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ON CHARACTERIZING THE GAMMA AND THE BETA q-DISTRIBUTIONS

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Abstract. In this paper, our central focus is upon gamma and beta q-distributions from a probabilistic viewpoint. The gamma and the beta q-distributions are characterized by investing the nature of the joint q-probability density function through the q-independence property and the q-Laplace transform.

1. Introduction

Quantum calculus is the modern name for the investigation of calculus without limits. Recently, many researchers have focused on the q-calculus [11, 14, 16, 17, 24] which corresponds to the link between mathematics and physics. The quantum calculus began with F. H. Jackson [8] in the early twentieth century. The book of Quantum Calculus [19] published by Kac and Cheung covers many of the fundamental aspects of quantum calculus. Chung et al. [13] defined the q-addition operator and discussed its properties. They used it in the properties of the q-logarithmic function and q-exponential.

The quantum calculus has a lot of applications in different mathematical areas such as number theory, difference equation (see [7]), orthogonal polynomials, probability theory, ...

In mathematical physics and probability, the q-distribution is more general than classical distribution. It was introduced by Díaz [12,13] in the continuous case and by Charalambos [9] in the discrete case. The construction of a q-distribution is the construction of a q-analogue of ordinary distribution. Mathai in [23] introduced the q-analogue of the gamma distribution with respect to Lebesgue measure. In this paper, gamma q-distribution is studied with respect to Jackson q-integral. If q goes to 1, we obtain the ordinary calculus. This condition is the necessary condition in the theory of q-calculus.

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Charalambos was the pioneer to coin the notion of q-distribution in the discrete case [9]. As for the continuous case, Díaz et al. identified the Gaussian q-distribution [13].

A function $p_q(x)$ is a q-probability density provided that it satisfies $p_q(x) \geq 0, \forall\, x \in \mathbb{R}$, and $\int_{\mathbb{R}} p_q(x) d_q x = 1$. The q-cumulative distribution function of a real-valued random variable X, is the q-probability that X takes a value less than or equal to x. It gives the area under the probability q-density function from $-\infty$ to x. It is defined by

$$F_q(x) = \mathbb{P}_q(X \leq x) = \int_{-\infty}^{x} p_q(s) d_q s,\ x \in \mathbb{R}.$$

Díaz et al. in [12] defined the gamma q-distribution in terms of

$$\gamma_{q,a}(x) = \frac{1}{\Gamma_q(a)} x^{a-1} E_q^{-qx} \mathbf{1}_{[0, \frac{1}{1-q}]}(x).$$

In 1955 Lukacs [22] proved that X/Y and $X+Y$ are independent if and only if X and Y are gamma distributed with the same scale parameter. Using the moment, in 1978 Findeisen [15] characterized the gamma distribution. Also, in 1999 Hwang and Hu [18] proved a characterization of the gamma distribution by the independence of the sample mean and the sample coefficient of variation. In 1967 I. Kotlarski [21] characterized the gamma distribution by the nature of joint distribution of the two quotients $\frac{X_1}{X_3}$, $\frac{X_2}{X_3}$ for three identically gamma distributed random variables.

Our work stands for an extension to the results given by I. Kotlarski [21].

Let $X_1,\ X_2,\ X_3$ be three positives independent real random variables and let $Y_1 = \frac{X_1}{X_3}$ and $Y_2 = \frac{X_2}{X_3}$.

The necessary and sufficient condition for X_k to be q-gamma distributed with parameters p_k $(k = 1,\ 2,\ 3)$ is that the q-Laplace transform of the couple $(\mathrm{Log}_q Y_1, \mathrm{Log}_q Y_2)$ is given by

$$\frac{\Gamma_q(p_1 - \theta_1)}{\Gamma_q(p_1)} \times \frac{\Gamma_q(p_2 - \theta_2)}{\Gamma_q(p_2)} \times \frac{\Gamma_q(p_3 + \theta_1 \oplus_q \theta_2)}{\Gamma_q(p_3)}.$$

The beta q-distribution is also characterized in the same way.

This paper is structured as follows: In Section 2, some preliminary concepts related to q-derivative, q-integral, q-operator addition and some essential results are presented to build our work. In Section 3, we defined the joint q-density function by using the q-Fubini's theorem. Besides, we introduced the notion of independence and marginal q-distribution. In Section 4, the gamma q-distribution is characterized by the nature of the joint q-distribution of the two quotients $\frac{X_1}{X_3}$, $\frac{X_2}{X_3}$ for three identically q-gamma distributed random variables. Also, the beta q-distribution was characterized by the same way.

2. Preliminaries

In this section, some useful basic definitions [10, 11, 19] are introduced. We shall start with the q-derivative and the Jackson q-integral. Fixing a real number $0 < q < 1$, the q-derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at $x \in \mathbb{R} \setminus \{0\}$ is given by:

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

It is also known as the Jackson derivative.

It is manifestly linear,

$$D_q(f(x) + g(x)) = D_q f(x) + D_q g(x).$$

It has a product rule analogous to the ordinary ones, with two equivalent forms

$$D_q(f(x)g(x)) = g(x)D_q f(x) + f(qx)D_q g(x) = g(qx)D_q f(x) + f(x)D_q g(x).$$

Similarly, it satisfies a quotient rule,

$$D_q(\frac{f(x)}{g(x)}) = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(qx)g(x)}, \ g \neq 0.$$

For an integer $n \geq 1$, we have $D_q x^n = [n]_q x^{n-1}$, where

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}.$$

We also denote, for all $n \in \mathbb{N}$,

$$[n]_q! = \begin{cases} 1 & \text{if } n = 0, \\ [n]_q[n-1]_q \cdots [1]_q & \text{otherwise.} \end{cases}$$

For $x \in \mathbb{R}$,

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

If x goes to ∞, we obtain $[\infty]_q = \frac{1}{1-q}$ which is called a q-analogue of ∞.

Note that $[\infty]_q$ approaches 1 when q goes to 0 and goes to $+\infty$ when q approaches 1.

We recall some usual notations used in the q-theory.

$$(a+b)_q^n = \prod_{i=0}^{n-1}(a + q^i b), \ \forall\, n \in \mathbb{N},$$

$$(1+a)_q^\infty = \prod_{i=0}^{\infty}(1 + q^i a),$$

$$(1+a)_q^t = \frac{(1+a)_q^\infty}{(1+q^t a)_q^\infty}, \ \forall\, t \in \mathbb{R}.$$

A right inverse of the q-derivative is obtained via the Jackson integral.

For $a,\ b \in \mathbb{R}$ the Jackson integral or q-integral of $f : \mathbb{R} \to \mathbb{R}$ on $[a, b]$ is defined by

$$\int_a^b f(x) d_q x = (1-q) \sum_{n=0}^{\infty} q^n \left(b f(q^n b) - a f(q^n a) \right).$$

It is clear if one lets q approaches 1, then the q-derivative approaches the Newton derivative and the Jackson integral approaches the Riemann integral. The q-analogue of the integration theorem by a variable change is given by

$$\text{(1)} \qquad \int_{u(a)}^{u(b)} f(u) d_q u = \int_a^b f(u(x)) d_{q^{1/\beta}} u(x), \quad \text{where} \quad u(x) = \alpha x^{\beta}.$$

The q-analogue of the rule of integration by parts is

$$\text{(2)} \qquad \int_a^b g(x) D_q f(x) d_q x = [f(x) g(x)]_a^b - \int_a^b f(qx) D_q g(x) d_q x.$$

For any function $f(x)$ continuous at $x = 0$, we have

$$\text{(3)} \qquad \int_0^a D_q f(x) d_q x = f(a) - f(0) \text{ and } D_q \int_0^x f(t) d_q t = f(x).$$

Jackson in [19] proposed the q-analogue of the exponential function e^x given by

$$e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}.$$

It is clear that $e_q^0 = 1$ and $D_q e_q^x = e_q^x$.

The q-analogue of the identity $e^x e^{-x} = 1$ is $e_q^x E_q^{-x} = 1$, where the function E_q^x defined by $e_{1/q}^x$ is given also by

$$e_{1/q}^x = E_q^x = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{x^n}{[n]_q!}.$$

The q-logarithm function $\log_q(x)$ is the inverse of the q-exponential function e_q^x, and the function $\mathrm{Log}_q(x)$ is the inverse function of E_q^x.

In 1994 Chung et al. [10] proposed the q-addition operator and discussed its properties. The q-addition operator is defined by

$$\begin{cases} (a \oplus_q b)^n = \displaystyle\sum_{k=0}^{n} {}_qC_k^n a^k b^{n-k}, \ \forall\ n \in \mathbb{N}, \quad (a \neq b,) \\ (a \oplus_q a)^n = (a + a)^n = 2^n a^n, \end{cases}$$

where

$${}_qC_k^n = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

From the above definition, we have the following property:

$$k(a \oplus_q b) = ka \oplus_q kb, \ \forall\ k \in \mathbb{R}.$$

It is easy to see that this operator is commutative, i.e., $a \oplus_q b = b \oplus_q a$. In addition, if we take $b = a$, then we get $a \oplus_q a = a + a = 2a$. Finally if we take $b = 0$, we obtain $a \oplus_q 0 = 0 \oplus_q a = a$.

The q-subtraction is defined as $a \ominus_q b = a \oplus_q (-b)$, $a \neq b$ and if we take $b = a$, then we have $a \ominus_q a = a - a = 0$.

This operator permits to express the properties of q-logarithm and q-exponential functions in a more compact form:

(i) $e_q^a e_q^b = e_q^{a \oplus_q b}$,
(ii) $e_q^{na} = (e_q^a)^n$, $\forall n \in \mathbb{N}$,
(iii) $\log_q(ab) = \log_q(a) \oplus_q \log_q(b)$,
(iv) $\log_q(a^n) = n \log_q(a)$, $\forall n \in \mathbb{N}$.

The only function f verified that $f(x \oplus_q y) = f(x)f(y)$ is the q-exponential function [7]. The function $\mathrm{Log}_q(x)$ is the inverse function to E_q^x, and it satisfies the same mentioned above properties as $\log_q(x)$.

2.1. The gamma and beta q-distributions

Jackson in [19] showed that the q-beta function has the q-integral representation, which is a q-analogue of Euler's formula:

$$\beta_q(t, s) = \int_0^1 x^{t-1}(1 - qx)_q^{s-1} d_q x, \ \forall\ t,\ s > 0. \tag{4}$$

The q-gamma function expressed as Γ_q is defined in [19] by

$$\Gamma_q(t) = \int_0^{\frac{1}{1-q}} x^{t-1} E_q^{-qx} d_q x, \ \forall\ t > 0. \tag{5}$$

Jackson [19] proved the following properties of the q-gamma function:

$$\Gamma_q(1) = 1, \tag{6}$$

$$\Gamma_q(t + 1) = [t]_q \Gamma_q(t), \ \forall\ t > 0, \tag{7}$$

$$\Gamma_q(n + 1) = [n]_q!, \ \forall n \in \mathbb{N}. \tag{8}$$

The relationship between the q-gamma and the q-beta functions is given by

$$\beta_q(t, s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t + s)}, \ \forall\ t,\ s > 0. \tag{9}$$

Díaz et al. in [12] defined the gamma q-distribution in $[0, \frac{1}{1-q}]$ by

$$\gamma_{q,a}(x) = \frac{x^{a-1} E_q^{-qx}}{\Gamma_q(a)} \mathbf{1}_{[0, \frac{1}{1-q}]}(x), \tag{10}$$

and the beta q-distribution in $[0, 1]$ by

$$\beta_q(a, b)(x) = \frac{x^{a-1}(1 - qx)_q^{b-1}}{\beta_q(a, b)} \mathbf{1}_{[0,1]}(x). \tag{11}$$

Díaz et al. in [13] defined the q-moment of a random variable X with q-density function f on $[a, b]$, by

$$_qM(n) = \int_a^b x^n f(x) d_q x.$$

In particular, the q-expectation of X is given by

$$\mathbb{E}_q(X) = \int_a^b x f(x) d_q x = {}_qM_1.$$

3. Joint q-density function and independence marginal q-distribution

3.1. Joint q-density functions

In the univariate continuous case, Díaz et al. [13] identified the Gaussian q-distribution. Now, we introduce the bivariate q-density function of two random variables by using Fubini's Theorem. Al-Ashwal in [6] proved the Leibniz's Rule and Fubini's Theorem in quantum calculus.

Theorem 3.1. *Let f be a function defined on the closed rectangle $R = [0, a] \times [0, b]$. Assuming that $f(t, s)$ is continuous at $t = 0$ uniformly and continuous at $s = 0$ uniformly. Then, the double q-integrals*

$$\int_0^b \int_0^a f(t, s) d_q t d_q s \text{ and } \int_0^a \int_0^b f(t, s) d_q s d_q t$$

exist and they are equal, that is

$$\int_0^b \int_0^a f(t, s) d_q t d_q s = \int_0^a \int_0^b f(t, s) d_q s d_q t$$

with

$$\int_0^a \int_0^b f(t, s) d_q s d_q t = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} q^{j+k} ab(1 - q)^2 f(q^j a, q^k b).$$

According to Theorem 3.1, we introduce the notion of joint q-density function.

Definition 1. Let (X, Y) be a couple of positive random variables with continuous joint q-density function $f_{X,Y}(x, y)$. Then the q-cumulative function is given by

$$\mathbb{P}_q(X \le s, Y \le t) = \int \int_{x \le s, y \le t} f_{X,Y}(x, y) d_q x d_q y.$$

Using Fubini's Theorem, we can also write the q-integral as

$$\begin{aligned} \mathbb{P}_q(X \le s, Y \le t) &= \int_0^s \left(\int_0^t f_{X,Y}(x, y) d_q y \right) d_q x \\ &= \int_0^t \left(\int_0^s f_{X,Y}(x, y) d_q x \right) d_q y. \end{aligned}$$

Note that as in the classical case, the q-joint density function $f_{X,Y}(x,y)$ verifies

$$\begin{cases} f_{X,Y}(x,y) \geq 0, \\ \displaystyle\iint f_{X,Y}(x,y) d_qx d_qy = 1. \end{cases}$$

3.2. Independence and marginal q-distributions

Now, we are able to provide some properties of q-independence and marginal q-distributions.

Definition 2. Let (X,Y) be a couple of a random variables with joint q-density function $f_{X,Y}(x,y)$. Then the marginal q-densities of X and Y are given respectively by

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y) d_qy,$$
$$f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y) d_qx.$$

Definition 3. Let (X,Y) be a couple of random variables with joint q-density function $f_{X,Y}(x,y)$. The random variables X and Y are called independent if

$$f_{X,Y}(x,y) = f_X(x) f_Y(y), \ \forall x, \ y.$$

Definition 4. The q-expectation of XY (if it exists) is given by

$$\mathbb{E}_q(XY) = \int\int xy f_{X,Y}(x,y) d_qx d_qy.$$

Note that if X and Y are independent, then

$$\mathbb{E}_q(XY) = \mathbb{E}_q(X)\mathbb{E}_q(Y).$$

In fact, since X and Y are independent, then $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. Hence,

$$\begin{aligned} \mathbb{E}_q(XY) &= \int\int xy f_X(x) f_Y(y) d_qx d_qy \\ &= \int x f_X(x) d_qx \int y f_Y(y) d_qy \\ &= \mathbb{E}_q(X)\mathbb{E}_q(Y). \end{aligned}$$

Studies on q-Laplace transform go back to W. H. Abdi in [1]. Research in the area was pursued in many works by W. H. Abdi in [2–4] and more recently in [7]. The q-version of the Laplace transform consists in choosing a q-version of the exponential function $e^{-\theta x}$ and then replacing the integral by the corresponding q-integral.

Definition 5. Let f be a q-density function with bounded support $[0, A]$ and let X be a random variable with q-density function f, the q-Laplace transform of X is defined by

$$
\begin{aligned}
{}_qL_X(\theta) &= \int_0^A E_q^{-\theta x} f(x) d_q x \\
&= \mathbb{E}_q(E_q^{-\theta X}).
\end{aligned}
$$

Now, we extend the notion of q-Laplace transform in bivariate case.

Definition 6. Let (X,Y) be a couple of a positive random variables with joint q-density function $f(x,y)$ defined on $[0,A]\times[0,B]$.

The q-Laplace transform of f is given by

$$
\begin{aligned}
{}_qL_{(X,Y)}(\theta_1,\theta_2) &= \int_0^A \int_0^B E_q^{-(\theta_1 x \oplus_q \theta_2 y)} f(x,y) d_q x d_q y \\
&= \mathbb{E}_q(E_q^{-(\theta_1 X \oplus_q \theta_2 Y)}).
\end{aligned}
$$

4. Characterizing the gamma and the beta q-distributions

Díaz et al. [12] identified the gamma and the beta q-distributions. The gamma q-density function [4] is defined by

$$
\gamma_{q,a}(x) = \frac{1}{\Gamma_q(a)} x^{a-1} E_q^{-qx} \mathbf{1}_{[0,\frac{1}{1-q}]}(x),
$$

with q-moment

$$
M_q(n) = \frac{\Gamma_q(a+n)}{\Gamma_q(a)}.
$$

Now, we will characterize the gamma and the beta q-distributions. This characterization is based on the following two technical lemmas:

Lemma 4.1. *Let $X_1,\ X_2,\ X_3$ be three independent real random variables, with common bounded support $[0,A]$ and let*

$$
Z_1 = X_1 \ominus_q X_3 \text{ and } Z_2 = X_2 \ominus_q X_3.
$$

Then, the q-distribution of (Z_1,Z_2) determines the q-distributions of X_1, X_2, X_3.

Proof. Let ${}_qL_{(Z_1,Z_2)}(\theta_1,\theta_2)$ be the q-Laplace transform function of the pair (Z_1,Z_2), and let ${}_qL_{X_k}(\theta)$ be the q-Laplace transform function of X_k $(k=1,2,3)$.

Assuming that X_1, X_2, X_3 are independent, we obtain

$$
\begin{aligned}
{}_qL_{(Z_1,Z_2)}(\theta_1,\theta_2) &= \mathbb{E}_q(E_q^{-(\theta_1 Z_1 \oplus_q \theta_2 Z_2)}) \\
&= \mathbb{E}_q(E_q^{-\theta_1(X_1\ominus_q X_3)\ominus_q \theta_2(X_2 \ominus_q X_3)}) \\
&= {}_qL_{X_1}(\theta_1)\,{}_qL_{X_2}(\theta_2)\,{}_qL_{X_3}(-\theta_1 \ominus_q \theta_2).
\end{aligned}
$$

Let U_1, U_2 and U_3 be three independent random variables.

If we take $V_1 = U_1 \ominus_q U_3$ and $V_2 = U_2 \ominus_q U_3$, then

$$
{}_qL_{(V_1,V_2)}(\theta_1,\theta_2) =_q L_{U_1}(\theta_1)\,{}_qL_{U_2}(\theta_2)\,{}_qL_{U_3}(-\theta_1 \ominus_q \theta_2).
$$

If (Z_1, Z_2) and (V_1, V_2) have the same distribution, then their q-Laplace transform coincides.

$$\begin{aligned} (12) \qquad & {}_qL_{X_1}(\theta_1)L_{q,X_2}(\theta_2)L_{q,X_3}(-\theta_1 \ominus_q \theta_2) \\ &= {}_qL_{U_1}(\theta_1)L_{q,U_2}(\theta_2)L_{q,U_3}(-\theta_1 \ominus_q \theta_2). \end{aligned}$$

Let $p_1,\ p_2,\ p_3$ be three continuous functions such that

$$(13) \qquad {}_qL_{U_1}(\theta_1) =_q L_{X_1}(\theta_1)p_1(\theta_1),$$

$$(14) \qquad {}_qL_{U_2}(\theta_2) =_q L_{X_2}(\theta_2)p_2(\theta_2), \text{ and}$$

$$(15) \qquad {}_qL_{U_3}(-\theta_1 \ominus_q \theta_2) =_q L_{X_3}(-\theta_1 \ominus_q \theta_2)p_3(-\theta_1 \ominus_q \theta_2).$$

Then, equation (12) becomes

$$\begin{aligned} (16) \qquad & {}_qL_{X_1}(\theta_1)_qL_{X_2}(\theta_2)_qL_{X_3}(-\theta_1 \ominus_q \theta_2) \\ &= {}_qL_{X_1}(\theta_1)p_1(\theta_1)_qL_{X_2}(\theta_2)p_2(\theta_2)_qL_{X_3}(-\theta_1 \ominus_q \theta_2)p_3(-\theta_1 \ominus_q \theta_2). \end{aligned}$$

Which gives

$$(17) \qquad p_1(\theta_1)p_2(\theta_2)p_3(-\theta_1 \ominus_q \theta_2) = 1,$$

with $p_k(\theta)$ is any function.

Note that

$$(18) \qquad {}_qL_{U_k}(0) = {}_qL_{X_k}(0)p_k(0),\ k \in \{1,\ 2,\ 3\}.$$

Hence,

$$(19) \qquad p_k(0) = 1 \quad \text{for } k \in \{1,\ 2,\ 3\}.$$

In order to solve (17) we take $\theta_1 = \theta$ and $\theta_2 = 0$. So, we obtain

$$(20) \qquad p_1(\theta)p_3(-\theta) = 1.$$

Setting $\theta_2 = \theta$ and $\theta_1 = 0$, we obtain

$$(21) \qquad p_2(\theta)p_3(0 \ominus_q \theta) = p_2(\theta)p_3(-\theta) = 1.$$

Therefore

$$\begin{cases} p_1(\theta) = \frac{1}{p_3(-\theta)}, \\ p_2(\theta) = \frac{1}{p_3(-\theta)}. \end{cases}$$

By inserting them in (17), we have

$$(22) \qquad p_3(\theta_1 \oplus_q \theta_2) = p_3(\theta_1)p_3(\theta_2).$$

The only function that checks (19) and (22) is the q-exponential function. Hence, $p_k(\theta) = E_q^{C\theta},\ k \in \{1,\ 2,\ 3\}$, with C is a constant.

Based on the equation (13), we obtain

$$(23) \qquad {}_qL_{U_k}(\theta) = E_q^{C\theta}\,{}_qL_{X_k}(\theta) = {}_qL_{X_k \oplus_q C}(\theta),\ k \in \{1,\ 2,\ 3\}.$$

Since X_k and U_k are concentrated on the bounded interval $[0, A]$, then it's necessary that $C = 0$.

Therefore X_k and U_k have the same q-density function. As matter of fact, we deduce that

$$ {}_qL_{U_k}(\theta) = {}_qL_{X_k}(\theta), \ k \in \{1,\ 2,\ 3\}. \tag{24} $$ □

Lemma 4.2. *Let $X_1,\ X_2,\ X_3$ be three positive independent real random variables with common bounded support $[0, A]$.*

Let (Y_1, Y_2) be a couple of random variables such that $Y_1 = \frac{X_1}{X_3}$ and $Y_2 = \frac{X_2}{X_3}$. Then, the q-Laplace transform of the couple $(\mathrm{Log}_q(Y_1), \mathrm{Log}_q(Y_2))$ determines the distribution of $X_1,\ X_2,\ X_3$.

Proof. The proof of Lemma 4.2 is based on the proof of Lemma 4.1 because $\mathrm{Log}_q(X_k)(k = 1,\ 2,\ 3)$ satisfies the condition of Lemma 4.1. □

Theorem 4.3. *Let $X_1,\ X_2,\ X_3$ be three positive independent real random variables with common bounded support $[0, \frac{1}{1-q}]$ and let $Y_1 = \frac{X_1}{X_3}$ and $Y_2 = \frac{X_2}{X_3}$.*

The necessary and sufficient condition for X_k to be q-gamma distributed, $\gamma_q(p_k)$, with parameters p_k $(k = 1,\ 2,\ 3)$ is that the q-Laplace transform of the couple $(\mathrm{Log}_q Y_1, \mathrm{Log}_q Y_2)$ has the following expression

$$ \frac{\Gamma_q(p_1 - \theta_1)}{\Gamma_q(p_1)} \times \frac{\Gamma_q(p_2 - \theta_2)}{\Gamma_q(p_2)} \times \frac{\Gamma_q(p_3 + \theta_1 \oplus_q \theta_2)}{\Gamma_q(p_3)}. $$

Proof. " $\Rightarrow$ " Let $X_k \sim \gamma_q(p_k)$ $(k = 1,\ 2,\ 3)$. The q-Laplace transform of $\mathrm{Log}_q X_k$ is:

$$ \begin{aligned} {}_qL_{\mathrm{Log}_q X_1}(\theta_1) &= \int_0^{\frac{1}{1-q}} x_1^{-\theta_1} \frac{x_1^{p_1-1} E_q^{-qx_1}}{\Gamma_q(p_1)} d_q x_1 \\ &= \int_0^{\frac{1}{1-q}} x_1^{-\theta_1+p_1-1} \frac{1}{\Gamma_q(p_1)} E_q^{-qx_1} d_q x_1 \\ &= \frac{1}{\Gamma_q(p_1)} \int_0^{\frac{1}{1-q}} x_1^{p_1-\theta_1-1} E_q^{-qx_1} d_q x_1 \\ &= \frac{\Gamma_q(p_1 - \theta_1)}{\Gamma_q(p_1)}. \end{aligned} $$

In the same way, we show that:

$$ {}_qL_{\mathrm{Log}_q X_2}(\theta_2) = \frac{\Gamma_q(p_2 - \theta_2)}{\Gamma_q(p_2)}, $$

and

$$ {}_qL_{\mathrm{Log}_q X_3}(-\theta_1 \ominus_q \theta_2) = \frac{\Gamma_q(p_3 + \theta_1 \oplus_q \theta_2)}{\Gamma_q(p_3)}. $$

The q-Laplace transform of the couple $(\mathrm{Log}_q Y_1, \mathrm{Log}_q Y_2)$ is

$$ \begin{aligned} {}_qL_{(\mathrm{Log}_q Y_1, \mathrm{Log}_q Y_2)}(\theta_1, \theta_2) &= {}_qL_{\mathrm{Log}_q X_1}(\theta_1)\, {}_qL_{\mathrm{Log}_q X_2}(\theta_2)\, {}_qL_{\mathrm{Log}_q X_3}(-\theta_1 \ominus_q \theta_2) \\ &= \frac{\Gamma_q(p_1 - \theta_1)}{\Gamma_q(p_1)} \times \frac{\Gamma_q(p_2 - \theta_2)}{\Gamma_q(p_2)} \times \frac{\Gamma_q(p_3 + \theta_1 \oplus_q \theta_2)}{\Gamma_q(p_3)} \end{aligned} $$

" $\Leftarrow$ " By using Lemma 4.2 and the q-Laplace transform of the pair ($\mathrm{Log}_q Y_1$, $\mathrm{Log}_q Y_2$), we obtain the desired result. □

Several characterizations of the beta distribution have been determined [5, 20]. At this level, we are able to characterize the beta q-distribution by the same way as the gamma q-distribution. The beta q-density function [12] is expressed as

$$f(x) = \frac{x^{a-1}(1-qx)_q^{b-1}}{\beta_q(a,b)} \mathbf{1}_{[0,1]}(x).$$

Theorem 4.4. *Let X_1, X_2, X_3 be three independent random variables with support $[0,1]$ and let $Y_1 = \frac{X_1}{X_3}$ and $Y_2 = \frac{X_2}{X_3}$.*

The necessary and sufficient condition for X_1, X_2, X_3 to be q-beta distributed with parameters (p_i, p_{i+1}) for $\{i = 1, 3, 5\}$ is that the q-Laplace transform of the couple $(\mathrm{Log}_q Y_1, \mathrm{Log}_q Y_2)$ has the following expression

$$\frac{\beta_q(p_1-\theta_1, p_2)}{\beta_q(p_1, p_2)} \times \frac{\beta_q(p_3-\theta_2, p_4)}{\beta_q(p_3, p_4)} \times \frac{\beta_q(p_5+\theta_1 \oplus_q \theta_2, p_6)}{\beta_q(p_5, p_6)}.$$

Proof. " $\Rightarrow$ " Let $X_1 \sim \beta_q(p_1, p_2)$, the q-Laplace transform of $\mathrm{Log}_q X_1$ is

$$\begin{aligned} {}_qL_{\mathrm{Log}_q X_1}(\theta_1) &= \frac{1}{\beta_q(p_1,p_2)} \int_0^1 x^{p_1-\theta_1}(1-qx)_q^{p_2-1} d_q x \\ &= \frac{\beta_q(p_1-\theta_1, p_2)}{\beta_q(p_1, p_2)}. \end{aligned}$$

In the same way, we show that

$${}_qL_{\mathrm{Log}_q X_2}(\theta_2) = \frac{\beta_q(p_3-\theta_2, p_4)}{\beta_q(p_3, p_4)},$$

and

$${}_qL_{\mathrm{Log}_q X_3}(-\theta_1 \ominus_q \theta_2) = \frac{\beta_q(p_5+\theta_1 \oplus_q \theta_2, p_6)}{\beta_q(p_5, p_6)}.$$

Now, we compute the q-Laplace transform of the couple $(\mathrm{Log}_q Y_1, \mathrm{Log}_q Y_2)$ with $Y_1 = \frac{X_1}{X_3}$ and $Y_2 = \frac{X_2}{X_3}$.

$$\begin{aligned} &{}_qL_{(\mathrm{Log}_q Y_1, \mathrm{Log}_q Y_2)}(\theta_1, \theta_2) \\ &= {}_qL_{\mathrm{Log}_q X_1}(\theta_1)\, {}_qL_{\mathrm{Log}_q X_2}(\theta_2)\, {}_qL_{\mathrm{Log}_q X_3}(-\theta_1 \ominus_q \theta_2) \\ &= \frac{\beta_q(p_1-\theta_1, p_2)}{\beta_q(p_1, p_2)} \times \frac{\beta_q(p_3-\theta_2, p_4)}{\beta_q(p_3, p_4)} \times \frac{\beta_q(p_5+\theta_1 \oplus_q \theta_2, p_6)}{\beta_q(p_5, p_6)}. \end{aligned}$$

" $\Leftarrow$ " At this stage, if we take the q-distribution of the couple (Y_1, Y_2) with q-Laplace transform of the pair $(\mathrm{Log}_q Y_1, \mathrm{Log}_q Y_2)$ given by

$$\begin{aligned} &{}_qL_{(\mathrm{Log}_q Y_1, \mathrm{Log}_q Y_2)}(\theta_1, \theta_2) \\ &= \frac{\beta_q(p_1-\theta_1, p_2)}{\beta_q(p_1, p_2)} \times \frac{\beta_q(p_3-\theta_2, p_4)}{\beta_q(p_3, p_4)} \times \frac{\beta_q(p_5+\theta_1 \oplus_q \theta_2, p_6)}{\beta_q(p_5, p_6)}, \end{aligned}$$

and using Lemma 4.2, then the proof is complete. □

5. Conclusion

In this work, we have characterized the gamma and the beta q-distributions families. This characterization is based on the q-Laplace transform and q-independence properties. Departing from this result, we estimate that the gamma q-distribution would be characterized with the bivariate beta q-distribution in subsequent research. Having explored the gamma q-distribution, our work is a step that may be taken further. In future works, we aspire to introduce the Dirichlet q-distribution with their two kinds as well as to explore its characterization.

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