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# ON THE MATCHING NUMBER AND THE INDEPENDENCE NUMBER OF A RANDOM INDUCED SUBHYPERGRAPH OF A HYPERGRAPH

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Abstract. For $r \geq 2$, let $\mathcal{H}$ be an $r$-uniform hypergraph with $n$ vertices and $m$ hyperedges. Let $R$ be a random vertex set obtained by choosing each vertex of $\mathcal{H}$ independently with probability $p$. Let $\mathcal{H}[R]$ be the subhypergraph of $\mathcal{H}$ induced on $R$.

We obtain an upper bound on the matching number $\nu(\mathcal{H}[R])$ and a lower bound on the independence number $\alpha(\mathcal{H}[R])$ of $\mathcal{H}[R]$. First, we show that if $mp^r \geq \log n$, then $\nu(\mathcal{H}[R]) \leq 2e^\ell mp^r$ with probability at least $1 - 1/n^\ell$ for each positive integer $\ell$. It is best possible up to a constant factor depending only on $\ell$ if $m \leq n/r$. Next, we show that if $mp^r \geq \log n$, then $\alpha(\mathcal{H}[R]) \geq np - \sqrt{3\ell np \log n} - 2re^\ell mp^r$ with probability at least $1 - 3/n^\ell$.

## 1. Introduction

For an integer $r \geq 2$, an $r$-uniform hypergraph $\mathcal{H}$ is a pair $(V, E)$ in which the set $V$ is a set of vertices and the set $E$ is a family of $r$-subsets of $V$ called hyperedges. A *matching* of $\mathcal{H}$ is a family of pairwise disjoint hyperedges. The *matching number* $\nu(\mathcal{H})$ of $\mathcal{H}$ is the size of a maximum matching of $\mathcal{H}$. There is a polynomial time algorithm that gives a maximum matching for a given graph. However, for $r \geq 3$, the problem of finding a maximum matching contained in an $r$-uniform hypergraph $\mathcal{H}$ is known as an NP-hard problem. On the other hand, an *independent set* of a hypergraph $\mathcal{G}$ is a vertex set which does not contain a hyperedge. The *independence number* $\alpha(\mathcal{G})$ of $\mathcal{G}$ is the size of a maximum independent set of $\mathcal{G}$.

A well-known question suggested by Erdős [2] was to determine the largest possible number of hyperedges in any $r$-uniform hypergraph $\mathcal{H}$ with given matching number. There are recent results by Huang, Loh and Sudakov [3] and by Alon, Frankl, Huang, Rödl, Ruciński and Sudakov [1].

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One may ask a question about the converse: What is the possible matching number for a hypergraph $\mathcal{H}$ with given numbers of vertices and edges? In the paper, we deal with the random subset version of the question. Let $\mathcal{H}$ be a uniform hypergraph with $n$ vertices and $m$ edges. We consider a random vertex set $R \subset V$ and estimate the matching number of the subhypergraph of a uniform hypergraph $\mathcal{H}$ induced on $R$ with high probability.

We define the random set $R$ more precisely and introduce notation. Let $R$ be a random vertex set obtained by choosing each vertex of $\mathcal{H}$ independently with probability $p$. Let $\mathcal{H}[R]$ be the sub-hypergraph of $\mathcal{H}$ induced on $R$. Recall that $\nu(\mathcal{H}[R])$ denotes the matching number of $\mathcal{H}[R]$. The goal of this paper is to estimate the least upper bound on $\nu(\mathcal{H}[R])$ with high probability, that is, with probability $1 - o(1)$, where $o(1)$ goes to 0 as $n \to \infty$.

We will consider a stronger version of high probability. That is a probability at least $1 - 1/n^\ell$ for a given positive integer $\ell$. This version is useful in many situations, especially when the union bound or the Borel–Cantelli Lemma is applied. In order to apply the union bound, we hope to have that the sum of the probabilities of the compliments of the given events is less than 1. If the number of events is $n^k$, we have

$$\sum_{i=1}^{n^k} \frac{1}{n^\ell} = \frac{1}{n} = o(1)$$

by taking $\ell = k + 1$. Next, in order to use the Borel–Cantelli lemma, we hope to have that the sum of the probabilities of the complements of infinitely many events is finite. We have

$$\sum_{n=1}^{\infty} \frac{1}{n^\ell} < \infty$$

for $\ell \geq 2$.

Now we formalize our goal. For integers $\ell \geq 1, r \geq 2, n, m$ and a real number $p$, let

$$f_\ell = f_\ell(r, n, m, p)$$

be the minimum value of $k$ such that, for any $r$-uniform hypergraph $\mathcal{H}$ with $n$ vertices and $m$ edges,

$$\nu(\mathcal{H}[R]) \leq k$$

with probability at least $1 - 1/n^\ell$. The goal of this paper is to estimate $f_\ell = f_\ell(r, n, m, p)$. It turns out that an important parameter for a bound on the matching number of $\mathcal{H}[R]$ is the expected number of hyperedges in $\mathcal{H}[R]$, that is,

$$\mathbb{E}(|\mathcal{H}[R]|) = mp^r.$$

We first show two propositions. One is about a lower bound on $f_\ell$, and the other is about an upper bound on $f_\ell$.

**Proposition 1.** *Let $\ell \geq 1$, $r \geq 2$, $m \leq n/r$ and $mp^r \geq 3/\varepsilon^2$ for a positive constant $\varepsilon$. We have that*

$$f_\ell > (1-\varepsilon)mp^r.$$

*Proof.* Let $\mathcal{G}$ be an $r$-uniform hypergraph with $n$ vertices and $m$ 'vertex-disjoint' hyperedges ($m \leq n/r$). Then, clearly

$$\nu\left(\mathcal{G}[R]\right) = |\mathcal{G}[R]|,$$

and hence, it suffices to estimate $|\mathcal{G}[R]|$. We clearly have that

$$\mathbb{P}\Big[|\mathcal{G}[R]| \leq (1-\varepsilon)mp^r\Big] \leq \mathbb{P}\Big[\big||\mathcal{G}[R]| - mp^r\big| \geq \varepsilon mp^r\Big].$$

Since each hyperedge is contained in $R$ independently with probability $p^r$, a version of Chernoff's bound (see Lemma 6) implies that

$$\mathbb{P}\Big[|\mathcal{G}[R]| \leq (1-\varepsilon)mp^r\Big] \leq 2\exp\left(-\frac{\varepsilon^2 mp^r}{3}\right) \leq \frac{2}{e} < 0.9,$$

where the second inequality follows from $mp^r \geq 3/\varepsilon^2$. This yields Proposition 1. $\square$

**Proposition 2.** *For $\ell \geq 1$, $r \geq 2$ and $n, m, p > 0$, we have that*

$$f_\ell \leq n^\ell mp^r.$$

*Proof.* For an arbitrary $r$-uniform hypergraph $\mathcal{H}$, we clearly have that

$$\nu(\mathcal{H}[R]) \leq |\mathcal{H}[R]|,$$

and hence,

$$\mathbb{P}\Big[\nu(\mathcal{H}[R]) \geq n^\ell mp^r\Big] \leq \mathbb{P}\Big[|\mathcal{H}[R]| \geq n^\ell mp^r\Big].$$

Since $n^\ell mp^r > 0$, Markov's inequality gives that

$$\mathbb{P}\Big[\nu(\mathcal{H}[R]) \geq n^\ell mp^r\Big] \leq \frac{1}{n^\ell},$$

which completes the proof of Proposition 2. $\square$

The lower and upper bounds in Propositions 1 and 2 have a large multiplicative gap as $n^\ell$. The following main result gives an improved upper bound to match the lower bound in Proposition 1 up to a constant factor depending only on $\ell$.

**Theorem 3.** *Let $\ell \geq 1$, $r \geq 2$, $n, m, p > 0$ and $mp^r \geq \log n$. We have that*

$$f_\ell \leq 2e^\ell mp^r.$$

*In other words, for an $r$-uniform hypergraph $\mathcal{H}$ with $n$ vertices and $m$ edges and a random subset $R \subset V$, we have that*

$$\nu(\mathcal{H}[R]) \leq 2e^\ell mp^r$$

*with probability at least $1 - 1/n^\ell$.*

The proof of Theorem 3 will be provided in Section 2.

The result on the matching number $\nu(\mathcal{H}[R])$ can be applied to estimating the independence number of $\mathcal{H}[R]$. Theorem 3 yields the following.

**Theorem 4.** *Let $\ell \geq 1$, $r \geq 2$, $n, m, p > 0$ and $mp^r \geq \log n$. For an $r$-uniform hypergraph $\mathcal{H}$ with $n$ vertices and $m$ edges and a random subset $R \subset V$, we have that*

$$\alpha(\mathcal{H}[R]) \geq np - \sqrt{3\ell np \log n} - 2re^{\ell}mp^r$$

*with probability at least $1 - 3/n^{\ell}$.*

The proof of Theorem 4 will be given in Section 3.

## 2. Proof of Theorem 3

For a proof of Theorem 3, we first show the following lemma.

**Lemma 5.** *Let $r \geq 2$ and $n, m, p > 0$. Let $\mathcal{H}$ be an $r$-uniform hypergraph with $n$ vertices and $m$ edges. For any real number $\lambda > 1$, we have that*

$$\nu(\mathcal{H}[R]) < 2\lambda mp^r$$

*with probability at least $1 - \lambda^{-mp^r}$.*

*Proof.* For convenience, let $\nu_{_R} := \nu(\mathcal{H}[R])$. We clearly have that

$$\mathbb{P}\Big[\nu_{_R} \geq 2\lambda mp^r\Big] \leq \mathbb{P}\Big[\nu_{_R} \geq 2\lambda mp^r \;\Big|\; \nu_{_R} \geq 2mp^r\Big].$$

Hence, in order to show Lemma 5, it suffices to show that

$$\mathbb{P}\Big[\nu_{_R} \geq 2\lambda mp^r \;\Big|\; \nu_{_R} \geq 2mp^r\Big] \leq \frac{1}{\lambda^{mp^r}}. \tag{1}$$

To deal with the conditional probability, from now on, we assume that

$$\nu_{_R} \geq 2mp^r.$$

Let $X = X(\mathcal{H}[R])$ denote the number of all matchings in $\mathcal{H}[R]$ of size

$$\mu = mp^r.$$

We will consider lower and upper bounds on $X$ separately. First, we consider a lower bound on $X$. By the definition of $\nu_{_R}$, the hypergraph $\mathcal{H}[R]$ contains a matching $M$ of size $\nu_{_R}$. If we arbitrarily choose $\mu$ hyperedges from $M$, it forms a matching of size $\mu$. Hence, we have that

$$X \geq \binom{\nu_{_R}}{\mu} \geq \frac{(\nu_{_R} - \mu)^{\mu}}{\mu!} = \frac{\nu_{_R}^{\mu}}{\mu!}\left(1 - \frac{\mu}{\nu_{_R}}\right)^{\mu}.$$

The assumption $\nu_{_R} \geq 2\mu$ gives that $1 - \mu/\nu_{_R} \geq 1/2$, and hence,

$$X \geq \frac{\nu_{_R}^{\mu}}{\mu!\, 2^{\mu}}. \tag{2}$$

Next, we consider an upper bound on $X$. The number of matchings in $\mathcal{H}$ of size $\mu$ is clearly at most $\binom{m}{\mu}$, and each matching is contained in $R$ with probability $p^{r\mu}$. Hence,

$$\mathbb{E}\left[X \;\middle|\; \nu_{_R} \geq 2mp^r\right] \leq \binom{m}{\mu} p^{r\mu} \leq \frac{m^\mu}{\mu!} p^{r\mu} = \frac{(mp^r)^\mu}{\mu!}.$$

Since $mp^r > 0$, Markov's inequality gives that

$$\mathbb{P}\left[X \geq \frac{(\lambda mp^r)^\mu}{\mu!} \;\middle|\; \nu_{_R} \geq 2mp^r\right] \leq \frac{1}{\lambda^\mu} = \frac{1}{\lambda^{mp^r}}.$$

In other words,

$$X < \frac{(\lambda mp^r)^\mu}{\mu!} \tag{3}$$

with probability at least $1 - \lambda^{-mp^r}$.

Therefore, combining (2) and (3) gives that under the assumption $\nu_{_R} \geq 2\mu$ we have that

$$\nu_{_R} < 2\lambda mp^r$$

with probability at least $1 - \lambda^{-mp^r}$. It is equivalent to (1), which completes the proof of Lemma 5. $\square$

Lemma 5 immediately yields Theorem 3.

*Proof of Theorem 3.* We set $\lambda = e^\ell$, and then

$$\lambda^{mp^r} \geq (e^\ell)^{\log n} = n^\ell,$$

where the first inequality follows from the assumption $mp^r \geq \log n$. Therefore, Lemma 5 implies Theorem 3. $\square$

## 3. Proof of Theorem 4

Let $M$ be a maximum matching of $\mathcal{H}[R]$. Observe that each hyperedge in $E(\mathcal{H}) \setminus M$ shares at least a vertex to a hyperedge of $M$. Thus the vertex set $R \setminus V(M)$ is an independent set of $\mathcal{H}[R]$. Therefore,

$$\alpha(\mathcal{H}[R]) \geq |R| - r \cdot \nu(\mathcal{H}[R]). \tag{4}$$

For an estimate of $|R|$, we use the following version of Chernoff's bound.

**Lemma 6** (**Chernoff's bound**, Corollary 4.6 in [4])**.** *Let $X_i$ be independent random variables such that*

$$\Pr[X_i = 1] = p_i \quad \textit{and} \quad \Pr[X_i = 0] = 1 - p_i,$$

*and let $X = \sum_{i=1}^n X_i$. For $0 < \lambda < 1$,*

$$\mathbb{P}\Big[|X - \mathbb{E}(X)| \geq \lambda \mathbb{E}(X)\Big] \leq 2\exp\Big(-\frac{\lambda^2}{3}\mathbb{E}(X)\Big).$$

Lemma 6, with $X = |R|$ and $\lambda = \sqrt{3\ell \log n/(np)}$, gives that

$$\mathbb{P}\Big[\big||R| - np\big| \geq \sqrt{3\ell np \log n}\Big] \leq \frac{2}{n^{\ell}}.$$

In view of (4), this, together with Theorem 3, completes the proof of Theorem 4.

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