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BEYOND THE CACTUS RANK OF TENSORS

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Abstract. We study additive decompositions (and generalized additive decompositions with a zero-dimensional scheme instead of a finite sum of rank 1 tensors), which are not of minimal degree (for sums of rank 1 tensors with more terms than the rank of the tensor, for a zero-dimensional scheme a degree higher than the cactus rank of the tensor). We prove their existence for all degrees higher than the rank of the tensor and, with strong assumptions, higher than the cactus rank of the tensor. Examples show that additional assumptions are needed to get the minimally spanning scheme of degree cactus +1.

1. introduction

Fix positive integers, k and n_i, $1 \le i \le k$. Set $Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and $r := -1 + \prod_{i=1}^{k}(n_i+1)$. Let $\nu : Y \to \mathbb{P}^r$ be the Segre embedding of the multiprojective space Y, i.e., the embedding of Y induced by the complete linear system $|\mathcal{O}_Y(1,\dots,1)|$. Set $X := \nu(Y)$.

Definition 1.1. The rank $r_X(q)$ (resp. cactus rank $c_X(q)$) of the point $q \in \mathbb{P}^r$ is the minimal integer $|A|$ (resp. $\deg(A)$), with $A \subset Y$ a finite set (resp. a zero-dimensional scheme) and $q \in \langle \nu(A)\rangle$, where $\langle\ \ \rangle$ denote the linear span.

The rank of q is the tensor rank of any tensor $T \in \mathbb{A}^{r+1} \setminus \{0\}$ with q as its equivalence class. See [28] for a general reference for the tensor decomposition with a strong bent on geometry and algebra and an extensive bibliography. For symmetric tensors from the point of view of algebraic geometry and commutative algebra, see [26]. See [7,8,30] for the notion of cactus rank (called *scheme rank* in [26]). Any scheme evincing a cactus rank is Gorenstein ([11, Lemma 2.4]).

Notation 1.2. If $q \in \mathbb{P}^r$ and $B \subset Y$ is a zero-dimensional scheme we say that B *minimally spans* q if $q \in \langle \nu(B)\rangle$ and $q \notin \langle \nu(B')\rangle$ for any $B' \subsetneq B$.

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If B evinces the rank or the cactus rank of $q \in \mathbb{P}^r$, then B minimally spans q. We have $\langle \nu(B)\rangle \neq (\cup_{B' \subsetneq B} \nu(B'))\rangle$ if and only if B minimally spans some point of q; in this case $\langle \nu(B)\rangle \setminus (\cup_{B' \subsetneq B} \nu(B'))\rangle$ is the set of all points of $\mathbb{P}^r$ minimally spanned by B. If B is a finite set we have $\langle \nu(B)\rangle \neq (\cup_{B' \subsetneq B} \nu(B'))\rangle$ if and only if $\nu(B)$ is linearly independent. There are zero-dimensional schemes B, even schemes with one connected component of degree 1 or with all components of degree at most 2 with $\langle \nu(B)\rangle \neq \cup_{B' \subsetneq B} \nu(B'))\rangle$ and $\nu(B)$ linearly dependent (Examples 3.1, 3.2 and 3.3). Now take a zero-dimensional scheme $A \subset Y$ such that $\nu(A)$ is linearly independent. If A has only finitely many subschemes of degree $\deg(A)-1$ (say e subschemes of degree $\deg(A)-1$), then $\nu(A) \neq \cup_{A' \subsetneq A} \langle \nu(A')\rangle$ and $\cup_{A' \subsetneq A} \langle \nu(A')\rangle$ is the union of e distinct hyperplanes of $\langle \nu(A)\rangle$; if A is curvilinear we have $e = |A_{\mathrm{red}}|$.

The first aim of this paper is to raise 3 questions, the first one (Question 1.3) being not always true (see Proposition 3.6 for a counterexample for some zero-dimensional scheme A), but being true in some interesting cases.

Question 1.3. *Fix $q \in \mathbb{P}^r$ and a zero-dimensional scheme $A \subset Y$ minimally spanning q. Set $w := \dim\langle \nu(A)\rangle$.*

(1) *Fix an integer x such that $w+1 \leq x \leq r$. Is there a zero-dimensional scheme B minimally spanning q with* $\dim\langle \nu(B)\rangle = x$ *and* $\deg(B) = \deg(A)+x-w$*?*
(2) *Is it* (1) *true at least if $x = w+1$?*
(3) *Is it true if in* (1) *we drop the condition* $\deg(B) = \deg(A)+x-w$*?*
(4) *Are* (1) *or* (2) *or* (3) *true if we only require it for a general $q \in \langle \nu(A)\rangle$?*

Case (4) of Question 1.3 is known to be true if A is a finite set ([6, Theorem 3.8]).

We prove the strong form (1) of Question 1.3 if either A has at least one connected component of degree 1 (i.e., A has a single point as a connected component) or if A is a disjoint union of points and tangent vectors (Theorem 1.4 and Proposition 1.6). In the general case (i.e., for schemes for which either it is false or we do not know if it is true) we propose a measure of its failure (see Definition 1.9).

Theorem 1.4. *Let $A \subset Y$ be a zero-dimensional scheme such that at least one connected component of A has degree* 1*. Assume $\langle \nu(A)\rangle \neq \cup_{A' \subsetneq A} \langle nu(A')\rangle$ and fix $q \in \langle \nu(A)\rangle \setminus (\cup_{A' \subsetneq A} \langle nu(A')\rangle)$. Set $w := \dim\langle \nu(A)\rangle$. Then for each integer x with $w < x \leq r$ there is a zero-dimensional scheme $B \subset Y$ such that B minimally spans q,* $\deg(B) = \deg(A)+x-w$ *and* $\dim\langle \nu(B)\rangle = x$*.*

Remark 1.5. Take the set-up of Theorem 1.4 and write $A = A_1 \sqcup \{p\}$ with $p \in Y$. The proof of Theorem 1.4 given below constructs B with $B = A_1 \sqcup S$ and S a finite set. In particular if A is a finite set we get as B a finite set, improving [6, Theorem 3.8] from a statement like part (4) to a statement like part (1) of Question 1.3.

Proposition 1.6. *Let $A \subset Y$ be a zero-dimensional scheme such that $\nu(A)$ is linearly independent and each connected component of A has degree at most 2. Fix $q \in \langle \nu(A) \rangle$ such that $q \notin \langle \nu(A') \rangle$ for any $A' \subsetneq A$. Then for each integer z such that $\deg(A) < z \leq r+1$ there is a zero-dimensional scheme $B \subset Y$ such that $\deg(B) = z$, B minimally spans q and each connected component of B has degree at most 2.*

Notation 1.7. For any $q \in \mathbb{P}^r$ let $\mathcal{S}(q)$ (resp. $\mathcal{Z}(q)$) denote the set of all finite sets (resp. zero-dimensional schemes) $A \subset Y$ such that $|A| = r_X(q)$ (resp. $\deg(A) = c_X(q)$) and $q \in \langle \nu(A) \rangle$.

Note that $\mathcal{S}(q) \neq \emptyset$, $\mathcal{Z}(q) \neq \emptyset$ and that if $A \in (\mathcal{S}(q) \cup \mathcal{Z}(q))$, then $q \notin \langle \nu(A') \rangle$ for any $A' \subsetneq A$.

For each positive integer m let $\sigma_m(X) \subseteq \mathbb{P}^r$ denote the *m-secant variety* of X, i.e., the closure in $\mathbb{P}^r$ of the union of all linear spaces $\langle B \rangle$ with B a finite subset of X with $|B| = m$. Each $\sigma_m(X)$ is an irreducible variety with $\dim \sigma_m(X) \leq \min\{m(n+1)-1, r\}$, where $n := n_1 + \cdots + n_k = \dim X$. The *border rank* $b_X(q)$ of $q \in \mathbb{P}^r$ is the minimal integer m such that $q \in \sigma_m(X)$. When $\dim \sigma_m(X) = m(n+1)-1$ many papers studied if $|\mathcal{S}(q)| = 1$ for a general $q \in \sigma_m(X)$ ([5, 9, 10, 14–25, 27, 29, 31]). This problem is called *uniqueness* or *generic uniqueness* for $\sigma_m(X)$. If $m(n+1)-1 = r$ (and so $\sigma_m(X) = \mathbb{P}^r$), almost never $|\mathcal{S}(q)| = 1$ for a general $q \in \sigma_m(X)$ and the consensus is that the few cases with $|\mathcal{S}(q)| = 1$ have some interesting geometry. If $m(n+1)-1 < r$ the case with $|\mathcal{S}(q)| > 1$ are usually very interesting and many papers proved (for certain $k, n_1, \ldots, n_k$) that if $r \gg m(n+1)$, then generic uniqueness holds. We may also ask if generic uniqueness holds for the set $\mathcal{Z}(q)$, but we do not have evidence for it, except a few cases with very low m. We need these cases to construct a counterexample to Question 1.3 (see Proposition 3.6).

Question 1.8. *Let $X \subset \mathbb{P}^r$ be the Segre variety of a product of k projective spaces. Is $r_X(q) \leq k b_X(q)$ for all $q \in \mathbb{P}^r$?*

Question 1.8 is true for all $q \in \mathbb{P}^r$ with $b_X(q) \leq 3$ ([12, Proposition 1.1 and Theorem 1.2], [1, Theorem 1], [4]). The question is motivated by a similar question concerning symmetric tensors, i.e., degree d homogeneous polynomials in $n+1$ variables, i.e., for the order d Veronese embedding of $\mathbb{P}^n$. For a Segre variety the number k of its positive dimensional factors is a weak substitute for the order of the Veronese embedding as evinced by the similarities between the classification of the ranks for tensors and symmetric tensors with very low border rank ([1,2,4,12]). Question 1.8 is in general false if we use $c_X(q)$ instead of $b_X(q)$, even for symmetric tensors when $k = 3$ and for general homogeneous degree d polynomials in a high number of variables ([8]), but it may be true if instead of the cactus rank we consider the *smoothable rank* (i.e., in the definition of cactus rank we only allow smoothable zero-dimensional schemes) or the *curvilinear rank* (i.e., we only allow curvilinear zero-dimensional schemes in the definition ([3, Questions 1 and 2, Theorem 1]).

Definition 1.9. For any $q \in \mathbb{P}^r$ let $\delta_X(q)$ be the minimal integer $t > c_X(q)$ such that there is a degree t zero-dimensional scheme $A \subset Y$ with $q \in \langle \nu(A) \rangle$ and $A \not\supseteq Z$ for any $Z \in \mathcal{Z}(q)$.

Any scheme A as in Definition 1.9 minimally spans q. Proposition 3.4 gives an example with $\delta_X(q) - c_X(q) \ge 2$.

Question 1.10. *What is the maximum of all $\delta_X(q) - c_X(q)$, $q \in \mathbb{P}^r$?*

Remark 1.11. Theorem 3.6 shows that $\delta_X(q) \le r_X(q) + 1$ for all $q \in \mathbb{P}^r$.

In our proofs we focus on one connected component E of A, say $A = A_1 \sqcup E$; E is a point in Theorem 1.4. We think about $A_1 \sqcup E$ as a partial additive decomposition of the tensor associated to $q \in \mathbb{P}^r$ (e.g. [26, Def. 1.30], [13] and references therein) (although it is not canonically associated to q, because in our proofs not even A is canonically associated to q) and we hope that this approach and the use of zero-dimensional schemes may be used for other additive decompositions.

2. The proofs

Recall that $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ and that $X = \nu(Y)$, where $\nu : Y \to \mathbb{P}^r$, $r = -1 + \prod_{i=1}^{k}(n_i + 1)$, is the Segre embedding of the multiprojective space Y. Let $\pi_i : Y \to \mathbb{P}^{n_i}$ denote the projection onto the i-th factor of Y. Let $\eta_i : Y \to \prod_{j \ne i} \mathbb{P}^{n_j}$ denote the projection onto all factors of Y, except the i-th one.

Remark 2.1. Let $Z \subset Y$ be a zero-dimensional scheme, which is curvilinear, i.e., for each connected component E of Z the Zariski tangent space of E at the point E_{red} has either dimension 0 (i.e., E is a point) or dimension 1 (and hence E is contained in the germ of a smooth curve). Then Z has finitely many subschemes and in particular it has finitely many subschemes of degree $\deg(Z) - 1$. Thus if A is as in Proposition 1.6, then $\langle \nu(A) \rangle \supsetneq \cup_{A' \subsetneq A} \langle \nu(A') \rangle$.

This is our main lemma.

Lemma 2.2. *Fix a linear space $V \subset \mathbb{P}^r$ such that* $\dim V \le r - 2$ *and $p \in X$ such that $p \notin V$. Then there is a line $L \subset X$ such that $L \cap V = \emptyset$ and $p \in \langle L \cup V \rangle$.*

Proof. Set $V' := \langle V \cup \{p\} \rangle$. Let $o = (o_1, \dots, o_k)$ be the point such that $\nu(o) = p$. Set $T_i := \eta_i^{-1}(o)$ and $T = T_1 \cup \cdots \cup T_k$. Note that $\nu(T_i)$ is a projective space of dimension n_i, that $\nu(T_i) \cap \nu(T_j) = \{p\}$ for all $i \ne j$ and that T is the union of all lines of X passing through p. Since $\nu(T_i)$ is a linear space, $W_i := V \cap \nu(T_i)$ is a linear subspace of V. Since $o \notin V$, we have $\dim W_i \le n_i - 1$. Let $M_i \subset Y$ be the set such that $\nu(M_i) = W_i$.
Claim 1. If there is $i \in \{1, \dots, k\}$ such that $\dim W_i \le n_i - 2$ (with the convention $W_i = \emptyset$ if $n_i = 1$), then there is a line $L \subset X$ such that $p \in L$ and $L \cap V = \emptyset$.

Proof of Claim 1. Take as L any line of $\nu(T_i)$ containing p and not intersecting W_i, e.g., a general line L of $\nu(T_i)$ containing p.

By Claim 1 we may assume $\dim W_i = n_i - 1$ for all i. Since $\dim W_i = n_i - 1$ and $p \notin W_i$, then V' contains $\nu(T)$ and V' is spanned by the union of V and any point of $\nu(T_i) \setminus W_i$ (for a single index i). Since $\dim V \le r-2$, we have $V' \ne \mathbb{P}^r$. Fix a general $x_1 \in \mathbb{P}^{n_1}$ and set $x := (x_1, o_2, \dots, o_k)$. We have $x \in T_1 \setminus M_1$. Set $R'' := \eta_1^{-1}(x_1) \cong \mathbb{P}^{n_1}$. Take a line (up to the identification of R'' with $\mathbb{P}^{n_1}$) $R' \subseteq R''$ with $x \in R'$ and set $R := \nu(R')$. R is a line containing $\nu(x)$. Since $x \notin M_1$, $\langle V \cup R\rangle$ contains T_1 and hence contains p. Since $p \notin V$ and $\dim R = 1$, either $R \cap V = \emptyset$ (and in this case we may take R as L) or $\langle R \cup V\rangle = V'$. Thus we may assume that we are in the latter case for every line $R' \subseteq R''$ with $x \in R'$. Thus $\nu(R'') \subset V'$. Take a general $x_2 \in \mathbb{P}^{n_2}$ and set $x' := (x_1, x_2, o_3, \dots, o_k)$. Set $D'' := \eta_2^{-1}(x_2) \cong \mathbb{P}^{n_2}$. Take a line (up to the identification of D'' with $\mathbb{P}^{n_2}$) $D' \subseteq D''$ with $x' \in D'$ and set $D := \nu(D')$. D is a line containing $\nu(x')$. Since $x' \notin M_2$, $\langle V \cup DR\rangle$ contains $\nu(T_2)$ and hence contains p. Since $p \notin V$ and $\dim D = 1$, either $D \cap V = \emptyset$ (and in this case we may take D as L) or $\langle D \cup V\rangle = V'$. Thus we may assume that we are in the latter case for every line $D' \subseteq D''$ with $x' \in D'$. Thus $\nu(D'') \subset V'$. We see that V' contains all points $\nu((x_1, x_2, o_3, \dots, o_k))$. If $k = 2$ we get $V' = \mathbb{P}^r$, a contradiction. Now assume $k > 2$. Take a general $x_3 \in \mathbb{P}^{n_3}$ and set $y = (x_1, x_2, x_3, o_4, \dots, o_k) \in T_3 \setminus M_3$. Set $R_1 := \eta_3^{-1}(x_3)$. We use R_1 as we used D''. Since W_3 is a hyperplane of $\nu(T_3)$ we get that V' contains all $\nu(y_1, y_2, y_3, o_k, \dots, o_k)$ with $(y_1, y_2, y_3) \in \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$. If $k = 3$ we get $V' = \mathbb{P}^r$, a contradiction. If $k > 3$ we use first η_4, then η_5, and so on. □

Proof of Theorem 1.4. Write $A = A_1 \sqcup \{o\}$ with $o \notin (A_1)_{\mathrm{red}}$. Set $V := \langle \nu(A_1)\rangle$. Since A minimally spans q, we have $\nu(o) \notin V$. By Lemma 2.2 there is a line $L \subset X$ such that $V \cap L = \emptyset$ and $o \in \langle V \cup L\rangle$. Fix two general points $p_1, p_2 \in L$, write $p_i = \nu(q_i)$, $i = 1, 2$, and set $B := A_1 \cup \{q_1, q_2\}$. Since $\nu(A_1) \subseteq V$ and $o \in \langle V \cup L\rangle$, we have $q \in \langle \nu(B)\rangle$. To conclude the proof when $x = w + 1$ it is sufficient to prove that for each $B' \subsetneq B$ we have $q \notin \langle \nu(B')\rangle$. Since for each $B' \subsetneq B$ there is a scheme B'' with $B' \subseteq B'' \subset B$ and $\deg(B'') = \deg(B) - 1$, it is sufficient to test all $B' \subset B$ with $\deg(B') = \deg(B) - 1 = \deg(A)$.

First assume that $B' \supset A_1$, i.e., $B' = A_1 \sqcup \{q_i\}$ for some i. Since p_1, p_2 are general in L, we have $o \ne q_i$ for all i. Since $\dim\langle \nu(A_1) \cup L\rangle = \dim\langle A_1\rangle + 2$, we have $\dim\langle \nu(A_1) \cup \{a\}\rangle = \dim\langle A_1\rangle + 1$ if $a \in \{\nu(o), p_1, p_2\}$ and the linear spaces $\langle \nu(A_1) \cup \{a\}\rangle$, $a \in \{\nu(o), p_1, p_2\}$, are different. Thus $\langle \nu(A_1) \cup \{p_i\}\rangle \cap \langle \nu(A_1) \cup \{\nu(o)\}\rangle = \langle \nu(A_1)\rangle$. Hence $q \notin \langle \nu(B')\rangle$.

Now assume $A_1 \nsubseteq B'$, i.e., $B' = A_2 \sqcup \{q_1, q_2\}$ with $A_2 \subset A_1$ and $\deg(A_2) = \deg(A_1) - 1$. Since $L = \langle \{p_1, p_2\}\rangle$ and $V \cap L = \emptyset$, we get $\langle \nu(B')\rangle \cap \langle \nu(A)\rangle = \langle \nu(A_2 \cup \{o\})\rangle$ and so $q \notin \langle \nu(B')\rangle$.

The scheme B solves the case $x = w + 1$ and it has a degree 1 connected component. If $r > w+1$ we just apply the proof to B instead of A and conclude by induction on the integer $x - w$. □

For any zero-dimensional scheme $Z \subset Y$ set $Y[Z] := \prod_{i=1}^{k} \langle \pi_i(Z) \rangle$, where the linear span $\langle \pi_i(Z) \rangle$ is taken in $\mathbb{P}^{n_i}$. Note that $Y[Z] \subseteq Y$ is the minimal multiprojective subspace of Y containing Z and that the i-th factor of $Y[Z]$ has dimension at most $\min\{n_i, \deg(Z) - 1\}$.

To prove Proposition 1.6 we first need the case $\deg(A) = 2$ and A connected, i.e., when A is a tangent vector of Y.

Lemma 2.3. *Let $\tau(X) \subset \mathbb{P}^r$ denote the tangential variety of X. Since X is smooth, we have $\tau(X) = \cup_{p \in X} T_pX$. Since each element of $T_pX \setminus \{p\}$ is contained in a line through p and each such a line is spanned by a tangent vector of X at p, v exists. Fix any $q \in \tau(X) \setminus X$ with $r := r_X(q) > 1$. Let $v \subset Y$ be a connected degree 2 scheme such that $q \in \langle \nu(v) \rangle$. Then there is a degree 3 scheme $B = u \sqcup \{p\}$ with u a connected degree 2 scheme, $u_{\mathrm{red}} = v_{\mathrm{red}}$, $\langle \nu(v) \rangle \subset \langle \nu(B) \rangle$ and $q \notin \langle \nu(B') \rangle$ for any $B' \subsetneq B$.*

Proof. It is sufficient to find $B \subset Y[v]$ and hence we may assume $Y = Y[v]$. Since $Y[v] \cong (\mathbb{P}^1)^r$ (by the proof of [1, Theorem 1]), in this proof we assume $k = r$ and $n_i = 1$ for all i. Set $\{o\} := v_{\mathrm{red}}$ and write $o = (o_1, \dots, o_k)$. Set $T_i := \eta_i^{-1}(o_i)$, $i = 1, \dots, k$, and $T := T_1 \cup \cdots \cup T_k$. Note that each $\nu(T_i)$ is a line through $\nu(o)$. Moreover, $\dim\langle \nu(T) \rangle = r$ and hence for each $p_i \in T_i \setminus \{o\}$, $i = 1, \dots, k$, the set $\nu(\{o, p_1, \dots, p_k\})$ is linearly independent and spans $\langle \nu(T) \rangle$. We have $\nu(v) \subset \langle \nu(T) \rangle$. For each $i \in \{1, \dots, k\}$ set $\hat{T}_i := \cup_{j \neq i} T_i$. We have $\dim\langle \nu(\hat{T}_i) \rangle = r - 1$, $\nu(v) \subset \langle \nu(T) \rangle$, $\nu(v) \cap \langle \nu(\hat{T}_i) \rangle = \nu(o)$ and $\langle \{q, \nu(o)\} \rangle = \langle \nu(v) \rangle$. The proof of [1, Theorem 1] also gave the existence of $p_i \in T_i \setminus \{o\}$ such that $q \in \langle \nu(\{p_1, \dots, p_k\}) \rangle$. Thus for each $i = 1, \dots, k$ the set $\langle \{q, \nu(p_i), \nu(o)\} \rangle$ is a plane contained in $\langle \nu(T) \rangle$ and hence the set $L_i := \langle \nu(\hat{T}_i) \rangle \cap \langle \{q, \nu(p_i), \nu(o)\} \rangle$ is a line through $\nu(o)$. Any such line is a tangent line to X at $\nu(o)$ and so there is a degree 2 connected zero-dimensional scheme $v_i \subset Y$ such that $\{o\} = (v_i)_{\mathrm{red}}$ and $L_i = \langle \nu(v_i) \rangle$. Fix any i and set $u := v_i$, $p := p_i$ and $B := v_i \cup \{p_i\}$. Since $u_{\mathrm{red}} = \{o\} \neq \{p\}$, we have $\deg(B) = 3$. Since $q \in \langle \nu(\{p_1, \dots, p_k\}) \rangle$ and $\{p_1, \dots, p_k\} \setminus \{p_i\} \subset \hat{T}_i$, we have $q \in \langle L_i \cup \{p_i\} \rangle = \langle \nu(B) \rangle$. Since $\nu(o) \in \langle \nu(v_i) \rangle \subset \langle \nu(B) \rangle$, we have $\langle \nu(v) \rangle \subset \langle \nu(B) \rangle$. Since $r > 1$ and $\nu(\{o, p_i\}) \subset \nu(T_i)$, we have $q \notin \langle \nu(\{o, p_i\}) \rangle$. Since $\langle \nu(v_i) \rangle \subset \langle \nu(\hat{T}_i) \rangle$ and $q \notin \langle \nu(\hat{T}_i) \rangle$, we have $q \notin \langle \nu(B') \rangle$ for any $B' \subsetneq B$. □

Proof of Proposition 1.6. We use induction on the integer $\deg(A)$, the starting point of the induction, i.e., the case $\deg(A) = 1$, being true by Theorem 1.4.

If A has at least one degree 1 connected component, then we may apply Theorem 1.4. Now assume that each connected component of A has degree 2. If $\deg(A) = 2$, then Lemma 2.3 gives the case $w = 3$ and we apply Theorem 1.4 to the degree 3 scheme given by Lemma 2.3. Now assume $\deg(A) \geq 4$ and write $A = A_1 \sqcup v$ with v connected $\deg(v) = 2$. Set $\{o\} := v_{\mathrm{red}}$. Since $\nu(A)$ is linearly independent, we have $\langle \nu(A_1) \rangle \cap \langle \nu(v) \rangle = \emptyset$ and hence there are unique $q_1 \in \langle \nu(A_1) \rangle$ and $q_2 \in \langle \nu(v) \rangle$ such that $q \in \langle \{q_1, q_2\} \rangle$. By [12, Proposition 1.1] or [1, Theorem 1] we have $1 \leq r_X(q_2) \leq k$.

First assume $r_X(q_2) = 1$. Thus there is $a \in X$ with $\nu(a) = q_2$. Since $q_2 \in \langle \nu(v)\rangle$ and $\langle \nu(A_1)\rangle \cap \langle \nu(v)\rangle = \emptyset$, $\{a\}$ is a connected component of $A_1 \cup \{a\}$. Since $A_1 \cup \{a\}$ minimally spans q, we may apply Theorem 1.4 to $A_1 \cup \{a\}$ and conclude the proof of Proposition 2.3 in this case.

From now on we assume $r := r_X(q_2) > 1$. Set $\{o\} := v_{\mathrm{red}}$. We need the set-up of the proof of Lemma 2.3, which in turn uses the set-up of [1]. We have $Y[v] = (\mathbb{P}^1)^r$. Call $B_1 = u \sqcup \{p\}$ one of the schemes given by Lemma 2.3 for q_2 and v with u connected of degree 2. In particular $u_{\mathrm{red}} = \{o\}$ and $p \neq o$. Since $\langle \nu(v)\rangle \subset \langle \nu(B_1)\rangle$, we have $q \in \langle \nu(A_1 \cup B_1)\rangle$. We also know that for any $E \subsetneq B_1$, we have $q_2 \notin \langle \nu(E)\rangle$. To get u, p we made some choices and the main one was to fix one of the r non-trivial factors of $Y[v]$, say the one corresponding to the index $i \in \{1, \dots, k\}$. The points p and o have the same coordinates, except the i-th one. To stress that the pair (u, p) depends from the choice of i, we write it as $(u[i], p[i])$. We take the set-up of the proof of Lemma 2.3, but now we have $Y[v] = (\mathbb{P}^1)^r$. We permute the factors of Y so that the first r factors of Y are the r non-trivial factors of $Y[v]$. With this convention we have $T_i := \eta_i^{-1}(o_i)$, $1 \le i \le r$, and $T = T_1 \cup \cdots \cup T_r$ and we have defined $(u[i], p[i])$ if and only if $1 \le i \le r$. Note that $T_{\nu(o)}\nu(Y[v]) = \langle \nu(T)\rangle$, where $T_{\nu(o)}\nu(Y[v]) \subset \mathbb{P}^r$ is the Zariski tangent space at $\nu(o)$ to the r-dimensional smooth variety $\nu(Y[v])$ (the set $T_{\nu(o)}\nu(Y[v])$ is an r-dimensional linear subspace of $\mathbb{P}^r$).

Claim 1. Proposition 1.6 is true when $T_{\nu(o)}\nu(Y[v]) \subseteq \langle \nu(A)\rangle$.
Proof of Claim 1. Assume $T_{\nu(o)}\nu(Y[v]) \subseteq \langle \nu(A)\rangle$. Recall that

$$\langle \nu(v)\rangle \subset T_{\nu(o)}\nu(Y[v]) \text{ and } \langle \nu(A_1)\langle \cap \langle \nu(v)\rangle = \emptyset.$$

Thus $\langle \nu(A_1)\rangle \cap T_{\nu(o)}\nu(Y[v])$ is an $(r-2)$-dimensional linear subspace E of $T_{\nu(o)}\nu(Y[v])$ with $E \cap \langle \nu(v)\rangle = \emptyset$ and $F := \langle \{\nu(o)\} \cup E\rangle$ is an $(r-1)$-dimensional linear subspace of $T_{\nu(o)}\nu(Y[v])$ with $F \cap \langle \nu(v)\rangle = \{\nu(o)\}$. Since $T_{\nu(o)}\nu(Y[v]) = \langle \nu(T)\rangle$, for any $a_i \in T_i \setminus \{o\}$, $i = 1, \dots, r$, the set $\{\nu(o), \nu(a_1), \dots, \nu(a_r)\}$ is linearly independent and it spans $T_{\nu(o)}\nu(Y[v])$. Thus there is $i \in \{1, \dots, r\}$ such that $\langle \nu(A)\rangle = \langle \nu(A_1 \cup \{o, a_1\})\rangle$. Apply Theorem 1.4 to q and $A_1 \cup \{o, a_1\}$.

Let $B' \subseteq A_1 \cup B_1$ be the minimal subscheme of $A_1 \cup B_1$ such that $q \in \langle \nu(B)\rangle$. Every connected component of B' has degree at most 2.

(a) Assume $\deg(B') > \deg(A)$, i.e., $B' = A_1 \cup B_1$. B' solves the case $x = w + 1$. Since B' minimally spans q and B' has a degree 1 connected components, Theorem 1.4 applied to (q, B') gives all cases with $x > w + 1$.

(b) Assume $\deg(B') = \deg(A)$. If $B' = A_1 \cup \{o, p\}$, then we may apply Theorem 1.4 to (q, B') and get all $x > w$. Now assume $B' = E \cup B_1$ with $E \subset A_1$ and $\deg(E) = \deg(A_1) - 1$. The scheme E has 2 connected components of degree 1 and hence all cases with $x > w$ are true by Theorem 1.4 applied to the pair (q, B'). Now assume $B' = A_1 \cup u$. Since $u \neq v$ we have $B' \cap A = A \cup \{o\}$. Since $q \in \langle \nu(B')\rangle \cap \langle \nu(A)\rangle$ and $q \notin \langle \nu(A_1 \cup \{o\})\rangle$, we get $\langle \nu(A_1 \cup u)\rangle = \langle \nu(A)\rangle$. Since $u \neq v$ and $\nu(v) \subset \nu(B_1)$, we get $\nu(p) \in \langle \nu(A)\rangle$. Assume that this is true for all $i = 1, \dots, r$, i.e., that $\nu(p[i]) \in \langle \nu(A)\rangle$ for $i = 1, \dots, r$. Since

$p[i] \in T_i \setminus \{o\}$, the set $\{\nu(o), \nu(p_1), \dots, \nu(p_r)\}$ spans $\langle \nu(T) \rangle = T_{\nu(o)}(\nu(Y[v])$. Since $\nu(o) \in \langle \nu(A) \rangle$, it is sufficient to use Claim 1.

(c) Assume $\deg(B') < \deg(A)$. If B' has at least one connected component of degree 1, then we may apply Theorem 1.4 to (q, B'), because B' minimally spans q. Now assume that all connected components of B' have degree 2 (as the ones of A) and so $B' = A_2 \cup u$ with A_2 union of some of the connected components of A_1. Since $q \in \langle \nu(A) \rangle \setminus \langle \nu(A_1 \cup \{o\}) \rangle$, we get $\nu(u) \subset \langle \nu(A) \rangle$. We conclude as in step (b) using Claim 1. □

3. The examples

Example 3.1. Assume $n_1 \ge 2$ and take a plane $M \subseteq \mathbb{P}^{n_1}$. If $k > 1$ fix $a_i \in \mathbb{P}^{n_i}$ for all $i > 1$ and let $j : M \to Y$ be the embedding with $\pi_i(j(x)) = a_i$ for all $i > 1$ and $\pi_1(j(x)) = x$, $x \in M$. Let $C \subset M$ be a smooth plane cubic. Let $o \in C$ be one of its 9 flexes and $A = 4o \subset C$ the degree 4 effective divisor of C with o as its reduction. Note that A spans M, but that it is not linearly independent and that each proper subscheme of A is contained in the tangent line L of C at o. The scheme $\nu(j(A))$ is not linearly independent, but it minimally spans each point of $\nu(j(M \setminus L))$.

Example 3.2. Assume $k \ge 2$. Take $M := \mathbb{P}^1 \times \mathbb{P}^1$ and take any embedding $j :$ of $\mathbb{P}^1 \times \mathbb{P}^1$ into Y as a multiprojective subspace. Let $C \in |\mathcal{O}_M(3,3)|$ be a smooth curve such that there is $o \in C$ such that the line $L \in |\mathcal{O}_M(1,0)|$ intersects C at o with multiplicity 3. Let $A \subset C$ be the effective degree 4 divisor of C with o as its reduction. The scheme $\nu(j(A))$ is not linearly independent, but it minimally spans any point of $\langle \nu(j(A)) \rangle \setminus \nu(j(L))$.

Example 3.3. Assume $k \ge 2$ and $n_2 \ge 2$, so that Y contains the multiprojective space $Y' := \mathbb{P}^1 \times \mathbb{P}^2$. Fix $o \in \mathbb{P}^2$ and $a_1, a_2, a_3 \in \mathbb{P}^1$ with $a_i \ne a_j$ for all $i \ne j$. Let v_i be the general tangent vector of Y' at (a_i, o). We have $\dim \langle \nu(v_1 \cup v_2 \cup v_3) \rangle = 4$ (and so $\nu(v_1 \cup v_2 \cup v_3)$ is not linearly independent), $\dim \langle \nu(B) \rangle = 3$ for any degree 5 subscheme $B \subset v_1 \cup v_2 \cup v_3$, and $v_1 \cup v_2 \cup v_3$ has only 3 degree 5 subschemes. Thus $v_1 \cup v_2 \cup v_3$ minimally spans a general $q \in \langle \nu(v_1 \cup v_2 \cup v_3) \rangle$. This example shows that the linear independence assumed in Proposition 1.6 is not a consequence of the existence of some $q \in \langle \nu(A) \rangle$ such that A minimally spans q. Now assume $n_1 + \cdots + n_k > 3$, i.e., assume $Y' \ne Y$. Take a general $p \in Y \setminus Y'$ and set $A := v_1 \cup v_2 \cup v_3 \cup \{p\}$. A minimally spans a general $q \in \langle \nu(A) \rangle$, but $\nu(A)$ is not linearly independent; this is an example relevant to both Theorem 1.4 and Proposition 1.6.

Let $D \subset Y$ be an effective divisor. For any scheme $Z \subset Y$ let $\mathrm{Res}_D(Z)$ denote the residual scheme of Z with respect to D, i.e., the closed subscheme of Y with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. We have $\mathrm{Res}_D(Z) \subseteq Z$. If Z is zero-dimensional, then $\deg(Z) = \deg(\mathrm{Res}_D(Z)) + \deg(Z \cap D)$. Note that $Z \subset D$ if and only if $\mathrm{Res}_D(Z) = \emptyset$.

Remark 3.4. Assume $k \geq 3$. Then $\mathcal{Z}(q) = \mathcal{S}(q)$ and $|\mathcal{S}(q)| = 1$ for a general $q \in \sigma_2(X)$.

Proof. Since $k \geq 2$ we have $X \subsetneq \sigma_2(X)$ and so $c_X(q) = 2$. A general $u \in \tau(X)$ has rank k. Since $k \geq 3$, we get that $\tau(X)$ is a proper subvariety of $\sigma_2(X)$. Since q is general, we get $\mathcal{Z}(q) = \mathcal{S}(q)$. The generic uniqueness is very classical and true for $k = 3$, $n_1 = n_2 = n_3 = 1$ by the case $a = 2$ of [17, Theorem 1.2] and then true for all $k \geq 3$ and $n_i \geq 1$, by an inductive method as in [10]. $\square$

Proposition 3.5. *Take $q \in \tau(X) \setminus X$. We have $|\mathcal{Z}(q)| = 1$ if and only if $r_X(q) \geq 3$.*

Proof. Set $m := r_X(q)$. Since $q \notin X$, we have $r_X(q) > 1$. We have $c_X(q) = 2$. Fix any connected degree 2 zero-dimensional scheme Z such that $q \in \langle \nu(Z) \rangle$. Let $Y' \subseteq Y$ be a minimal multiprojective space such that $q \in \langle \nu(Y') \rangle$. The proof of [1, Theorem 1] gives $Y' = Y[Z] \cong (\mathbb{P}^1)^m$. If $m = 2$ we get that $\mathcal{S}(q)$ is infinite and 2-dimensional and (in characteristic $\neq 2$) $\mathcal{Z}(q) \setminus \mathcal{S}(q)$ is infinite and 1-dimensional. Now assume $m > 2$. Since $m > 2$, each element of $\mathcal{Z}(q)$ is connected. Assume the existence of $A \in \mathcal{Z}(q)$ such that $A \neq Z$. We saw that $Y[A] = Y[Z]$ and so it is sufficient to consider the case $Y = Y'$, i.e., the case $k = m$ and $n_i = 1$ for all i. Write $\{o\} := Z_{\mathrm{red}}$, $\{a\} := A_{\mathrm{red}}$, $o = (o_1, \dots, o_m)$ and $a = (a_1, \dots, a_m)$.

First assume $a = o$. In this case the two distinct lines $\langle \nu(Z) \rangle$ and $\langle \nu(A) \rangle$ meets only at $\nu(a)$ and so $q = \nu(a)$ has rank 1, contradicting our assumption.

Now assume $a \neq o$, say $a_1 \neq o_1$. Set $D := \pi_1^{-1}(a_1)$. We have $\mathrm{Res}_D(A) = \{a\}$ and $\mathrm{Res}_D(Z) = Z$. Assume for the moment $a_j \neq o_j$ for some $j > 1$, say $a_2 \neq o_2$. Set $D' := \pi_2^{-1}(a_2)$. We have $\mathrm{Res}_{D+D'}(A) = \emptyset$ and $\mathrm{Res}_{D+D'}(Z) = Z$. Let $\gamma : (\mathbb{P}^1)^m \to (\mathbb{P}^1)^{m-2}$ denote the projection onto the last $m-2$ factors of $(\mathbb{P}^1)^m$. Since $m > 2$ and Z is general, $\gamma_{|Z}$ is an embedding. Thus $h^1(\mathcal{I}_Z(0, 0, 1 \dots, 1)) = h^1((\mathbb{P}^1)^{m-2}, \mathcal{I}_{\gamma(Z)}(1, \dots, 1)) = 0$. Since $\mathrm{Res}_{D+D'}(A) = \emptyset$ and $\mathrm{Res}_{D+D'}(Z) \neq \emptyset$, [2, Lemma 5.1(a)] gives a contradiction. Now assume $a_j = o_j$ for all $j > 1$. Set $T := \pi_m^{-1}(a_m)$. We have $\mathrm{Res}_T(Z \cup A) = \{o, a\}$ and hence $\mathrm{Res}_T(A) \cap \mathrm{Res}_T(Z) = \emptyset$. Since $a_1 \neq o_1$, we have $a \neq o$ and $h^1(\mathcal{I}_{\{o,a\}}(1, \dots, 1, 0)) = 0$, contradicting [2, Lemma 5.1(a)]. $\square$

Proposition 3.6. *Assume $k \geq 5$ and $n_i \geq 2$ for all i. Let $Z \subset Y$ be a general degree 3 connected curvilinear scheme. Fix $q \in \langle \nu(Z) \rangle$ such that $q \notin \langle \nu(Z') \rangle$ for any $Z' \subsetneq Z$.*

(a) *$\mathcal{Z}(q) = \{Z\}$.*

(b) *$A \supset Z$ for any degree 4 scheme A such that $q \in \langle \nu(A) \rangle$.*

Proof. Since Y is smooth and connected, the set of all its connected curvilinear subschemes with a prescribed degree is irreducible. The proof of [4] gives $Y[Z] \cong (\mathbb{P}^2)^k$ (more precisely, this is the only case with $Y[Z] \cong (\mathbb{P}^2)^k$, i.e., it is the only case with $\dim Y[Z] = 2k$) and $r_X(q) = 2k - 1 > 3$. Thus no element of $\mathcal{Z}(q)$ is reduced. Since $2k - 1 > k$, we have $q \notin \tau(X)$ ([12, Proposition 1.1], [1, Theorem 1]). Thus $c_X(q) = 3$. Assume the existence of $B \in \mathcal{Z}(q) \setminus \{Z\}$. Since $2k-1 > k+1$ and any element of $\tau(X)$ has rank at most k ([12, Proposition 1.1], [1, Theorem 1]), B is connected. By the list of cases (i), (ii), (iii) and (v) in [12, Theorem 1.2] we see that B is curvilinear. Since $r_X(q) = 2k-1$, we see that $\langle \pi_i(B)\rangle = 2$ for all i. Take any $D \in |\mathcal{I}_B(1,1,0,\dots 0)|$ (it exists, because $h^0(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1)) = 9$. We have $\mathrm{Res}_D(B) = \emptyset$ and $\mathrm{Res}_D(Z) \subseteq Z$ and so $\mathrm{Res}_D(B \cup Z) \subseteq Z$. Let $\gamma : Y \to \prod_{i=3}^k \mathbb{P}^{n_i}$ (resp. $\alpha : Y \to \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$) denote the projection of Y onto the product of its last $k-2$ factors (resp. its first two factors). Since Z is general and $k \ge 4$, $\gamma_{|Z}$ is an embedding and so $h^1(\mathcal{I}_Z(0,0,1,\dots,1)) = h^1(\prod_{i=3}^k \mathbb{P}^{n_i}, \mathcal{I}_{\gamma(Z)}(1,\dots,1))$. Since Z is general, $\gamma(Z)$ is the general degree 3 curvilinear subscheme of $\prod_{i=3}^k \mathbb{P}^{n_i}$ and in particular $\pi_3(\gamma(Z)) = \pi_3(Z)$ spans a plane. Hence $h^1(\prod_{i=3}^k \mathbb{P}^{n_i}, \mathcal{I}_{\gamma(Z)}(1,\dots,1)) = 0$. Thus $h^1(\mathcal{I}_{\mathrm{Res}_D(B\cup Z)}(0,0,1,\dots,1)) = 0$. Since $\mathrm{Res}_D(B) = \emptyset$, [2, Lemma 5.1(a)] gives $Z \subset D$, i.e., $|\mathcal{I}_{\alpha(Z)}(1,1)| \subseteq |\mathcal{I}_{\alpha(A)}(1,1)|$. Since Z is general, $\alpha_{|Z}$ is an embedding and $\alpha(Z)$ is a general connected degree 3 zero-dimensional subscheme of $\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}$. Since $n_1 \ge 2$ and $n_2 \ge 2$, we see that $h^1(\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}, \mathcal{I}_{\alpha(Z)}(1,1)) = 0$ and that $\alpha(Z)$ is the scheme-theoretic base locus of $|\mathcal{I}_{\alpha(Z)}(1,1)|$. Thus $\alpha(Z) = \alpha(B)$, i.e., $\pi_i(Z) = \pi_i(B)$ for $i = 1,2$. Using the first and the i-th factor of Y, $i = 3,\dots,k$, of Y instead of the first and the second one, we get $\pi_i(B) = \pi_i(Z)$ for all i, i.e., $B = Z$, a contradiction.

Now we prove part (b). For every $i \in \{1,\dots,k\}$ let $\varepsilon_i \in \mathbb{N}^k$ be the multiindex $(d_1,\dots,d_k)$ with $d_i = 1$ and $d_j = 0$ for all $j \ne 1$. By part (a) we may assume $\deg(A) = 4$. We repeat the proof of (a) with A instead of B. We get $\deg(\alpha(A)) = 4$ (i.e., $\alpha_{|A}$ is an embedding) and $|\mathcal{I}_{\alpha(A)}(1,1)| \subsetneq |\mathcal{I}_{\alpha(Z)}(1,1)|$. Since $\dim|\mathcal{I}_{\alpha(Z)}(1,1)| = \dim|\mathcal{O}_{\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}}(1,1)|-3$, we get $h^1(\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}, \mathcal{I}_{\alpha(A)}(1,1)) = 0$. Since $\alpha_{|A}$ is an embedding, we get $h^1(\mathcal{I}_A\varepsilon_1+\varepsilon_2) = 0$. Using any two other factors of Y instead of the first two ones we get $h^1(\mathcal{I}_A\varepsilon_i+\varepsilon_j) = 0$ for all $i \ne j$ and hence $h^1(\mathcal{I}_A(a_1,\dots,a_k) = 0$ for all $(a_1,\dots,a_k) \in \mathbb{N}^k$ with at least 2 positive components. Since $k \ge 4$ we get $h^1(\mathcal{I}_{A'}(1,\dots,1,0,0) = 0$ for any $A' \subseteq A$. Take $T \in |\mathcal{I}_Z\varepsilon_{k-1}+\varepsilon_k|$. We have $\mathrm{Res}_T(Z) = \emptyset$ and so $\mathrm{Res}_T(Z\cup A) \subseteq A$. Since $\mathrm{Res}_T(Z) = \emptyset$ and $h^1(\mathcal{I}_{\mathrm{Res}_T(Z\cup A)}(1,\dots,1,0,0)) = 0$, [2, Lemma 5.1(a)] gives $A \subset T$. Thus $|\mathcal{I}_Z(0,\dots,0,1,1)| \subseteq |\mathcal{I}_A(0,\dots,0,1,1)|$. Since we proved that $\dim|\mathcal{I}_A(0,\dots,0,1,1)| = \dim|\mathcal{O}_Y(0,\dots,0,1,1)| - 4$, we get a contradiction. □

3.1. A toy example

Here we give a toy example, which shows why we think that our approach works in many cases. Suppose you have $q \in \mathbb{P}^r$ and a zero-dimensional scheme $A \subset Y$ minimally spanning q with $A = A_1 \sqcup A_2$. Assume that we may control A_2 so that for any q_2 minimally spanned by A_2 we may find a "nice" $B_1 \subset Y$ with $q_2 \in \langle \nu(B_1)\rangle$; here "nice" may mean that it has al least one degree 1 component, so that $A_1 \cup B_1$ has at least one degree 1 connected component, so that we may apply Theorem 1.4 to $A_2 \cup B_1$. There are $q_1 \in \langle \nu(A_1)\rangle$ and $q_2 \in \langle \nu(A_2)\rangle$

such that $q \in \langle \{q_1, q_2\} \rangle$ and the points q_1, q_2 are uniquely determined by q if and only if $\langle \nu(A_1) \rangle \cap \langle \nu(A_2) \rangle = \emptyset$. Using q_2 and A_2 we get B_1. We have $q \in \langle \nu(A_1 \cup B_1) \rangle$. If $A_1 \cup B_1$ minimally span q (this is always the case if $\langle \nu(A_1) \rangle \cap \langle \nu(B_1) \rangle = \emptyset$), then we may use $A_1 \cup B_1$, except that $\dim \langle \nu(A_1 \cup B_1) \rangle$ and $\deg(A_1 \cup B_1)$ depends on B_1 (you need to get $\deg(B_1)$ as small as possible); this is exactly how we used Lemma 2.2 to prove Theorem 1.4. If $A_1 \cup B_1$ does not minimally span q, we take $B' \subset A_1 \cup B_1$ which minimally spans q. If $A_1 \cap B_1 = \emptyset$ and $B' \supseteq B_1$, we use B', because B_1 is a union of some of the connected components of B'. The practical implementation of this approach (and the technical problems that may arise in its use) are mirrored in our use of Lemma 2.3 in the proof of Proposition 1.6.

References

[1] E. Ballico, *Tensor ranks on tangent developable of Segre varieties*, Linear Multilinear Algebra **61** (2013), no. 7, 881–894.

[2] E. Ballico and A. Bernardi, *Stratification of the fourth secant variety of Veronese varieties via the symmetric rank*, Adv. Pure Appl. Math. **4** (2013), no. 2, 215–250.

[3] ———, *Curvilinear schemes and maximum rank of forms*, Matematiche (Catania) **72** (2017), no. 1, 137–144.

[4] ———, *On the ranks of the third secant variety of Segre-Veronese embeddings*, arXiv: 1701.06845.

[5] E. Ballico, A. Bernardi, and L. Chiantini, *On the dimension of contact loci and the identifiability of tensors*, arXiv: 1706.02746.

[6] E. Ballico, A. Bernardi, L. Chiantini, and E. Guardo, *Bounds on the tensor rank*, arXiv: 1705.02299.

[7] A. Bernardi, J. Brachat, and B. Mourrain, *A comparison of different notions of ranks of symmetric tensors*, Linear Algebra Appl. **460** (2014), 205–230.

[8] A. Bernardi and K. Ranestad, *On the cactus rank of cubics forms*, J. Symbolic Comput. **50** (2013), 291–297.

[9] C. Bocci and L. Chiantini, *On the identifiability of binary Segre products*, J. Algebraic Geom. **22** (2013), no. 1, 1–11.

[10] C. Bocci, L. Chiantini, and G. Ottaviani, *Refined methods for the identifiability of tensors*, Ann. Mat. Pura Appl. (4) **193** (2014), no. 6, 1691–1702.

[11] W. Buczyńska and J. Buczyński, *Secant varieties to high degree veronese reembeddings, catalecticant matrices and smoothable Gorenstein schemes*, J. Algebraic Geom. **23** (2014), 63–90.

[12] J. Buczyński and J. M. Landsberg, *On the third secant variety*, J. Algebraic Combin. **40** (2014), no. 2, 475–502.

[13] M. V. Catalisano, L. Chiantini, A. V. Geramita, and A. Oneto *Waring-like decompositions of polynomials, 1*, Linear Algebra Appl. **533** (2017), 311–325.

[14] L. Chiantini and C. Ciliberto, *On the concept of k-secant order of a variety*, J. London Math. Soc. (2) **73** (2006), no. 2, 436–454.

[15] ———, *On the dimension of secant varieties*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 5, 1267–1291.

[16] L. Chiantini, M. Mella, and G. Ottaviani, *One example of general unidentifiable tensors*, J. Algebr. Stat. **5** (2014), no. 1, 64–71.

[17] L. Chiantini and G. Ottaviani, *On generic identifiability of 3-tensors of small rank*, SIAM J. Matrix Anal. Appl. **33** (2012), no. 3, 1018–1037.

[18] L. Chiantini, G. Ottaviani, and N. Vannieuwenhoven, *An algorithm for generic and low-rank specific identifiability of complex tensors*, SIAM J. Matrix Anal. Appl. **35** (2014), no. 4, 1265–1287.

[19] ———, *On generic identifiability of symmetric tensors of subgeneric rank*, Trans. Amer. Math. Soc. **369** (2017), no. 6, 4021–4042.

[20] ———, *Effective criteria for specific identifiability of tensors and forms*, SIAM J. Matrix Anal. Appl. **38** (2017), no. 2, 656–681.

[21] H. Derksen, *Kruskal's uniqueness inequality is sharp*, Linear Algebra Appl. **438** (2013), no. 2, 708–712.

[22] I. Domanov and L. De Lathauwer, *On the uniqueness of the canonical polyadic decomposition of third-order tensors—Part I*: *Basic results and uniqueness of one factor matrix*, SIAM J. Matrix Anal. Appl. **34** (2013), no. 3, 855–875.

[23] ———, *On the uniqueness of the canonical polyadic decomposition of third-order tensors—Part II*: *Uniqueness of the overall decomposition*, SIAM J. Matrix Anal. Appl. **34** (2013), no. 3, 876–903.

[24] ———, *Generic uniqueness conditions for the canonical polyadic decomposition and INDSCAL*, SIAM J. Matrix Anal. Appl. **36** (2015), no. 4, 1567–1589.

[25] F. Galuppi and M. Mella, *Identifiability of homogeneous polynomials and Cremona Transformations*, Preprint arXiv:1606.06895, 2016; To appear in J. Reine Angew. Math.; DOI: https://doi.org/10.1515/crelle-2017-0043.

[26] A. Iarrobino and V. Kanev, *Power Sums, Gorenstein Algebras, and Determinantal Loci*, Lecture Notes in Mathematics, **1721**, Springer-Verlag, Berlin, 1999.

[27] J. B. Kruskal, *Three-way arrays*: *rank and uniqueness of trilinear decompositions, with application to arithmetic complexity and statistics*, Linear Algebra and Appl. **18** (1977), no. 2, 95–138.

[28] J. M. Landsberg, *Tensors*: *geometry and applications*, Graduate Studies in Mathematics, **128**, American Mathematical Society, Providence, RI, 2012.

[29] M. Mella, *Singularities of linear systems and the Waring problem*, Trans. Amer. Math. Soc. **358** (2006), no. 12, 5523–5538.

[30] K. Ranestad and F.-O. Schreyer, *On the rank of a symmetric form*, J. Algebra **346** (2011), 340–342.

[31] N. D. Sidiropoulos and R. Bro, *On the uniqueness of multilinear decomposition of N-way arrays*, J. Chemometrics **14** (2000), 229–239.

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