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UNIQUENESS OF SOLUTIONS OF A CERTAIN NONLINEAR ELLIPTIC EQUATION ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we prove that if every bounded $\mathcal{A}$-harmonic function on a complete Riemannian manifold M is asymptotically constant at infinity of p-nonparabolic ends of M, then each bounded $\mathcal{A}$-harmonic function is uniquely determined by the values at infinity of p-nonparabolic ends of M, where $\mathcal{A}$ is a nonlinear elliptic operator of type p on M. Furthermore, in this case, every bounded $\mathcal{A}$-harmonic function on M has finite energy.

1. Introduction

In this paper, we consider a Liouville type theorem for a certain nonlinear elliptic operator on a complete Riemannian manifold. The classical Liouville theorem states that any bounded harmonic function on $\mathbf{R}^2$ must be constant. In 1975, Yau [17] proved a remarkable result that any positive harmonic function on a complete Riemannian manifold with nonnegative Ricci curvature must be constant. Later, Li and Tam [11] pointed out that every positive harmonic function on a complete Riemannian manifold with nonnegative sectional curvature outside a compact subset is asymptotically constant at infinity on each end. By the result of Sung, Tam, and Wang [16], if every bounded harmonic function on a complete Riemannian manifold is asymptotically constant at infinity of each nonparabolic end of the manifold, then there exists a unique bounded harmonic function taking the given values at infinity of each nonparabolic end of the manifold.

On the other hand, in the case of nonlinear elliptic operators on a complete Riemannian manifold, one cannot use the well known procedure, based on linearity, for harmonic functions. In [10], the present author gave an barrier argument at infinity of ends to overcome the obstacle due to the nonlinearity

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and proved that if every bounded solution for a certain nonlinear elliptic operator of type p on each p-nonparabolic end of a complete Riemannian manifold is asymptotically constant at infinity of the end, then there exists a bounded solution for the operator taking the given values at infinity of each p-nonparabolic end of the manifold.

In this paper, we prove the uniqueness of bounded solutions for the problem of a certain nonlinear elliptic operator $\mathcal{A}$ in the following setting: Let M be a complete Riemannian manifold and Ω be an open subset of M. Let $W^{1,p}(\Omega)$ be the Sobolev space of all functions $u \in L^p(\Omega)$ whose distributional gradient ∇u also belongs to $L^p(\Omega)$. We equip $W^{1,p}(\Omega)$ with the norm $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$. The space $W_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

Let $T\Omega = \cup_{x\in\Omega} T_xM$. We will consider functionals associated with $\mathbf{F} : T\Omega \to \mathbf{R}$ and a constant $1 < p < \infty$, where

($A1$) the mapping $\mathbf{F}_x = \mathbf{F}|_{T_xM} : T_xM \to \mathbf{R}$ is strictly convex and differentiable for all $x \in \Omega$, and the mapping $x \mapsto \mathbf{F}_x(\xi)$ is measurable whenever ξ is;

($A2$) there exist constants $0 < C_1 \le C_2 < \infty$ such that
$$C_1|\xi|^p \le \mathbf{F}_x(\xi) \le C_2|\xi|^p$$
for all $x \in \Omega$ and $\xi \in T_xM$.

We will write
$$\mathbf{J}(u,\Omega) = \int_\Omega \mathbf{F}_x(\nabla u).$$
Let $\mathcal{A}_x(\xi) = (A^1(\xi), A^2(\xi), \ldots, A^n(\xi))$ be defined by
$$A^i(\xi) = \frac{\partial}{\partial \xi^i}\,\mathbf{F}_x(\xi)$$
for $i = 1, 2, \ldots, n$. Then $\mathcal{A}$ also the following properties: (See [14])

($A3$) the mapping $\mathcal{A}_x = \mathcal{A}|_{T_xM} : T_xM \to T_xM$ is continuous for a.e. $x \in \Omega$, and the mapping $x \mapsto \mathcal{A}_x(\xi)$ is a measurable vector field whenever ξ is;

for a.e. $x \in \Omega$ and for all $\xi \in T_xM$,

($A4$) $\langle \mathcal{A}_x(\xi), \xi\rangle \ge C_1|\xi|^p$;

($A5$) $|\mathcal{A}_x(\xi)| \le C_2|\xi|^{p-1}$;

($A6$) $\langle \mathcal{A}_x(\xi_1) - \mathcal{A}_x(\xi_2), \xi_1 - \xi_2\rangle > 0$ whenever $\xi_1 \ne \xi_2$.

A function u in $W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution (supersolution, subsolution, respectively) of the equation

$$-\operatorname{div}\mathcal{A}_x(\nabla u) = 0 \ (\ge 0, \le 0, \text{respectively}) \tag{1}$$

in Ω if
$$\int_\Omega \langle \mathcal{A}_x(\nabla u), \nabla\phi\rangle = 0 \ (\ge 0, \le 0, \text{respectively})$$
for any (nonnegative, respectively) $\phi \in C_0^\infty(\Omega)$. We say that a function u is $\mathcal{A}$-harmonic (of type p) if u is a continuous solution of the equation (1). In the

typical case $\mathcal{A}_x(\xi) = \xi|\xi|^{p-2}$, $\mathcal{A}$-harmonic functions are called p-harmonic and, in particular, if $p = 2$, we obtain harmonic functions. An important property of $\mathcal{A}$-harmonic functions is the comparison principle as follows: If $u \in W^{1,p}(\Omega)$ is a supersolution and $v \in W^{1,p}(\Omega)$ is a subsolution of (1), respectively, in an open set Ω and $\min\{u - v, 0\} \in W_0^{1,p}(\Omega)$, then $u \geq v$ a.e. in Ω. In particular, if both u and v are $\mathcal{A}$-harmonic in a bounded set Ω and $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω. (See [4].) A Green's function $G = G(x, \cdot)$ for the elliptic operator $\mathcal{A}$ on M denotes (if exists) a positive solution of the equation

$$-\operatorname{div}\mathcal{A}(\nabla G) = \delta_x$$

for each $x \in M$, in the sense of distributions, i.e.,

$$\int_M \langle \mathcal{A}(\nabla G), \nabla \phi \rangle = \phi(x)$$

for any $\phi \in C_0^\infty(M)$. It is known that a Green's function G for the elliptic operator $\mathcal{A}$ on M exists if and only if M has positive p-capacity, i.e., there exists a compact subset $K \subset M$ such that

$$\mathrm{Cap}_p(K, \infty, M) = \inf_\phi \int_M |\nabla \phi|^p > 0,$$

where the infimum is taken over all functions $\phi \in C_0^\infty(M)$ with $\phi = 1$ on K. We say that a complete Riemannian manifold M is p-parabolic if M does not admit any Green's function G. Otherwise, M is called p-nonparabolic.

We now introduce additional assumptions on $\mathbf{F}$ as follows:

($A7$) $\mathcal{A}_x(\lambda\xi) = \lambda|\lambda|^{p-2}\mathcal{A}_x(\xi)$ whenever $\lambda \in \mathbf{R}, \lambda \neq 0$;
for any $\xi_1, \xi_2 \in T_xM$,

($A8$) in case $2 \leq p < \infty$,

$$\mathbf{F}_x\Big(\frac{\xi_1 + \xi_2}{2}\Big) + \mathbf{F}_x\Big(\frac{\xi_1 - \xi_2}{2}\Big) \leq \frac{1}{2}(\mathbf{F}_x(\xi_1) + \mathbf{F}_x(\xi_2));$$

in case $1 < p \leq 2$,

$$\mathbf{F}_x\Big(\frac{\xi_1 + \xi_2}{2}\Big)^{\tilde{p}} + \mathbf{F}_x\Big(\frac{\xi_1 - \xi_2}{2}\Big)^{\tilde{p}} \leq \Big(\frac{1}{2}(\mathbf{F}_x(\xi_1) + \mathbf{F}_x(\xi_2))\Big)^{\tilde{p}},$$

where $\tilde{p} = 1/(p-1)$.

Using Clarkson's inequality, the assumption ($A8$) is satisfied in the typical case $\mathbf{F}(\xi) = \frac{1}{p}|\xi|^p$, i.e., p-harmonic case. (See [5].)

In the above setting, we can prove that bounded $\mathcal{A}$-harmonic functions on a complete Riemannian manifold M uniquely are determined in terms of p-nonparabolic ends of M as follows:

Theorem 1.1. *Let M be a complete Riemannian manifold having p-nonparabolic ends, explained in Section 2, $E_1, E_2, \ldots, E_l$. Suppose that every bounded $\mathcal{A}$-harmonic function on E_i is asymptotically constant at infinity of*

E_i. Then given real numbers $a_1, a_2, \ldots, a_l \in \mathbf{R}$, there exists a unique bounded $\mathcal{A}$-harmonic function f on M such that for each $i = 1, 2, \ldots, l$,

$$\lim_{x\to\infty, x\in E_i} f(x) = a_i. \tag{2}$$

Furthermore, every bounded $\mathcal{A}$-harmonic function on M has finite energy.

There are various examples that every bounded $\mathcal{A}$-harmonic function on ends is asymptotically constant at infinity of the ends. An example satisfying the asymptotically constant property is ends satisfying the volume doubling condition, the Poincaré inequality and the finite covering condition near infinity of the ends, expounded later. In fact, these conditions are valid on ends of a complete Riemannian manifold with nonnegative Ricci curvature outside a compact subset and finite Betti number. (See [13] and [15].) On the other hand, the asymptotically constant property also holds the class being roughly isometric to ends satisfying the volume doubling condition, the Poincaré inequality and the finite covering condition near infinity of the ends. Moreover, the number of ends and the p-parabolicity of ends are preserved under rough isometries between complete Riemannian manifolds. (See [6] and [9].) Therefore, as corollaries of our result, we have the following results:

Corollary 1.2. *Let M be a complete Riemannian manifold being roughly isometric to a complete Riemannian manifold having ends, each of which satisfies the volume doubling condition, the Poincaré inequality and the finite covering condition near infinity. Let M have p-nonparabolic ends $E_1, E_2, \ldots, E_l$. Then given real numbers $a_1, a_2, \ldots, a_l \in \mathbf{R}$, there exists a unique bounded $\mathcal{A}$-harmonic function f on M satisfying* (2)*. Furthermore, every bounded $\mathcal{A}$-harmonic function on M has finite energy.*

Corollary 1.3. *Let M be a complete Riemannian manifolds being roughly isometric to a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set and finite first Betti number. Let M have p-nonparabolic ends $E_1, E_2, \ldots, E_l$. Then given real numbers $a_1, a_2, \ldots, a_l \in \mathbf{R}$, there exists a unique bounded $\mathcal{A}$-harmonic function f on M satisfying* (2)*. Furthermore, every bounded $\mathcal{A}$-harmonic function on M has finite energy.*

Corollary 1.4. *Let $M_1, M_2, \ldots, M_k$ be complete Riemannian manifolds with nonnegative Ricci curvature and M be a complete Riemannian manifold being roughly isometric to a connected sum of $M_1, M_2, \ldots, M_k$. Then the set of all bounded $\mathcal{A}$-harmonic functions on M is one to one corresponding to $\mathbf{R}^l$, where l is the number of M_i's being p-nonparabolic. Furthermore, every bounded $\mathcal{A}$-harmonic function on M has finite energy.*

2. Proof of main results

We begin with defining ends of a complete Riemannian manifold M: Let o be a fixed point in M and $B_r(o)$ be the metric r-ball centered at o. We denote by

$\sharp(r)$ the number of unbounded components of $M \setminus B_r(o)$, and each unbounded components of $M \setminus B_r(o)$ is called an end of M corresponding to $B_r(o)$. It is easy to prove that $\sharp(r)$ is nondecreasing in $r > 0$. Let $\lim_{r\to\infty}\sharp(r) = k$, where k may be infinity, then we say that the number of ends of M is k. In the case when $k < \infty$, we can choose $r_0 > 0$ such that $\sharp(r) = k$ for all $r \geq r_0$.

We classify all ends of a complete Riemannian manifold by the p-parabolicity as follows:

Definition 2.1. We say that an end E of a complete Riemannian manifold M is p-nonparabolic if E has positive p-capacity at infinity, i.e., for some $r_1 > 0$,

$$\mathrm{Cap}_p(E \setminus B_{r_1}(o)) = \inf_u \int_{E\setminus B_{r_1}(o)} |\nabla u|^p > 0,$$

where the infimum is taken over all continuous functions u on $E \setminus B_{r_1}(o)$ such that u is compactly supported, smooth on $E\setminus \overline{B}_{r_1}(o)$ and $u = 1$ on $\partial B_{r_1}(o) \cap E$. Otherwise, E is called p-parabolic.

On the other hand, $\mathcal{A}$-harmonic functions are quasi-minimizers of p-Dirichlet integral, i.e., if $u \in W^{1,p}(\Omega)$ is a solution of (1) and $u - \phi \in W_0^{1,p}(\Omega)$, then there exists a constant $C < \infty$ such that

$$\int_\Omega |\nabla u|^p \leq C \int_\Omega |\nabla \phi|^p.$$

Therefore, the p-parabolicity of an end E is equivalent to the existence of a continuous function u_E, called an $\mathcal{A}$-harmonic measure of E, on E such that

$$\begin{cases} \mathcal{A}\, u_E = 0 & \text{in} \quad E \setminus \overline{B}_{r_1}(o); \\ \quad u_E = 0 & \text{on} \quad M \setminus (E \setminus \overline{B}_{r_1}(o)); \\ \sup_{E\setminus B_{r_1}(o)} u_E = 1. & \end{cases}$$

In Theorem 1.1, the ends $E_1, E_2, \ldots, E_l$ mean the ends of a complete Riemannian manifold corresponding to $B_{r_0}(o)$ for some $r_0 > 0$. In the case when the manifold M has no p-nonparabolic ends, every bounded $\mathcal{A}$-harmonic function on M must be constant. Hence the case is the trivial case of our problem. In this paper, we assume that each manifold has at least one p-nonparabolic end, unless otherwise specified.

As mentioned above, the existence of bounded $\mathcal{A}$-harmonic function satisfying (2) was already proved in [10]. Therefore, we have only to prove the uniqueness of the solution as follows:

Proof of Theorem 1.1. Without loss of generality, we may assume that f is a bounded $\mathcal{A}$-harmonic function such that for each $i = 1, 2, \ldots, l$,

$$\lim_{x\to\infty, x\in E_i} f(x) = a_i > 0.$$

Then f is nonnegative on M. Let $m = \max\{a_i : i = 1, 2, \ldots, l\}$.

Since f and the $\mathcal{A}$-harmonic measure u_{E_i} of E_i are asymptotically constant at infinity of the end E_i, for any ϵ with $0 < \epsilon < 1$, there exists $r_2 > \max\{r_0, r_1\}$ such that for each $i = 1, 2, \ldots, l$,

$$|f(x) - a_i| < a_i\epsilon \quad \text{whenever} \quad x \in E_i \setminus B_{r_2}(o),$$

where $r_0 > 0$ and $r_1 > 0$ are given above.

Choose a sequence $\{u_r\}_{r>r_2}$ of continuous functions such that

$$\begin{cases} \mathcal{A}u_r = 0 & \text{in} \quad B_r(o); \\ \quad u_r = a_i u_{E_i} & \text{on} \quad E_i \setminus B_r(o); \\ \quad u_r = 0 & \text{on} \quad M \setminus ((\cup_{i=1}^l E_i) \cup B_r(o)), \end{cases}$$

where $i = 1, 2, \ldots, l$. Obviously, u_r has finite energy on M. Since $f \geq 0$ on M and $(1-\epsilon)a_i u_{E_i} \leq f$ on $\partial B_r(o) \cap E_i$ for each $i = 1, 2, \ldots, l$, by the comparison principle,

$$0 \leq (1 - \epsilon)u_r \leq f \ \text{ in } \ B_r(o).$$

On the other hand, there exists a constant $\alpha < \infty$ such that

$$\alpha = \inf_{\eta} \mathbf{J}(\sum_{i=1}^{l} a_i u_{E_i} - \eta, M),$$

where the infimum is taken over all compactly supported smooth functions η on M. Since u_r is $\mathcal{A}$-harmonic function on $B_r(o)$ and $u_r = \sum_{i=1}^l a_i u_{E_i}$ on $M \setminus B_r(o)$, u_r minimizes $\mathbf{J}(w, M)$ where w's are smooth functions such that $w = \sum_{i=1}^l a_i u_{E_i}$ on $M \setminus B_r(o)$. (See Theorem 2.96 of [14].) Thus $\mathbf{J}(u_r, M)$ is decreasing in $r > 0$, hence we have

$$\lim_{r\to\infty} \mathbf{J}(u_r, M) = \alpha.$$

Note that $u_r - \sum_{i=1}^l a_i u_{E_i} = 0$ on $\partial B_r(o)$. In case $2 \leq p < \infty$, by $(A8)$,

$$\begin{aligned} \alpha &\leq \mathbf{J}((u_r + u_{r'})/2, M) \\ &\leq \mathbf{J}((u_r + u_{r'})/2, M) + \mathbf{J}((u_r - u_{r'})/2, M) \\ &\leq 2^{-1}(\mathbf{J}(u_r, M) + \mathbf{J}(u_{r'}, M)) \to \alpha \quad \text{as} \quad r, r' \to \infty. \end{aligned}$$

In case $1 < p \leq 2$, by $(A8)$,

$$\begin{aligned} \alpha^{\tilde{p}} &\leq \mathbf{J}((u_r + u_{r'})/2, M)^{\tilde{p}} \\ &\leq \mathbf{J}((u_r + u_{r'})/2, M)^{\tilde{p}} + \mathbf{J}((u_r - u_{r'})/2, M)^{\tilde{p}} \\ &\leq 2^{-\tilde{p}}(\mathbf{J}(u_r, M) + \mathbf{J}(u_{r'}, M))^{\tilde{p}} \to \alpha^{\tilde{p}} \quad \text{as} \quad r, r' \to \infty, \end{aligned}$$

where $\tilde{p} = 1/(p - 1)$. These imply that

$$\mathbf{J}(u_r - u_{r'}, M) \to 0 \quad \text{as} \quad r, r' \to \infty.$$

Since the sequence $\{u_r\}$ is uniformly bounded, it is equicontinuous and by Ascoli's theorem, there exists a subsequence $\{u_{r_k}\}$ converging uniformly to the limit function u on any compact subset of M. Then by the result [4], the limit function u is an $\mathcal{A}$-harmonic function such that

$$(1-\epsilon)u \le f \text{ on } M, \tag{3}$$

and u has finite energy.

On the other hand, since $(1-\epsilon)a_i u_{E_i} \le (1-\epsilon)u_r \le f$ in $B_r(o) \cap E_i$ for each $i = 1, 2, \ldots, l$, we have

$$(1-\epsilon)a_i u_{E_i} \le (1-\epsilon)u \le f \text{ in } E_i. \tag{4}$$

Since $\epsilon > 0$ is arbitrarily chosen, by (3) and (4), $u \le f$ on M and for each $i = 1, 2, \ldots, l$,

$$\lim_{x\to\infty, x\in E_i} u(x) = a_i.$$

Now choose another sequence $\{v_r\}_{r>r_2}$ of continuous functions such that

$$\begin{cases} \mathcal{A}v_r = 0 & \text{in} \quad B_r(o); \\ v_r = m - (m-a_i)u_{E_i} & \text{on} \quad E_i \setminus B_r(o); \\ v_r = m & \text{on} \quad M \setminus ((\cup_{i=1}^{l} E_i) \cup B_r(o)), \end{cases}$$

where $i = 1, 2, \ldots, l$. Similarly arguing as in the case of the sequence $\{u_r\}_{r>r_2}$, there exists an energy finite $\mathcal{A}$-harmonic function v on M such that

$$f \le (1+\epsilon)v \text{ on } M. \tag{5}$$

Furthermore, for each $i = 1, 2, \ldots, l$,

$$f \le (1+\epsilon)v \le (1+\epsilon)(m - (m-a_i)u_{E_i}) \text{ in } E_i. \tag{6}$$

Since $\epsilon > 0$ is arbitrarily chosen, by (5) and (6), $f \le v$ on M and for each $i = 1, 2, \ldots, l$,

$$\lim_{x\to\infty, x\in E_i} v(x) = a_i.$$

From the above construction, u and v are energy finite bounded $\mathcal{A}$-harmonic functions on M such that

$$\lim_{x\to\infty, x\in E_i} u(x) = \lim_{x\to\infty, x\in E_i} v(x)$$

for each $i = 1, 2, \ldots, l$. By Theorem 1.1 of [10], each energy finite bounded $\mathcal{A}$-harmonic function is uniquely determined by the values (2). Since both u and v are energy finite bounded $\mathcal{A}$-harmonic functions and

$$\lim_{x\to\infty, x\in E_i} u(x) = \lim_{x\to\infty, x\in E_i} v(x)$$

for each $i = 1, 2, \ldots, l$, we have $u = v$ on M. Since $u \le f \le v$ on M, $u = f = v$ on M, hence f has also finite energy. Consequently, each f is uniquely determined by the values in (2). $\square$

As mentioned above, let us consider some examples of ends satisfying the asymptotically constant at infinity of the ends:

Example. (i) Let E be an end of a complete Riemannian manifold M corresponding to $B_{r_0}(o)$. Suppose that E satisfies the volume doubling condition and the Poincaré inequality near infinity of E as follows:

$(D)_E$ for given $0 < \alpha < 1/2$, there is a constant $C < \infty$ depending only on α such that for any point $x \in \partial B_R(o) \cap E$ and any $r_0 < r < R/2$,

$$\mathrm{vol} B_r(x) \le C \,\mathrm{vol} B_{\alpha r}(x);$$

$(P)_E$ there exist a constant $C < \infty$ and an integer $k \in \mathbf{N}$ such that for any point $x \in \partial B_R(o) \cap E$, any $r_0 < r < R/2$ and all $f \in C^\infty(B_r(x))$,

$$\int_{B_{r/k}(x)} |f - \overline{f}| \le Cr(\mathrm{vol} B_r(x))^{1-1/p} \Big(\int_{B_r(x)} |\nabla f|^p \Big)^{1/p},$$

where $\overline{f} = (\mathrm{vol} B_{r/k}(x))^{-1} \int_{B_{r/k}(x)} f$.

By the result of [7], there exists a constant $C < \infty$ such that for any nonnegative $\mathcal{A}$-harmonic function f in $B_r(x)$,

$$\sup_{B_{r/2}(x)} f \le C \inf_{B_{r/2}(x)} f.$$

We now add the finite covering condition near infinity of E as follows:

$(FC)_E$ for given $0 < \alpha < 1/4$, there exist a positive integer $m = m(\alpha)$ and points $x_1, x_2, \ldots, x_m$ in $\partial C_{E,R}$ such that $\partial C_{E,R} \subset \bigcup_{i=1}^m B_{\alpha R}(x_i)$ and $\bigcup_{i=1}^m B_{\alpha R}(x_i)$ is connected, where $C_{E,R}$ denotes the unbounded component of $E \setminus B_R(o)$.

This condition gives the Harnack inequality near infinity of E, i.e., there exists a constant $C < \infty$ such that for any nonnegative $\mathcal{A}$-harmonic function f on E,

$$\sup_{\partial C_{E,R}} f \le C \inf_{\partial C_{E,R}} f.$$

The Harnack inequality near infinity of E forces each bounded $\mathcal{A}$-harmonic function to be asymptotically constant at infinity of E. (See [7] for the proof.)

(ii) Let M be a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set and E be an end of M. Then the volume doubling condition $(D)_E$ and the Poincaré inequality $(P)_E$ are satisfied on E. In addition, if the manifold M has finite first Betti number, then the finite covering condition $(FC)_E$ also holds on E. (See [12] and [13].)

(iii) As the simplest case, let M be a complete Riemannian manifold with nonnegative Ricci curvature everywhere. Then by the splitting theorem of Cheeger and Gromoll [2], M has at most two ends. Furthermore, for each end E of M, $\partial B_R(o) \cap E$ is connected. (See [1].) Since M satisfies the volume doubling condition globally, the finite covering condition $(FC)_E$ is valid. The Poincaré inequality also holds globally on M.

(iv) Let E be an end being roughly isometric to an end satisfying the volume doubling condition, the Poincaré inequality and the finite covering condition near infinity of the end. By the results of Kanai [8] and of Coulhon and Saloff-Coste [3], the volume doubling conditions and the Poincaré inequality near infinity of ends is invariant under rough isometries. Applying the program of [10] together with these invariance, there exist a sequence $\{H_R\}$ of hypersurfaces in E and a constant $C < \infty$ such that for any nonnegative $\mathcal{A}$-harmonic function f in E,

$$\sup_{H_R} f \leq C \inf_{H_R} f,$$

where $d(o, H_R) \to \infty$ as $R \to \infty$, and each H_R divides E into a bounded subset and the unbounded component of $E \setminus H_R$. Thus by [7], each bounded $\mathcal{A}$-harmonic function is asymptotically constant at infinity of E.

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