generally speaking, it is convenient by introducing the scalar and vector potential. Recently the relativistic Vlasov-Darwin system (RVDS) when replacing v with $\hat{v}$ in (1.1) have been studied in [5, 24, 30–32]. In [32], without help of the energy conservation, Sospedra-Alfonso, Agueh and Illner show that classical solutions to the relativistic Vlasov-Darwin system with generalized variables (RVDG) exists globally for the small datum by introducing the scalar potential and the generalized variables to obtain RVDG from RVDS and establishing the equivalence between the two ones in the sense of classical solutions. In the following, we set all physical constant except the light speed c to 1. We will follow the method in [32] and define the electromagnetic field (E_L, E_T, B) by the scalar and vector potential:

$$E_L(t,x) = -\nabla_x \Phi(t,x), \quad E_T(t,x) = -c^{-1}\partial_t A(t,x), \quad B(t,x) = \nabla \times A(t,x),$$

where (E_L, E_T, B) formally solves (1.2)-(1.3) (see [32, Page 837]) when choosing the Coulomb or transverse gauge $\nabla \cdot A = 0$. Then for fixed $(\Phi, A) \in C^1(I, C^2(\mathbb{R}^3); \mathbb{R} \times \mathbb{R}^3)$, the characteristic system of the Vlasov equations (1.1) is as follows:

$$\dot{X}(s,t,x,v) = V(s,t,x,v),$$
$$\dot{V}(s,t,x,v) = [-\nabla\Phi - c^{-1}\partial_t A + c^{-1}V(s)(\nabla \times A)](s,t,X(s,t,x,v),V(s,t,x,v)).$$

We define $P(s) = V(s,t,x,v) + c^{-1}A(s, X(s,t,x,v))$ and then $v_A(s) = P(s) - c^{-1}A(s, X(s,t,x,v)$. Now the characteristic equation with the generalized variables (x,p) have

$$\dot{X}(s,t,x,p) = v_A(s,t,x,p), \tag{1.4}$$

$$\dot{P}(s,t,x,p) = [-\nabla\Phi + \sum_{i=1}^{3} c^{-1} v_A^i \nabla A^i](s, X(s))$$
$$:= [-\nabla\Phi + c^{-1} v_A^i \nabla A^i](s, X(s)). \tag{1.5}$$

Noting by the direct calculating that

$$\nabla_x \cdot v_A + \nabla_p \cdot (-\nabla\Phi + c^{-1} v_A^i \nabla A^i) = 0,$$

which shows that the field in (1.4)-(1.5) is an incompressible vector field. Hence we obtain the equivalent representation of non-relativistic VDS: the three dimensional Vlasov-Darwin system with generalized variables (VDG):

$$\partial_t f + v_A \cdot \nabla_x f - [\nabla\Phi - c^{-1} v_A^i \nabla A^i] \cdot \nabla_p f = 0, \tag{1.6}$$

$$f(0,x,p) = \overset{\circ}{f}(x,p), \tag{1.7}$$

$$\Phi(t,x) = \int_{\mathbb{R}^3} \frac{\rho_f(t,y)}{|y-x|} dy, \tag{1.8}$$

$$A(t,x) = \frac{1}{2c} \int_{\mathbb{R}^3} [id + \omega \otimes \omega] j_{A_f}(t,y) \frac{dy}{|y-x|}, \tag{1.9}$$

$$v_A = p - c^{-1} A(t,x). \tag{1.10}$$

Here $(x,p) \in \mathbb{R}^3 \times \mathbb{R}^3, t > 0$, id is the 3×3 identity matrix, ω is the unit vector $\frac{y-x}{|y-x|}$, the symbol $\otimes$ represents the tensor product, i.e., $\omega \otimes \omega$ is the 3×3 matrix with entries $\omega^i \omega^j$, where $\omega = (\omega^1, \omega^2, \omega^3)$. $\rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,p)dp$ and $j_{A_f}(t,x) = \int_{\mathbb{R}^3} v_A f(t,x,p)dp$ denote the macroscopic density and current density respectively. As usual, the repeated indexes means summation, that is, $v_A^i \nabla A^i = \sum_{i=1}^3 v_A^i \nabla A^i$. Hence, we can consider the Cauchy problem of non-relativistic VDG such that the well-posedness problem for non-relativistic VDS is also solved. We give the first result for the Cauchy problem of non-relativistic VDG where we need to utilize the deduced new energy conservation because we loss the virtue of the relativistic velocity being less than 1.

Before introducing the structure of our paper, we briefly recall the classical approximation of RVMS and VMS-the Vlasov-Poisson system (VPS). It has been received a great deal of investigation, including global existence and growth estimates of classical solutions ([2, 3, 18, 19, 22, 25, 27]; see also [26] and the references therein) and global existence and uniqueness of weak solutions ([20], [23]). This paper is organized as follows. In the following section, we give some invariants for non-relativistic VDG and present our main results. In Section 3, the local existence of classical solution are proved by using some a prior estimates and the blow-up criterion are presented. In Section 4, we prove that in the pointwise sense the solutions to non-relativistic VDG converges to the one of VPS with the same initial data as the light speed c tends to infinity.

2. Main results

Firstly we fix some notation. The Hölder space $C_c^{k,\alpha}(\mathbb{R}^n;\mathbb{R}^m)$ consists of compactly supported continuously differentiable vector valued functions whose k-th order partial derivatives are locally Hölder continuous in $\mathbb{R}^n$ with exponent $\alpha \in (0,1)$ (see, e.g., [8, Chapter 4]) and when $m = 1$, we write $C_c^{k,\alpha}(\mathbb{R}^n)$ instead of $C_c^{k,\alpha}(\mathbb{R}^n;\mathbb{R})$. For $p, q \in [1,\infty]$, $\|f\|_{L_x^q(L_y^p)}$ denotes the norm $(\int_{\mathbb{R}^3} | \int_{\mathbb{R}^3} |f(t,y)|^p dy|^{\frac{q}{p}} dx)^{\frac{1}{q}}$ where $q, p = \infty$ is standard if no confusion is possible. $C(a)$ denotes a positive constant only depending on the parameter

a and the constant M stands for generic constants whose values may change from line to line, but not depending on the light speed c.

In the following, we assume that $f(t,x,p)$ is a classical solution to non-relativistic VDG and then investigate its conservation properties. The mapping $\mathbb{R}^3 \times \mathbb{R}^3 \ni (x,p) \mapsto (X_f(s,t,x,p), P_f(s,t,\ x,p)) \in \mathbb{R}^3 \times \mathbb{R}^3$ defined by the characteristic equations (1.4), (1.5) is a measure preserving C^1-diffeomorphism and thus f is constant along the characteristics. In addition, for $1 \le p \le \infty$

$$\|f(t)\|_{L^p} = \| \overset{\circ}{f} \|_{L^p}.$$

In particular, we have conservation of mass ($p = 1$) and conservation of non-negativity. We also have the conservation law of charges

$$\partial_t \rho + \mathrm{div}_x j_{A_f} = 0.$$

For the derivation of these above properties, we refer the readers to [32, Lemma 2]. Now we present conservation of total energy for the non-relativistic VDG. For this purpose we define local energy and local momentum respectively by

$$\begin{aligned} e(t,x) = &\int_{\mathbb{R}^3} |p - c^{-1}A|^2 f(t,x,p)dp + \frac{1}{2\pi}(|\nabla_x \Phi|^2 + |\nabla_x A|^2),\\ m(t,x) = &-\int_{\mathbb{R}^3} K(t,x,p)f(t,x,p)dp + \int_{\mathbb{R}^3} p^2 p f(t,x,p)dp\\ &- c^{-1}\int_{\mathbb{R}^3} p^2 A(t,x)f(t,x,p)dp, \end{aligned}$$

where $K(t,x,p) = (p - c^{-1}A(t,x))[c^{-1}p \cdot A(t,x) + c^{-1}(p - c^{-1}A(t,x)) \cdot A(t,x)]$. By complicated calculation, we can obtain

$$\partial_t e(t,x) + \mathrm{div}_x m(t,x) = 0,$$

which by integrating over $[0,t] \times \mathbb{R}^3$ leads to the conservation of total energy $\varepsilon(t) = \varepsilon(0)$, where the total energy $\varepsilon(t)$ at time t is defined by $\varepsilon(t) = \int_{\mathbb{R}^3} e(t,x)dx$, namely

$$\begin{aligned} \varepsilon(t) = &\int_{\mathbb{R}^3 \times \mathbb{R}^3} |p - c^{-1}A(t,x)|^2 f(t,x,p)dpdx\\ &+ \frac{1}{2\pi}\int_{\mathbb{R}^3} |\nabla_x \Phi|^2(t,x) + |\nabla_x A|^2(t,x)dx. \end{aligned}$$

Now we state main results of this paper. The first one concerns existence and uniqueness of a local classical solution to non-relativistic VDG and its continuation criterion.

Theorem 2.1. *For any nonnegative* $\overset{\circ}{f} \in C_c^{1,\alpha}(\mathbb{R}^3 \times \mathbb{R}^3)$ *there exists* $T^* \ge 0$ *not depending on c such that the non-relativistic VDG with initial datum* $\overset{\circ}{f}$ *has a unique classical and nonnegative solution* $f^c \in C^1([0,T] \times \mathbb{R}^3 \times \mathbb{R}^3)$ *on any time interval* $[0,T] \subset [0,T^*[$ *for any* $c \ge \max\{1, \sqrt{M^*}\}$ *with* $M^* =$

$\frac{3}{4}(\frac{\pi}{2})^{\frac{1}{3}}P(T)\|\overset{\circ}{f}\|^{\frac{2}{3}}_{L^1_{x,p}}\|\overset{\circ}{f}\|^{\frac{1}{3}}_{L^\infty_{x,p}}$ *where the nondecreasing function* $P(t):[0,T^*[\to R$ *is independent of the light speed* c *and satisfies*:

$$f^c(t,x,p)=0\ \ for\ \ |p|\geq P(t),$$

and the induced potentials $(\Phi^c,A^c)\in C^1([0,T^*[,C^2(\mathbb{R}^3;\mathbb{R}\times\mathbb{R}^3))$. *Moreover, the mappings*

$$[0,T^*[\ni t\longmapsto f^c(t)\in C^{1,\alpha}_c(\mathbb{R}^3\times\mathbb{R}^3)$$

and

$$[0,T^*[\ni t\longmapsto(\nabla\Phi^c,v^i_{A^c}\nabla(A^c)^i)\in C^1_b(\mathbb{R}^3;\mathbb{R}^6),$$

are well defined and uniformly bounded with respect to the Sobolev norm $\|\cdot\|_{W^{1,\infty}_{x,p}}$ *on compact subintervals of* $[0,T^*[$. *In addition, if* $T>0$ *is the life span of* f^c (*namely if* $[0,T[$ *is the maximal existing time interval of the solution*), *then* T *is independent of* c *and*

$$\bar{P}:=\sup\{|p|:\ \exists\ 0\leq t<T,x\in\mathbb{R}^3\ such\ that\ f^c(t,x,p)\neq 0\}<\infty,$$

implies that for any $c\geq\max\{1,\sqrt{\frac{3}{4}(\frac{\pi}{2})^{\frac{1}{3}}\bar{P}\|\overset{\circ}{f}\|^{\frac{2}{3}}_{L^1_{x,p}}\|\overset{\circ}{f}\|^{\frac{1}{3}}_{L^\infty_{x,p}}}\}$ *the solution is global in time, that is,* $T=\infty$.

Based on the above statement, we are in the position to discuss the classical limit of the non-relativistic VDG and in the following we assume that $(f^c(t,x,p),\Phi^c(t,x),A^c(t,x))$ is the solution constructed in Theorem 2.1 in an interval $[0,T[$ with T not depending on c.

Theorem 2.2. *Let* $\overset{\circ}{f}\in C^{1,\alpha}_c(\mathbb{R}^3\times\mathbb{R}^3)$ *be nonnegative, and Let* (f^∞,E^∞) *be the unique global classical solution to the Cauchy problem of the Vlasov-Poisson system* (*VPS*)

$$\begin{cases}\partial_t f+p\cdot\nabla_x f+E\cdot\nabla_p f=0, & f(0,x,p)=\overset{\circ}{f}(x,p),\\ U(t,x)=-V(\cdot)*_x\rho_f(t,\cdot), & V(x)=-(4\pi|x|)^{-1},\\ E(t,x)=-\nabla_x U(t,x), & \\ \rho_f(t,x)=\int_{\mathbb{R}^3}f(t,x,p)dp, & j_f(t,x)=\int_{\mathbb{R}^3}pf(t,x,p)dp,\end{cases}$$

in the plasma physics case, then for any $[0,\bar{T}]\subseteq[0,T[$ *there exists a constant* $M>0$ *depending on the initial datum and* $\bar{T}$ *such that*

$$|f^c(t,x,p)-f^\infty(t,x,p)|+|\nabla\Phi^c(t,x)-E^\infty(t,x)|+|A^c(t,x)|+|\nabla_x A^c(t,x)|\leq Mc^{-1}$$

for all $x\in\mathbb{R}^3,p\in\mathbb{R}^3,t\in[0,\bar{T}]$ *and* $c\geq\max\{1,\sqrt{\frac{3}{4}(\frac{\pi}{2})^{\frac{1}{3}}P(\bar{T})\|\overset{\circ}{f}\|^{\frac{2}{3}}_{L^1_{x,p}}\|\overset{\circ}{f}\|^{\frac{1}{3}}_{L^\infty_{x,p}}}\}$.

Remark 2.1. The existence and uniqueness of the classical solution (f^∞,E^∞) to the Cauchy problem of the VPS was proved in [25, 28] (see also [26]).

3. Proof of Theorem 2.1

In this section, we follow the same argument presented in [32, Theorem 1] to prove local-in-time existence for non-relativistic VDG and establish continuation criterion for enough large $c \geq 1$.

For non-relativistic VDG, the different part of the proof mainly have two points. On one hand, the current density $j_{A_f}(t,x) = \int_{\mathbb{R}^3} |p - c^{-1}A(t,x)| f(t,x,p)dp$ no longer meet the inequality $|j_{A_f}| \leq |\rho_f|$ compared with the relativistic version (see [32] for details). Here we obtain the a prior bound for the vector potential $A(t,x)$ by means of the energy conservation, which is sufficient to establish boundedness of the velocity support and $j_A(t,x)$. On the other hand, the well-posedness for integral equation (1.9) rely on the light speed. We show that the integral equation (1.9) is well-posedness with the light speed c enough large. Hence when carrying out the proof for Theorem 2.1 , we mainly discuss the boundness of velocity support and convergence of the approximate solutions with the light speed c enough large. For the discussion of the uniqueness and regularity for the solution, it is similar to [32, Proof of Theorem 1: Steps 5-8]. The difference point is that [32, Proof of Theorem 1: Steps 5-8] gives the proof with the light speed $c = 1$. We apply it by scaling property, i.e., if f is a solution of non-relativistic VDG with $c \neq 1$, then

$$\begin{aligned} \bar{f}(t,x) &= f(c^{-\frac{3}{2}}t, c^{-\frac{1}{2}}x, cp), \\ \bar{\Phi}(t,x) &= c^{-2}\Phi(c^{-\frac{3}{2}}t, c^{-\frac{1}{2}}x), \\ \bar{A}(t,x) &= c^{-2}A(c^{-\frac{3}{2}}t, c^{-\frac{1}{2}}x), \end{aligned} \tag{3.1}$$

is a solution of non-relativistic VDG with the speed of light normalized to unity. So we omit its proof here.

Since the vector potential $A(t,x)$ is implicitly defined by integral equation (1.9), to proceed further we need to show existence and uniqueness of solutions to this integral equation in suitable function spaces.

Lemma 3.1. *Let $f(t) \in C^1([0,T], C_c^{1,\alpha}(\mathbb{R}^3 \times \mathbb{R}^3; \mathbb{R})), 0 \leq \alpha \leq 1, T \geq 0$ and we assume that there exists an non-decreasing continuous function $R(t) : [0,T] \to [0,\infty[$ such that for $t \in [0,T]$*

$$\sup\{|p| : (x,p) \in \text{supp} f(t)\} \leq R(t).$$

Then there exists constant $M := \frac{3}{4}(\frac{\pi}{2})^{\frac{1}{3}} R(T) \|f\|^{\frac{2}{3}}_{L_t^\infty(L^1_{x,p})} \|f\|^{\frac{1}{3}}_{L^\infty_{t,x,p}} > 0$ such that if $c > \sqrt{M}$ the integral equation (1.9) has a unique solution $A^c \in C_b^1([0,T] \times \mathbb{R}^3; \mathbb{R}^3)) \cap C^1([0,T], C^{3,\alpha}(\mathbb{R}^3; \mathbb{R}^3))$.

Proof. The operator $\mathscr{T}$ is defined by

$$\begin{aligned} \mathscr{T}(A)(t,x) = &\frac{1}{2c} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} [id + \omega \otimes \omega] p f(t,y,p) \frac{dpdy}{|y-x|} \\ &- \frac{1}{2c^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} [id + \omega \otimes \omega] A(t,y) f(t,y,p) \frac{dpdy}{|y-x|} \end{aligned}$$

$$:= \frac{1}{c}g(t,x) - \frac{1}{c^2}\mathscr{K}(A)(t,x).$$

Thus the linear integral equation (1.9) can rewritten by $A = \mathscr{T}(A)$ and $\mathscr{T}$ is an bounded linear operator from $C_b([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$ to $C_b([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$. Indeed, the linearity is obvious. For any $A(t,x)\in C_b([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$, we have by Proposition A.2 and the assumption that

$$\begin{aligned}
\|g\|_{L^\infty_{t,x}} &\le \| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}[id+\omega\otimes\omega]pf(t,y,p)\frac{dpdy}{|y-x|}\|_{L^\infty_{t,x}} \\
&\le \| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|pf(t,y,p)|\frac{dpdy}{|y-x|}\|_{L^\infty_{t,x}} \\
&\le \frac{3}{2}(\frac{\pi}{2})^{\frac{1}{3}}\|\int_{\mathbb{R}^3}|pf(t,x,p)|dp\|^{\frac{2}{3}}_{L^\infty_t(L^1_x)}\|\int_{\mathbb{R}^3}|pf(t,x,p)|dp\|^{\frac{1}{3}}_{L^\infty_t(L^\infty_x)} \\
&\le \frac{3}{2}(\frac{\pi}{2})^{\frac{1}{3}}R^2(T)\|f\|^{\frac{2}{3}}_{L^\infty_t(L^1_{x,p})}\|f\|^{\frac{1}{3}}_{L^\infty_{t,x,p}} < \infty, \qquad (3.2)
\end{aligned}$$

and

$$\begin{aligned}
\|\mathscr{K}\|_{L^\infty_{t,x}} &\le \| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}[id+\omega\otimes\omega]A(t,y)f(t,y,p)\frac{dpdy}{|y-x|}\|_{L^\infty_{t,x}} \\
&\le \| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|A(t,y)f(t,y,p)|\frac{dpdy}{|y-x|}\|_{L^\infty_{t,x}} \\
&\le \|A\|_{L^\infty_{t,x}}\| \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|f(t,y,p)|\frac{dpdy}{|y-x|}\|_{L^\infty_{t,x}} \\
&\le \frac{3}{2}(\frac{\pi}{2})^{\frac{1}{3}}R(T)\|f\|^{\frac{2}{3}}_{L^\infty_t(L^1_{x,p})}\|f\|^{\frac{1}{3}}_{L^\infty_{t,x,p}}\|A\|_{L^\infty_{t,x}} < \infty. \qquad (3.3)
\end{aligned}$$

Further, for any $y,z\in\mathbb{R}^3, t\in[0,T]$, by applying the estimate [31, estimate (30)] and Lemma A.1, we have

$$\begin{aligned}
&|\mathscr{T}(A)(t,y)-\mathscr{T}(A)(t,z)| \\
&\le 3|y-z|\|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|p-c^{-1}A(t,x)|f(t,x,p)\frac{dpdx}{|x-\cdot|^2}\|_{L^\infty_\cdot} \\
&\le \frac{9\cdot(2\pi)^{2/3}}{2c}|y-z|\|\int_{\mathbb{R}^3}|p-c^{-1}A(t,x)|f(t,x,p)dp\|^{\frac{1}{3}}_{L^1_x} \\
&\quad\cdot\|\int_{\mathbb{R}^3}|p-A(t,x)|f(t,x,p)dp\|^{\frac{2}{3}}_{L^\infty_x} \\
&\le \frac{9\cdot(2\pi)^{2/3}}{2c}(R(T)+c^{-1}\|A(t)\|_{L^\infty_x})^3\|f(t)\|^{\frac{1}{3}}_{L^1_{x,p}}\|f(t)\|^{\frac{2}{3}}_{L^\infty_{x,p}}|y-z| \\
&\le C(R(T),\|A\|_{L^\infty_{t,x}})|y-z|,
\end{aligned}$$

in addition, we have for $t,\tau\in[0,T]$, for all $\mathbb{R}>0, x\in\mathbb{R}^3$

$$\begin{aligned}
&|\mathscr{T}(A)(t,x)-\mathscr{T}(A)(\tau,x)| \\
&\le \frac{1}{2c}|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|A(t,y)-A(\tau,y)|f(\tau,y,p)\frac{dpdy}{|y-x|}|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2c}\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|f(t,y,p)-f(\tau,y,p)||p-c^{-1}A(t,y)|\frac{dpdy}{|y-x|}\Big|\\
&\le \frac{1}{2c}\|A(t)-A(\tau)\|_{L^\infty_x}\frac{3}{2}\Big(\frac{\pi}{2}\Big)^{\frac{1}{3}}\|f\|^{\frac{2}{3}}_{L^\infty_t(L^1_{x,p})}\|f\|^{\frac{1}{3}}_{L^\infty_t(L^\infty_{x,p})}\\
&+\frac{1}{2c}\Big|\int_{|y-x|\le R}\int_{\mathbb{R}^3}|f(t,y,p)-f(\tau,y,p)||p-c^{-1}A(t,y)|\frac{dpdy}{|y-x|}\Big|\\
&+\frac{1}{2c}\Big|\int_{|y-x|>R}\int_{\mathbb{R}^3}|f(t,y,p)-f(\tau,y,p)||p-c^{-1}A(t,y)|\frac{dpdy}{|y-x|}\Big|\\
&\le \frac{1}{2c}\|A(t)-A(\tau)\|_{L^\infty_x}\frac{3}{2}\Big(\frac{\pi}{2}\Big)^{\frac{1}{3}}\|f\|^{\frac{2}{3}}_{L^\infty_t(L^1_{x,p})}\|f\|^{\frac{1}{3}}_{L^\infty_t(L^\infty_{x,p})}\\
&+\frac{2\pi R^2}{c}(R(T)+c^{-1}\|A(t)\|_{L^\infty_x})R^3(T)\\
&\sup\{|f(t,y,p)-f(\tau,y,p)|,|y-x|\le R,p\le R(T)\}\\
&+\frac{1}{c}(R(T)+\|A(t)\|_{L^\infty_x})\|f\|_{L^\infty_t(L^1_{x,p})}R^{-1},
\end{aligned}
\tag{3.4}
$$

which implies by $A\in C_b([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$ and $f(t)\in C^1([0,T],C_c^{1,\alpha}(\mathbb{R}^3\times\mathbb{R}^3;\mathbb{R}))$ that $\mathscr{T}(A)(t,x)\to\mathscr{T}(A)(\tau,x)$ uniformly on every compact set in $\mathbb{R}^3$ as $t\to\tau$ in $[0,T]$. Hence $\mathscr{T}(A)(t,x)\in C_b([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$. Second, we prove that the operator $\mathscr{T}$ is the contraction mapping. For any $A_1,A_2\in C_b([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$, we have

$$
\begin{aligned}
\|\mathscr{T}(A_1)-\mathscr{T}(A_2)\|_{L^\infty_{t,x}} &\le c^{-2}\frac{3}{4}\Big(\frac{\pi}{2}\Big)^{\frac{1}{3}}R(T)\|f\|^{\frac{2}{3}}_{L^\infty_t(L^1_{x,p})}\|f\|^{\frac{1}{3}}_{L^\infty_t(L^\infty_{x,p})}\|A_1-A_2\|_{L^\infty_{t,x}}\\
&:= c^{-2}M\|A_1-A_2\|_{L^\infty_{t,x}}.
\end{aligned}
$$

Since $c^2>M$, there exists real number $\lambda=\frac{M}{2c^2}<1$ such that

$$\|\mathscr{T}(A_1)-\mathscr{T}(A_2)\|_{L^\infty_{t,x}}<\lambda\|A_1-A_2\|_{L^\infty_{t,x}},$$

which implies that the operator $\mathscr{T}$ is the contraction mapping. So there exists a unique fixed point $A^c\in C_b([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$ by the Banach fixed point theorem, which satisfies

$$A^c(t,x)=\frac{1}{2c}\int_{\mathbb{R}^3}[id+\omega\otimes\omega]j_{A^c}(t,y)\frac{dy}{|y-x|},\tag{3.5}$$

where the current density $j_{A^c}(x)=\int_{\mathbb{R}^3}|p-c^{-1}A^c(t,x)|f(t,x,p)dp$. If we define the starting iteration $A_0^c(t,x)$ by

$$A_0^c(t,x)=\frac{1}{2c}\int_{\mathbb{R}^3}[id+\omega\otimes\omega]j_{A^c}(0,y)dp\frac{dy}{|y-x|},$$

then by the above statement the iterative sequence $\{A_n^c(t,x)\}_{n=0,1,2,\dots}$ with $A_n^c=\mathscr{T}^n(A_0^c)$ converges to A^c in $C_b([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$. In the following we further discuss the regularity for A^c. Noticing that the kernel $K(x,y):=\frac{id+\omega\otimes\omega}{|y-x|}$ in (3.5) satisfies:

$$|K(x,y)|\le C|y-x|^{-1},\quad |\partial_x K(x,y)|\le C|y-x|^{-2},$$

we can apply the theory for regularity of solutions of Poisson's equation to A^c. Firstly, because of $j_{A^c}(0,x) \in C_c([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$, it is obvious by assumption and [8, Lemma 4.1] that $A^c_0(t,x) \in C^1_b([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$. Now if we assume that $A^c_{n-1}(t,x) \in C^1_b([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$, [21, Theorem 10.2(iii)] show that the derivative about each variable exists for $A^c_n(t,x)$ with

$$\partial_t A^c_n(t,x) = \frac{1}{2c}\int_{\mathbb{R}^3}[id+\omega\otimes\omega]\partial_t j_{A^c_{n-1}}(t,y)dp\frac{dy}{|y-x|},$$
$$\partial_{x_i} A^c_n(t,x) = \partial_{x_i}\frac{1}{2c}\int_{\mathbb{R}^3}[id+\omega\otimes\omega] j_{A^c_{n-1}}(t,y)dp\frac{dy}{|y-x|},\quad i=1,2,3.$$

We can follow estimate (3.2)-(3.4) to obtain $\partial_t A^c_n(t)\in C_b([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$, and $\partial_{x_i}A^c_n(t)\in C_b(\mathbb{R}^3;\mathbb{R}^3)$ for any $t\in[0,T]$. Further by Proposition A.2, we have for $t,\tau\in[0,T]$, for all $d,R>0, x\in\mathbb{R}^3$ with $0<d\le R$,

$$\begin{aligned}&|\partial_{x_i}A^c_{k,n}(t,x)-\partial_{x_i}A^c_{k,n}(\tau,x)|\\ \le\ & C[R^{-3}\|j_{n-1}\|_{L^\infty_t(L^1_x)}+d\|\partial_x j_{n-1}\|_{L^\infty_t(L^1_x)}\\ &+(1+\ln(R/d))\sup\{|j_{n-1}(t,y)-j_{n-1}(\tau,y)|,|y-x|\le R\}].\end{aligned}$$

Under the assumption of function $f(t,x,p)\in C^1([0,T],C^{1,\alpha}_c(\mathbb{R}^3\times\mathbb{R}^3;\mathbb{R}))$ and $A^c_{n-1}(t,x)\in C^1_b([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$, we have $j_{n-1}(t,x)\to j_{n-1}(\tau,x)$ for $t\to\tau$ in $[0,T]$ uniformly on every compact set $\mathbb{R}^3$, which implies that $\partial_{x_i}A^c_n(t,x)\in C_b([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$ with $i=1,2,3$. Hence $A^c_n(t,x)\in C^1_b([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$ and then by the completeness we have $A^c(t,x)\in C^1_b([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$. Further, the obtained regularity on A^c conclude that $j_{A^c}\in C^1([0,T];C^1_c(\mathbb{R}^3;\mathbb{R}^3))$ and then $j_{A^c}\in C^1([0,T];C^\alpha_c(\mathbb{R}^3;\mathbb{R}^3))$. Thus it follows by [21, Theorem 10.3] that $A^c\in C^1([0,T];C^{2,\alpha}(\mathbb{R}^3;\mathbb{R}^3))$. We compete the proof. □

Proof of Theorem 2.1. We construct an iterative sequence of solutions to VDG as follows: Let $\overset{\circ}{f}\in C^{1,\alpha}_c(\mathbb{R}^3\times\mathbb{R}^3)$ with $\overset{\circ}{f}(x,p)=0$ for $|x|>R_0$ or $|p|>U_0$ and for $n=0$, we set

$$f^0(t,x,p)=\overset{\circ}{f}(x,p),\quad t\ge0,\quad x\in\mathbb{R}^3,\quad p\in\mathbb{R}^3.$$

Assuming that $f^n(t,x,p):[0,T]\times\mathbb{R}^3\times\mathbb{R}^3\to[0,\infty[$ is defined with any given $T>0$ and satisfies the assumed condition in [32, Lemma 3(a)] and Lemma 3.1. we give the definition of $(\Phi^n(t,x),A_n(t,x))$ as follows:

$$\begin{aligned}\Phi^n(t,x)&=\int_{\mathbb{R}^3}\rho^n(t,x)\frac{dy}{|y-x|},\ \rho^n(t,x)=\int_{\mathbb{R}^3}f^n(t,x,p)dp,\\ A_n(t,x)&=\frac{1}{2c}\int_{\mathbb{R}^3}[id+\omega\otimes\omega]j_n(t,x)\frac{dy}{|y-x|},\\ j_n(t,x)&=\int_{\mathbb{R}^3}|p-c^{-1}A_n|f^n(t,x,p)dp.\end{aligned}$$

By [32, Lemma 3(a)], the iterate $\Phi^n(t,x)$ is well defined and satisfies $\Phi^n(t,x)\in C^1([0,T],C^{2,\alpha}(\mathbb{R}^3))$. Then by Lemma 3.1, there exists an enough large $M^*_n>0$

such that $A_n(t,x)$ satisfying $C^1([0,T],\ C^{2,\alpha}(\mathbb{R}^3;\mathbb{R}^3))$ is well defined for $c \geq \sqrt{M_n^*}$. Further denote by $Z_n(s,t,z) = (X_n, P_n)(s,t,z)$ the solution of the characteristic system for the Vlasov equation (1.6)

$$\dot{X}_n(s,t,z) = v_{A_n}(s, X_n(s,t,z), P_n(s,t,z)), \tag{3.6}$$

$$\dot{P}_n(s,t,z) = -[\nabla\Phi^n - c^{-1}v^i_{A_n}\nabla A^i_n](s,t,X_n(s,t,z),P_n(s,t,z)), \tag{3.7}$$

$$Z_n(t,t,z) = z := (x,p).$$

We define the $(n+1)$-th iterate of the phase space density by

$$f^{n+1}(t,z) = \overset{\circ}{f}(Z_n(0,t,z)). \tag{3.8}$$

By [32, Lemma 2, Remark 1], the approximation sequence $\{f^n(t,x,p)\}$ is well defined and verifies the following regularity:

$$0 \leq f^n(t,x,p) \in C^1([0,T], C^{1,\alpha}(\mathbb{R}^6)).$$

If we further define the sequence of velocity support functions: for each $t \in [0,T], n \in \mathbb{N}$,

$$P^n(t) = \sup\{|p| : \text{there exist } s \in [0,t] \text{ and } x \in \mathbb{R}^3 \text{ such that } f^n(s,x,p) \neq 0\} + 1.$$

It is easy to see that when $n = 0$, we have $P^0(t) = U_0$ and by (3.8) we also have

$$P^n(t) = \sup\{|P^{n-1}(s,0,x,p)| : s \in [0,t],\ (x,p) \in \operatorname{supp} \overset{\circ}{f}\} + 1 \tag{3.9}$$

and

$$f^n(t,x,p) = 0 \quad \text{for} \quad |p| \geq P^n(t) \quad \text{or} \quad |x| \geq R_0 + \int_0^t P^n(s)ds,$$

which implies that f^n satisfies the assumption in Lemma 3.1 for any $n \in \mathbb{N}$.

Step 1. Boundedness of velocity support. In order to uniformly bound the sequence of velocity support functions, we firstly consider any time interval $[0,T[$ with $T > 0$ given. By the characteristic equation (3.7), we have

$$\begin{aligned} |P^n(s,0,x,p)| \leq |p| + \int_0^s (\|\partial_x\Phi^n(\tau)\|_{L^\infty_x} + c^{-1}(|P^n(\tau,0,x,p)| \\ + c^{-1}\|A_n(\tau)\|_{L^\infty_x})\|\partial_x A_n(\tau)\|_{L^\infty_x})d\tau. \end{aligned} \tag{3.10}$$

By Proposition A.2 and conservation of mass, we have

$$\begin{aligned} \|\partial_x\Phi^n(t)\|_{L^\infty_x} &\leq M\|\rho^n(t)\|^{\frac{1}{3}}_{L^1_x}\|\rho^n(t)\|^{\frac{2}{3}}_{L^\infty_x} \\ &\leq M\|f(t)\|^{\frac{1}{3}}_{L^1_{x,p}}\|\int_{|p|\leq P^n(t)} f^n(t,x,p)dp\|^{\frac{2}{3}}_{L^\infty_x} \\ &\leq M\|\overset{\circ}{f}\|^{\frac{1}{3}}_{L^1_{x,p}}\|\overset{\circ}{f}\|^{\frac{2}{3}}_{L^\infty_{x,p}}(P^n(t))^2. \end{aligned}$$

Because of the current density $j_n(t,x) = \int_{\mathbb{R}^3} |p - c^{-1}A_n| f^n(t,x,p)dp$, we need to make the careful estimate for the vector potential. By Proposition A.2, Proposition A.3, the energy conservation and noticing $c \geq 1$, it follows that

$$\begin{aligned}\|A_n(t)\|_{L^\infty_x} &\leq Mc^{-1}\|j_n(t)\|^{\frac{1}{3}}_{L^\infty_x}\|j_n(t)\|^{\frac{2}{3}}_{L^1_x}\\ &\leq Mc^{-1}(\|\overset{\circ}{f}\|_{L^1_{x,p}} + \varepsilon(0))^{\frac{2}{3}}(P^n(t) + c^{-1}\|A_n(t)\|_{L^\infty_x})^{\frac{1}{3}}P^n(t)\\ &\leq Mc^{-1}(\|\overset{\circ}{f}\|_{L^1_{x,p}} + \varepsilon(0))^{\frac{2}{3}}((P^n(t))^{\frac{4}{3}} + c^{-\frac{1}{3}}\|A_n(t)\|^{\frac{1}{3}}_{L^\infty_x}P^n(t))\\ &\leq Mc^{-1}(\|\overset{\circ}{f}\|_{L^1_{x,p}} + \varepsilon(0))^{\frac{2}{3}}((P^n(t))^{\frac{4}{3}} + \|A_n(t)\|^{\frac{1}{3}}_{L^\infty_x}P^n(t))\\ &:= C(\overset{\circ}{f})((P^n(t))^{\frac{4}{3}} + \|A_n(t)\|^{\frac{1}{3}}_{L^\infty_x}P^n(t)),\end{aligned}$$

by Young's inequality with $\varepsilon = 2^{\frac{1}{3}}$, we obtain that

$$\begin{aligned}\|A_n(t)\|_{L^\infty_x} &\leq C(\overset{\circ}{f})(P^n(t))^{\frac{4}{3}} + \frac{1}{3}(\|A_n(t)\|^{\frac{1}{3}}_{L^\infty_x})^3\varepsilon^3 + \frac{2}{3}(C(\overset{\circ}{f})P^n(t)))^{\frac{3}{2}}\varepsilon^{-\frac{3}{2}}\\ &\leq C(\overset{\circ}{f})(P^n(t))^{\frac{4}{3}} + \frac{2}{3}\|A_n(t)\|_{L^\infty_x} + \frac{\sqrt{2}}{3}(C(\overset{\circ}{f})P^n(t)))^{\frac{3}{2}},\end{aligned}$$

which implies that

$$\begin{aligned}\|A_n(t)\|_{L^\infty_x} &\leq 3C(\overset{\circ}{f})(P^n(t))^{\frac{4}{3}} + \sqrt{2}(C(\overset{\circ}{f})P^n(t)))^{\frac{3}{2}}\\ &\leq C(\overset{\circ}{f})(P^n(t))^{\frac{3}{2}}.\end{aligned} \tag{3.11}$$

Further, we estimate the first-order derivative for the potentials. Proposition A.2 shows that

$$\|\partial_x A_n(t)\|_{L^\infty_x} \leq Mc^{-1}\|j_n(t)\|^{\frac{2}{3}}_{L^\infty_x}\|j_n(t)\|^{\frac{1}{3}}_{L^1_x}.$$

In the same way, using (3.11), we have

$$\begin{aligned}\|\partial_x A_n(t)\|_{L^\infty_x} &\leq M(\|\overset{\circ}{f}\|_{L^1_{x,p}} + \varepsilon(0))^{\frac{1}{3}}[P^n(t) + \|A_n(t)\|_{L^\infty_x}]^{\frac{2}{3}}(P^n(t))^2\\ &\leq M(\|\overset{\circ}{f}\|_{L^1_{x,p}} + \varepsilon(0))^{\frac{1}{3}}[P^n(t) + C(\overset{\circ}{f})(P^n(t))^{\frac{3}{2}}]^{\frac{2}{3}}(P^n(t))^2\\ &\leq C(\overset{\circ}{f})(P^n(t))^3.\end{aligned} \tag{3.12}$$

Now we return back to (3.10) and then by using (3.11)-(3.12), we obtain that

$$\begin{aligned}|P^n(s,0,x,p)| \leq U_0 + C(\overset{\circ}{f})\int_0^s &((P^n(\tau))^2 + (|P^n(\tau,0,x,p)|\\ &+ (P^n(\tau))^{\frac{3}{2}})(P^n(\tau))^3 d\tau.\end{aligned}$$

Combining the definition (3.9) we have the inequality

$$P^{n+1}(t) \leq C_0 + C_0\int_0^t (P^n(\tau))^{\frac{9}{2}}P^{n+1}(\tau)d\tau,$$

where $C_0 := U_0 + C(\overset{\circ}{f})$. By induction we can show that there exists a non-negative, non-decreasing function $P(t) \in C([0, T^*[; \mathbb{R})$ with $T^* = \frac{2}{9} C_0^{-\frac{11}{2}}$ not depending on the light speed c such that for all $n \in \mathbb{N}$

$$P^n(t) \leq P(t), \quad t \in [0, T^*[, \tag{3.13}$$

and the function $P(t)$ is the maximal solution of

$$\dot{P}(t) = C_0 P^{\frac{11}{2}}(t), \quad P(0) = C_0,$$

which exists on the interval $[0, \frac{2}{9} C_0^{-\frac{11}{2}}[$. Then in any time interval $[0, \bar{T}] \subset [0, T^*[$, we take $M^* = \frac{3}{4}(\frac{\pi}{2})^{\frac{1}{3}} P(\bar{T}) \| \overset{\circ}{f} \|_{L^1_{x,p}}^{\frac{2}{3}} \| \overset{\circ}{f} \|_{L^\infty_{x,p}}^{\frac{1}{3}}$ such that for all $n \in \mathbb{N}$, $t \in [0, \bar{T}]$,

$$M_n^* < M^*.$$

Since $c \geq \max\{1, \sqrt{M^*}\}$, for any $n \in \mathbb{N}$, $A_n(t, x)$ satisfying $C^1([0, \bar{T}], C^{2,\alpha}(\mathbb{R}^3; \mathbb{R}^3))$ is well defined for $c \geq \max\{1, \sqrt{M^*}\}$. In addition we have

$$\begin{aligned} &\|\rho^n(t)\|_{L^\infty_x} + \|j_n(t)\|_{L^\infty_x} + \|\partial_x \Phi^n(t)\|_{L^\infty_x} + \|\partial_x A_n(t)\|_{L^\infty_x} + \|A_n(t)\|_{L^\infty_x} \\ &\leq C(\bar{T}, \overset{\circ}{f}). \end{aligned} \tag{3.14}$$

In addition, we also can obtain that for any $t \in [0, \bar{T}]$ and all $n \in \mathbb{N}$,

$$\|\partial_x \rho^n(t)\|_{L^\infty_x} + \|\partial_x j_n(t)\|_{L^\infty_x} + \|\partial_x^2 \Phi^n(t)\|_{L^\infty_x} + \|\partial_x^2 A_n(t)\|_{L^\infty_x} \leq C(\bar{T}, \overset{\circ}{f}), \tag{3.15}$$

which have the similar proof with the estimate [32, (82)] when substituting $v_{A_n} = P_n(s) - A_n(s, X_n(s))$ for $v_{A_n} = \frac{P_n(s) - A_n(s, X_n(s))}{\sqrt{1 + |P_n(s) - A_n(s, X_n(s))|^2}}$ and we shall neglect the proof.

Step 2. Convergence of the approximation solutions. Although we can apply the proof of [32, Theorem 1, step 4] to our case, the main difference still come from the current density. So we only point out the error. When we prove that f^n is Cauchy sequence on $[0, \bar{T}] \times \mathbb{R}^3 \times \mathbb{R}^3$, we shall meet that

$$\begin{aligned} &|f^{n+1}(t, x, p) - f^n(t, x, p)| \\ &\leq \|\partial_{(x,p)} \overset{\circ}{f} \|_{L^\infty_{x,p}} (|X_n(0, t, x, p) - X_{n-1}(0, t, x, p)| \\ &\quad + |P_n(0, t, x, p) - P_{n-1}(0, t, x, p)|) \\ &\leq M \int_0^t (\|\partial_x \Phi^n(\tau) - \partial_x \Phi^{n-1}(\tau)\|_{L^\infty_x} + \|\partial_x A_n(\tau) - \partial_x A_{n-1}(\tau)\|_{L^\infty_x} \\ &\quad + \|A_n(\tau) - A_{n-1}(\tau)\|_{L^\infty_x}) d\tau, \end{aligned} \tag{3.16}$$

where we use $c \geq 1$. Similarly, we need to establish a Gronwall's inequality for $|f^{n+1}(t, x, p) - f^n(t, x, p)|$. Because the scalar potential is independent of j_n, its proof is the same with [32, Theorem 1, step 4]. In the following we shall estimate the terms for the vector potential. According to (3.13), let $R = \max\{R_0 + C(\overset{\circ}{f})\bar{T}P^{\frac{3}{2}}(\bar{T}), P(\bar{T})\}$ and we have

$$\mathrm{supp} f^n(t) \subset B_R \times B_R$$

for all $n \in \mathbb{N}, t \in [0, \bar{T}]$. By definition, it follows that

$$\begin{aligned}
(3.17)\qquad & A_n(t,x) - A_{n-1}(t,x) \\
&= \frac{1}{2c}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}[id + \omega\otimes\omega](v_{A_n}f^n(t,y,p) - v_{A_{n-1}}f^{n-1}(t,y,p))\frac{dpdy}{|y-x|} \\
&= \frac{1}{2c^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}[id + \omega\otimes\omega](A_{n-1}(t,y) - A_n(t,y))f^n(t,y,p)\frac{dpdy}{|y-x|} \\
&\quad + \frac{1}{2c}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}[id + \omega\otimes\omega](p - c^{-1}A_{n-1}(t,y)) \\
&\quad (f^n(t,y,p) - f^{n-1}(t,y,p))\frac{dpdy}{|y-x|} \\
&:= J_1(t,x) + J_2(t,x).
\end{aligned}$$

For J_1, by $\mathrm{supp} f^n(t) \subset B_R \times B_R$, Lemma A.1 and Hölder inequality we have that

$$\begin{aligned}
J_1(t,x) &\le \frac{M}{2c^2}\Big\|\int_{B_R}|A_{n-1}(t,x) - A_n(t,x)|f^n(t,x,p)dp\Big\|^{\frac{1}{3}}_{L^1_x(B_R)} \\
&\quad \Big\|\int_{B_R}|A_{n-1}(t,x) - A_n(t,x)|f^n(t,x,p)dp\Big\|^{\frac{2}{3}}_{L^2_x(B_R)} \\
&\le \frac{M}{2c^2}\Big\|\int_{B_R}f^n(t,x,p)dp\Big\|^{\frac{1}{3}}_{L^2_x(B_R)}\Big\|\int_{B_R}f^n(t,x,p)dp\Big\|^{\frac{2}{3}}_{L^\infty_x(B_R)} \\
&\quad \|A_{n-1}(t) - A_n(t)\|_{L^2_x(B_R)} \\
(3.18)\qquad &\le C(R,\overset{\circ}{f})\|A_{n-1}(t) - A_n(t)\|_{L^2_x(B_R)}.
\end{aligned}$$

For J_2, by $\mathrm{supp} f^n(t) \subset B_R \times B_R$, (3.14) and Lemma A.1 we have

$$\begin{aligned}
J_2(t,x) &\le C(R,\overset{\circ}{f})\int_{B_R}\int_{B_R}|f^n(t,y,p) - f^{n-1}(t,y,p)|\frac{dpdy}{|y-x|} \\
&\le C(R,\overset{\circ}{f})\Big\|\int_{B_R}|f^n(t,x,p) - f^{n-1}(t,x,p)|dp\Big\|^{\frac{2}{3}}_{L^1_x(B_R)} \\
&\quad \cdot \Big\|\int_{B_R}|f^n(t,x,p) - f^{n-1}(t,x,p)|dp\Big\|^{\frac{1}{3}}_{L^\infty_x(B_R)} \\
(3.19)\qquad &\le C(R,\overset{\circ}{f})\|f(t) - g(t)\|_{L^\infty_{x,p}}.
\end{aligned}$$

Combining(3.17) with (3.18), (3.19), we have

$$\begin{aligned}
(3.20)\qquad & \|A_n(t) - A_{n-1}(t)\|_{L^\infty_x} \\
&\le C(R,\overset{\circ}{f})\|A_{n-1}(t) - A_n(t)\|_{L^2_x(B_R)} + C(R,\overset{\circ}{f})\|f^n(t) - f^{n-1}(t)\|_{L^\infty_{x,p}}.
\end{aligned}$$

In the following, we shall estimate the derivative of the vector potential. By

$$
\begin{aligned}
(3.21)\qquad &\partial_x A_n(t,x) - \partial_x A_{n-1}(t,x)\\
&\leq \frac{1}{2c}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v_{A_n} f^n(t,y,p) - v_{A_{n-1}} f^{n-1}(t,y,p)|\frac{dy}{|y-x|^2}\\
&\leq \frac{1}{2c}\int_{B_R}\int_{B_R} |p - c^{-1}A_n(t,y)||f^n(t,y,p) - f^{(n-1)}(t,y,p)|\frac{dy}{|y-x|^2}\\
&\quad + \frac{1}{2c^2}\int_{B_R}\int_{B_R} |A_n(t,y) - A_{n-1}(t,y)|f^{n-1}(t,y,p)\frac{dy}{|y-x|^2}\\
&\leq C(R,\mathring{f})\|f^n(t) - f^{n-1}(t)\|_{L^\infty_{x,p}}\\
&\quad + C(R,\mathring{f})\|\int_{B_R} |A_{n-1}(t,x) - A_n(t,x)|f^n(t,x,p)dp\|^{\frac{1}{3}}_{L^1_x(B_R)}\\
&\quad \cdot \|\int_{B_R} |A_{n-1}(t,x) - A_n(t,x)|f^n(t,x,p)dp\|^{\frac{2}{3}}_{L^\infty_x(B_R)}\\
&\leq C(R,\mathring{f})\|f^n(t) - f^{n-1}(t)\|_{L^\infty_{x,p}} + C(R,\mathring{f})\|A_n(t) - A_{n-1}(t)\|_{L^\infty_x}.
\end{aligned}
$$

Then we shall apply the crucial tools in [32, Lemma 8] to (3.20) such that we have

$$\|A_n(t) - A_{n-1}(t)\|_{L^\infty_x} \leq C(R,\mathring{f})\|f^n(t) - f^{n-1}(t)\|_{L^\infty_{x,p}}.$$

Indeed, if we define $h_\lambda = \lambda f^n + (1-\lambda)f^{n-1}$ for $0 \leq \lambda \leq 1$. It is easy to see that $h_\lambda \geq 0$ has compact support and $\partial_\lambda h_\lambda = f^n - f^{n-1}$. By [32, Lemma 3] we define the Darwin vector potential A_λ induced by h_λ by

$$\Delta A_\lambda(t,x) = -\frac{1}{c}\int_{B_R} v_{A_\lambda} h_\lambda(t,x,p)dp - \frac{1}{c}\nabla\int_{B_R}\int_{B_R}\nabla\cdot(v_{A_\lambda}h_\lambda)(t,x,p)\frac{dpdy}{|y-x|},$$

and $\nabla\cdot A_\lambda = 0$, where we have dropped a numerical factor. Notice that A_n (or A_{n-1}) solves the above equation when $\lambda = 1$ (or $\lambda = 0$). Then by the Jensen's inequality, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} |A_n(t,x) - A_{n-1}(t,x)|^2 dx &= \int_{\mathbb{R}^3} |\int_0^1 \partial_\lambda A_\lambda(t,x)d\lambda|^2 dx\\
&\leq \int_0^1\int_{\mathbb{R}^3} |\partial_\lambda A_\lambda(t,x)|^2 dx d\lambda\\
&\leq \sup_{0\leq\lambda\leq 1}\int_{\mathbb{R}^3} |\partial_\lambda A_\lambda(t,x)|^2 dx.
\end{aligned}
$$

So we need to estimate $\|\partial_\lambda A_\lambda(t)\|_{L^2_x(\mathbb{R}^3)}$ for all $0 \leq \lambda \leq 1$ and $t \in [0,\bar{T}]$. Here we give some notations. We denote by A^i_λ the component of the vector potential A_λ. From the equation satisfied by A_λ, we can obtain that

$$\Delta A^i_\lambda(t,x) = -\frac{1}{c}\int_{B_R} v^i_{A_\lambda} h_\lambda(t,x,p)dp - \frac{1}{c}\partial_{x_i}\int_{B_R}\int_{B_R}\nabla\cdot(v_{A_\lambda}h_\lambda)(t,x,p)\frac{dpdy}{|y-x|}.$$

Differentiating the equation with respect to the variable λ, we obtain that

$$\begin{aligned}\Delta\partial_\lambda A^i_\lambda(t,x) = & -\frac{1}{c}\int_{B_R}\partial_\lambda(v^i_{A_\lambda}h_\lambda)(t,x,p)dp \\ & -\frac{1}{c}\partial_{x_i}\int_{B_R}\int_{B_R}\nabla\cdot\partial_\lambda(v_{A_\lambda}h_\lambda)(t,y,p)\frac{dpdy}{|y-x|}.\end{aligned}$$

Then multiplying $\partial_\lambda A^i$ on both sides, it follows by integration by parts that

$$\begin{aligned}&\int_{R^3}|\partial_x\partial_\lambda A^i_\lambda|^2(t,x)dx \\ &= \int_{B_R}\int_{B_R}\partial_\lambda A^i_\lambda(t,x)\partial_\lambda(v^i_{A_\lambda}h_\lambda)(t,x,p)dpdx \\ &\quad -\int_{R^3}\int_{B_R}\partial_\lambda\partial_{x_i}A^i_\lambda(t,x)\nabla\cdot\partial_\lambda\int_{B_R}v_{A_\lambda}h_\lambda(t,y,p)dp\frac{dydx}{|y-x|},\end{aligned}\tag{3.22}$$

where $\partial_\lambda A^i_\lambda\partial_\lambda(v^i_{A_\lambda}h_\lambda)$ only represents the product of the two terms. Now the sum with respect to $i=1,2,3$ in (3.22) gives

$$\begin{aligned}&\int_{\mathbb{R}^3}|\partial_x\partial_\lambda A_\lambda|^2(t,x)dx \\ &= \frac{1}{c}\int_{B_R}\int_{B_R}\partial_\lambda A_\lambda(t,x)\partial_\lambda(v_{A_\lambda}h_\lambda)(t,x,p)dpdx \\ &\quad -\frac{1}{c}\int_{\mathbb{R}^3}\int_{B_R}\partial_\lambda\partial_x A_\lambda(t,x)\nabla\cdot\partial_\lambda\int_{B_R}v_{A_\lambda}h_\lambda(t,y,p)dp\frac{dydx}{|y-x|} \\ &= \frac{1}{c}\int_{B_R}\int_{B_R}h_\lambda(\partial_\lambda A_\lambda(t,x)\cdot\partial_\lambda v_{A_\lambda})dpdx \\ &\quad +\frac{1}{c}\int_{B_R}\int_{B_R}(\partial_\lambda A_\lambda(t,x)\cdot v_{A_\lambda})\partial_\lambda h_\lambda dpdx \\ &\quad -\frac{1}{c}\int_{\mathbb{R}^3}\int_{B_R}\partial_\lambda\nabla\cdot A_\lambda(t,x)\nabla\cdot\partial_\lambda\int_{B_R}v_{A_\lambda}h_\lambda(t,y,p)dp\frac{dydx}{|y-x|}.\end{aligned}\tag{3.23}$$

Using $\nabla\cdot A_\lambda=0$ and observing $\partial_\lambda v_{A_\lambda}=-\partial_\lambda A$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3}|\partial_x\partial_\lambda A_\lambda|^2(t,x)dx+\frac{1}{c}\int_{B_R}\int_{B_R}h_\lambda(t,x,p)|\partial_\lambda A_\lambda|^2(t,x)dpdx \\ &= \frac{1}{c}\int_{B_R}\int_{B_R}(\partial_\lambda A_\lambda(t,x)\cdot v_{A_\lambda})\partial_\lambda h_\lambda dpdx.\end{aligned}\tag{3.24}$$

Comparing with RVDG, (3.24) have the easier form. In addition, it is easy to see that

$$\begin{aligned}&\int_{\mathbb{R}^3}|\partial_x\partial_\lambda A_\lambda|^2(t,x)dx+\frac{1}{c}\int_{B_R}\int_{B_R}h_\lambda|\partial_\lambda A_\lambda|^2dpdx \\ &= \|\partial_x\partial_\lambda A_\lambda(t)\|^2_{L^2_x}+\frac{1}{c}\|(\int_{B_R}h_\lambda(t,x,p)dp)^{\frac{1}{2}}\partial_\lambda A_\lambda\|^2_{L^2_x(B_R)}.\end{aligned}\tag{3.25}$$

By the Hölder inequality and $c \geq 1$, we can deduce that

$$\frac{1}{c}\int_{B_R}\int_{B_R}(\partial_\lambda A_\lambda(t,x)\cdot v_{A_\lambda})\partial_\lambda h_\lambda dpdx$$
$$\leq C(R,\overset{\circ}{f})\|\partial_\lambda A_\lambda(t)\|_{L^2_x(B_R)}\|\partial_\lambda h_\lambda(t)\|_{L^2_{x,p}}. \tag{3.26}$$

The estimates (3.24), (3.25) and (3.26) imply that

$$\|\partial_x\partial_\lambda A_\lambda\|^2_{L^2_x} \leq C(R,\overset{\circ}{f})\|\partial_\lambda A_\lambda(t)\|_{L^2_x(B_R)}\|f^n(t)-f^{n-1}(t)\|_{L^2_{x,p}},$$

which implies by Poincaré inequality (see [8]) that

$$\begin{aligned}\|\partial_\lambda A_\lambda(t)\|^2_{L^2_x(B_R)} &\leq C(R,\overset{\circ}{f})\|\partial_x\partial_\lambda A_\lambda(t)\|^2_{L^2_x}\\ &\leq C(R,\overset{\circ}{f})\|\partial_\lambda A_\lambda(t)\|_{L^2_x(B_R)}\|f^n(t)-f^{n-1}(t)\|_{L^2_{x,p}}.\end{aligned}$$

So we have

$$\begin{aligned}\|\partial_\lambda A_\lambda(t)\|_{L^2_x(B_R)} &\leq C(R,\overset{\circ}{f})\|f^n(t)-f^{n-1}(t)\|_{L^2_{x,p}}\\ &\leq C(R,\overset{\circ}{f})\|f^n(t)-f^{n-1}(t)\|_{L^\infty_{x,p}},\end{aligned} \tag{3.27}$$

which implies by (3.14) that

$$\|A_n(t)-A_{n-1}(t)\|_{L^\infty_x} \leq C(R,\overset{\circ}{f})\|f^n(t)-f^{n-1}(t)\|_{L^\infty_{x,p}}. \tag{3.28}$$

Up to now, we have closed the Gronwall inequality by (3.16), (3.20), (3.21) and (3.28), which leads that the sequence $\{f^n\}$ are Cauchy sequences in the C^0 norm and it converge uniformly $f^c \in C^0([0,\bar{T}]\times\mathbb{R}^3\times\mathbb{R}^3)$. We complete the proof of Theorem 2.1. □

4. Proof of Theorem 2.2

In this section, we follow the argument in [29] and discuss the classical limit of the non-relativistic VDG.

Proof of Theorem 2.2. By the vlassov equation for non-relativistic VDG and VPS, we have

$$\begin{aligned}&\partial_t(f^c-f^\infty)+v_{A^c}\cdot\nabla_x(f^c-f^\infty)-[\nabla\Phi^c-c^{-1}v^i_{A^c}\nabla(A^c)^i]\cdot\nabla_p(f^c-f^\infty)\\ &=\partial_t f^c-\partial_t f^\infty+v_{A^c}\cdot\nabla_x f^c-v_{A^c}\cdot\nabla_x f^\infty-[\nabla\Phi^c-c^{-1}v^i_{A^c}\nabla(A^c)^i]\cdot\nabla_p f^c\\ &\quad+[\nabla\Phi^c-c^{-1}v^i_{A^c}\nabla(A^c)^i]\cdot\nabla_p f^\infty\\ &=-\partial_t f^\infty-v_{A^c}\cdot\nabla_x f^\infty+[\nabla\Phi^c-c^{-1}v^i_{A^c}\nabla(A^c)^i]\cdot\nabla_p f^\infty\\ &=p\cdot\nabla_x f^\infty+E^\infty\cdot\nabla_p f^\infty-v_{A^c}\cdot\nabla_x f^\infty+[\nabla\Phi^c-c^{-1}v^i_{A^c}\nabla(A^c)^i]\cdot\nabla_p f^\infty\\ &=c^{-1}A^c\cdot\nabla_x f^\infty+[E^\infty+\nabla\Phi^c-c^{-1}v^i_{A^c}\nabla(A^c)^i]\cdot\nabla_p f^\infty.\end{aligned}$$

For all $t\in[0,\bar{T}]\subseteq[0,T[$, $\nabla_x f^\infty, \nabla_p f^\infty$ are bounded on $[0,\bar{T}]$ and we define

$$d(t)=\sup\{|f^c(s,x,p)-f^\infty(s,x,p)| : x\in\mathbb{R}^3, p\in\mathbb{R}^3, \text{ and } s\in[0,t]\}.$$

In the following, we estimate the vector potential A^c and then by Young's inequality with $\varepsilon = 2^{\frac{1}{3}}$,

$$\begin{aligned}
\|A^c(t)\|_{L_x^\infty} &\le Mc^{-1}\|j_{A^c}(t)\|_{L_x^\infty}^{\frac{1}{3}}\|j_{A^c}(t)\|_{L_x^1}^{\frac{2}{3}} \\
&\le Mc^{-1}(1+c^{-1}\|A^c(t)\|_{L_x^\infty})^{\frac{1}{3}}\varepsilon^{\frac{2}{3}}(0) \\
&\le Mc^{-1} + Mc^{-\frac{4}{3}}\|A^c(t)\|_{L_x^\infty}^{\frac{1}{3}} \\
&\le Mc^{-1} + \frac{1}{3}(\|A^c(t)\|_{L_x^\infty}^{\frac{1}{3}})^3\varepsilon^3 + \frac{2}{3}(Mc^{-\frac{4}{3}})^{\frac{3}{2}}\varepsilon^{-\frac{3}{2}} \\
&\le Mc^{-1} + \frac{2}{3}\|A^c(t)\|_{L_x^\infty} + \frac{\sqrt{2}}{3}(Mc^{-\frac{4}{3}})^{\frac{3}{2}},
\end{aligned}$$

which leads to

$$\|A^c(t)\|_{L_x^\infty} \le Mc^{-1} + Mc^{-2} \le Mc^{-1}.$$

In the same way, we have

$$\|\partial_x A^c(t)\|_{L_x^\infty} \le Mc^{-1}\|j_{A^c}(t)\|_{L_x^\infty}^{\frac{2}{3}}\|j_{A^c}(t)\|_{L_x^1}^{\frac{1}{3}} \le Mc^{-1}.$$

By the definition of E^∞ and Φ^c, we have

$$\begin{aligned}
&E^\infty(t,x) + \nabla\Phi^c(t,x) \\
= &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{x-y}{|x-y|^3}f^\infty(t,y,p)dydp - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{x-y}{|x-y|^3}f^c(t,y,p)dydp \\
= &\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{x-y}{|x-y|^3}(f^\infty(t,y,p) - f^c(t,y,p))dydp.
\end{aligned}$$

By means of Lemma A.1, it follows that

$$\begin{aligned}
&|E^\infty(t,x) + \nabla\Phi^c(t,x)| \\
\le\ &M\|\int_{R^3}|f^\infty(t,x,p) - f^c(t,x,p)|dp\|_{L_x^1}^{\frac{1}{3}}\|\int_{R^3}|f^\infty(t,x,p) - f^c(t,x,p)|dp\|_{L_x^\infty}^{\frac{2}{3}} \\
\le\ &MP^3(\bar{T})(U_0 + \bar{T}P(\bar{T}))^{\frac{1}{3}}\|f^\infty(t) - f^c(t)\|_{L_{x,p}^\infty} \le Md(t).
\end{aligned}$$

Thus for any $x \in \mathbb{R}^3, p \in \mathbb{R}^3$ and $t \in [0,\bar{T}]$, by directly computing we have

$$\begin{aligned}
&|\frac{d}{ds}(f^\infty - f^c)(s, X_f(s), P_f(s))| \\
= &|\partial_t(f^c - f^\infty) + v_{A^c}\cdot\nabla_x(f^c - f^\infty) \\
&- [\nabla\Phi^c - c^{-1}v_{A^c}^i\nabla(A^c)^i]\cdot\nabla_p(f^c - f^\infty)|(s, X_{f^c}(s), P_{f^c}(s)) \\
\le\ &Mc^{-1} + Md(s),
\end{aligned}$$

where $s \in [0,\bar{T}]$ and $X_{f^c}(s), P_{f^c}(s)$ are defined by (3.6), (3.7). Noting that

$$f^\infty(0, X_{f^c}(0), P_{f^c}(0)) - f(0, X_{f^c}(0), P_{f^c}(0)) = \overset{\circ}{f}(x,p) - \overset{\circ}{f}(x,p) = 0.$$

Hence integrating from 0 to t, we have for all $t \in [0, \bar{T}]$

$$\begin{aligned}|f^\infty(t,x,p) - f^c(t,x,p)| &= |\int_0^t \frac{d}{ds}(f^\infty - f^c)(s, X_{f^c}(s), P_{f^c}(s))ds| \\ &\le M\int_0^t (c^{-1} + d(s))ds \le Mc^{-1} + M\int_0^t d(s)ds,\end{aligned}$$

which by definition of $d(s)$ leads to

$$d(s) \le Mc^{-1} + M\int_0^t d(s)ds.$$

Then by Gronwall's inequality

$$d(s) \le Mc^{-1}e^{Mt} \le Mc^{-1}.$$

So it follows that for all $x \in \mathbb{R}^3, p \in \mathbb{R}^3, c \ge 1$ and $t \in [0, \bar{T}]$.

$$\begin{aligned}&|f^\infty(t,x,p) - f^c(t,x,p)| + |E^\infty(t,x) + \nabla\Phi^c(t,x)| + |A^c(t,x)| + |\partial_x A^c(t,x)| \\ &\le d(t) + Md(t) + Mc^{-1} \le Mc^{-1}.\end{aligned}$$

Now we complete the proof of Theorem 2.2. □

A. Appendix

In this appendix, we collect some well known facts which have been used in the above discussion. We also provide details for the proof of some auxiliary results in Section 3 and Section 4.

Lemma A.1. *For any* $1 \le p < 3, r_0 = \frac{3}{3-p}$ *and* $r < r_0 < s$,

$$\|\int_{\mathbb{R}^3} \frac{\psi(y)}{|y-x|^p}dy\|_{L_x^\infty} \le C(p,r,s)\|\psi\|_{L_x^r}^{1-\theta}\|\psi\|_{L_x^s}^{\theta},$$

where $\theta = \frac{1-r/r_0}{1-r/s}$ *and the positive constant* $C(p,r,s)$ *depends only on* p, r *and* s*, in particular,* $C(p,1,\infty) = 3(4\pi/p)^{p/3}/(3-p)$.

Proof. See [24, Lemma 2.7]. □

Proposition A.2. *Let* $\rho \in C^1([0,T], C_c^1(\mathbb{R}^3))$ *and the vector* $j \in C^1([0,T], C_c^1(\mathbb{R}^3;\mathbb{R}^3))$*. Then the following hold*:

$$\|\Phi(t)\|_{L_x^\infty} \le C\|\rho(t)\|_{L_x^1}^{\frac{2}{3}}\|\rho(t)\|_{L_x^\infty}^{\frac{1}{3}},$$

$$\|\nabla\Phi(t)\|_{L_x^\infty} \le C\|\rho(t)\|_{L_x^1}^{\frac{1}{3}}\|\rho(t)\|_{L_x^\infty}^{\frac{2}{3}}. \tag{A-1}$$

Moreover, the second order derivative satisfies, for any $0 < d \le R$,

$$\|D_x^2\Phi(t)\|_{L_x^\infty} \le C[R^{-3}\|\rho(t)\|_{L_x^1} + d\|\partial_x\rho(t)\|_{L_x^\infty} + (1+\ln(R/d))\|\rho(t)\|_{L_x^\infty}]. \tag{A-2}$$

The Darwin vector potential

$$A(t,x) = \frac{1}{2}\int_{\mathbb{R}^3} [id + \omega\otimes\omega]j(t,y)\frac{dy}{|y-x|},$$

admits the same estimates as (A-1) *and* (A-2)*, but with* $\rho(t)$ *replaced by* $j(t)$*. In addition, we have*

$$\|D_x^2 A(t)\|_{L_x^\infty} \le C[R^{-3}\|j(t)\|_{L_x^1} + d\|\partial_x j(t)\|_{L_x^\infty} + (1+\ln(R/d)) \\ \sup\{|j(t,y)|, |y-x| \le R\}].$$

Proof. For the proof, we refer the readers to [26, Lemma P1] and [32, Lemma 6]. For the last inequality, we only check in the process in proving estimate(A-2) such that we can replace $\|j(t)\|_{L_x^\infty}$ with $\sup\{|j(t,y)|, |y-x| \le R\}$. □

Proposition A.3. *For any* $p \in [1, \frac{5}{4}]$*,* $t \in [0,T]$*,* $j_A(t) \in L_x^p(\mathbb{R}^3)$ *and*

$$\|j_A(t)\|_{L_x^{\frac{5}{4}}} \le \varepsilon^{\frac{5}{4}}(t)\|f(t)\|_{L_{x,p}^\infty}^{\frac{1}{5}}, \quad \|j_A(t)\|_{L_x^1} \le \frac{1}{2}(\|f(t)\|_{L_{x,p}^1} + \varepsilon(t)).$$

Proof. By the definition, we have

$$\begin{aligned} j_A(t,x) = &\int_{|p-c^{-1}A(t,x)|\le R} |p-c^{-1}A(t,x)| f(t,x,p)dp \\ &+ \int_{|p-c^{-1}A(t,x)|>R} |p-c^{-1}A(t,x)| f(t,x,p)dp \\ \le\ & R^4\|f(t)\|_{L^\infty} + R^{-1}\int_{\mathbb{R}^3} |p-c^{-1}A(t,x)|^2 f(t,x,p)dp \\ \le\ & (\int_{\mathbb{R}^3} |p-c^{-1}A(t,x)|^2 f(t,x,p)dp)^{\frac{4}{5}}\|f(t)\|_{L^\infty}^{\frac{1}{5}}, \end{aligned}$$

then we have

$$\|j_A(t)\|_{L^{\frac{5}{4}}} \le (\int_{\mathbb{R}^3\times\mathbb{R}^3} |p-c^{-1}A(t,x)|^2 f(t,x,p)dpdx)^{\frac{4}{5}}\|f(t)\|_{L^\infty}^{\frac{1}{5}}. \tag{A-3}$$

In addition, we have

$$\begin{aligned} \|j_A(t)\|_{L^1} &= \int_{\mathbb{R}^3\times\mathbb{R}^3} |p-c^{-1}A(t,x)|^2 f(t,x,p)dpdx \\ &\le \frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{R}^3} (1+|p-c^{-1}A(t,x)|^2) f(t,x,p)dpdx. \end{aligned} \tag{A-4}$$

Now we complete the proof. □

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